Wed. 10/9 Thurs 10/10 Fri., 10/11	*(C 17)* **12.1.1-.1.2, 12.3.1** E to B; **5.1.1-.1.2** Lorentz Force Law: fields and forces *(C 17)* **5.1.3** Lorentz Force Law: currents	 HW4
Mon. 10/14 Wed. 10/16 Thurs 10/17	**Study Day** *(C 17)* **5.2** Biot-Savart Law	 HW5

Big Picture

If you remember, on the first day of class, I said that the force between two charged particles was

$$\vec{F}_{Q\leftarrow q} = \frac{qQ}{4\pi\varepsilon_o}\frac{\mathscr{r}}{(\vec{\mathscr{r}}\cdot\vec{u})^3}\left\{\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right] + \frac{\vec{V}}{c}\times\left[\hat{\mathscr{r}}\times\left[(c^2 - v^2)\vec{u} + \vec{\mathscr{r}}\times(\vec{u}\times\vec{a})\right]\right]\right\}$$

Where $\vec{u} \equiv c\hat{\mathscr{r}} - \vec{v}$

For convenience, we break this into terms that do and do not depend on the sensing charge's velocity (V). the former we call Electric interaction, the latter we call Magnetic interaction.

<u>Summary</u>

- **This Semester thus far:**
    - **Electric Interaction**
        - **Fields.** First we did so in the context of the Momentum Principle – we found that rather than thinking of Forces directly between two charged objects, it was useful to think of an intermediary – one object establishes a field which in turn communicates the force to the other object.
        - **Potentials / Voltages:** Next, we did so in the context of the Energy Principle – we found that rather than thinking of changing Potential Energies of interacting charged objects, we'd think of one charged object establishing an electric potential, and the other object's motion through it constitutes a change in Potential Energy.
        - **Conceptual Abstractions, worth it:** Electric Field and Electric Potential were conceptual abstractions, and we seldom make life more complicated for ourselves than we have to – in fact, both of these prove powerful and even necessary.

- **Now:**
    - **Magnetic Interactions.** It too is an interaction between charged particles, but its dependent not just on charge, but on *velocity* (both magnitude and

direction). In that way, it is both related to and distinct from the Electric interaction.

- **Magnetic Field.** The magnetic interaction is a little more peculiar than the electric, it is therefore a little more important that we conceptually break it down into bite-sized parts: in the context of the Momentum Principle, again, one could think of the interaction in terms of forces directly between two moving charged particles. But it is again powerful to consider an intermediary – a *magnetic* field that is generated by one moving charge and that influences the other.

Theoretical Framework

We'll start today building the theoretical basics of the Magnetic Interaction. Now, it's a much less intuitive push-pull kind of interaction than is the electric, so we'll then start working to *get familiar* with it.

Magnetic Interaction: The Magnetic Force

- **Motivating Demo:** Two current-carrying wires attracting / repelling each other.

  From our perspective, both wires are electrically neutral, and so we don't see a good reason for their attraction / repulsion. Then again, we're just passive observers in this interaction; we might be able to glean some understanding if we looked at it from the perspective of one of the charges that's involved in the interaction. However, we need to be very careful with how we do this – the classical, Galilean transformation won't do.

- **Magnetism as Relativistic Electric Interaction.**
  - To transition from talking about the electric interaction to talking about the magnetic interaction, we'll actually *transform,* as in transform between reference frames. That will give us some insight into how Electric and Magnetic interactions are related.

Introduction.

- **Motivation – Better understand Magnetism.**
- **Intuition, proes and cons.** Before we start studying some really weird science: your intuition, sense of what's right (to be expected) and wrong (unexpected) is based on your experience. You, and everything you perceptibly interact with measure in the ranges of km to μm. And move from 0 m/s to about 344 m/s. So the rules of your intuition are built on observations in these ranges. They may not be applicable to smaller and faster objects.
- **Einstein's Relativity**

- Einstein revisited the *concept* and *math* of relativity in light of a new and perplexing *observation*. These three pieces proved irreconcilable. Something had to give, and that was the math. Fixing the *math* had profound repercussions on our *concepts* of distance and time measurements. This is the topic of the day.

- **Relativity**
  - **The Concept:** The *concept* of relativity is as old as the hills and more strongly believed than almost any other concept in physics. The basic idea is that the fundamental workings of the universe don't care if I'm standing still while watching them or moving at a constant velocity. For example, if I'm standing in the lounge car of a train, watching a pool game and you're standing on the station platform, watching the same game as the car goes by with some speed v, we both ought to be able to employ conservation of momentum and conservation of energy to accurately describe the collisions. We both should be able to correctly predict whether a ball will go in a pocket or not. The only difference is that I'm talking about the velocities and positions of the balls relative to me (moving with the car), and you're talking about them relative you (not moving with the car).
    - **Postulate 1:** Einstein had faith in this concept, and it is the first postulate of his new relativity – *the laws of physics are the same in any inertial reference frame.* (recall, inertial = not accelerating).
      - **"Special"** the "special" in special relativity denotes that the equations will only apply to the *special* case of inertial reference frames. Einstein went on to tackle the much more difficult *general* relativity – applicable even to non-inertial reference frames.
  - **The (old) Math: Galilean Relativity**
    - The classical *math* that goes along with this is simply that, comparing a pool ball velocity measured by me and that measured by you: $\vec{v}_{ball-you} = \vec{v}_{ball-me} + \vec{v}_{me-you}$.
      - For example, if I say it's moving 20mph ahead, and you say that I'm moving 10mph ahead, then you'd see the ball moving 20 + 10, or 30 mph ahead.
    - Similarly relating my measurement and your measurement of the ball's change in position: $\Delta\vec{x}_{ball-you} = \Delta\vec{x}_{ball-me} + \vec{v}_{me-you} \cdot \Delta t$.
    - These two equations allow us to translate or 'transform' between measurements made from my perspective and those made form your perspective.
    - This math works pretty well, but not perfectly.

- **The Observation: Light Speed in different reference frames**
  - Maxwell's connection between light and electric and magnetic fields struck many people as an *almost* complete description of the phenomenon of light. However, one piece appeared to be missing – a medium through which the light propagates.
  - **Speed of Sound & a medium.** Looking to sound for guidance: sound waves are the pressure fluctuations of air. On an average day, these pressure fluctuations move *through the air* at 344 m/s. Theory and observation agree on the speed of sound *relative to the air.* We'll consider a few examples for sound waves and then compare them with similar examples for light. Say you have a speaker and microphone set up opposite each other on a calm day. You could measure the speed with which sound waves travel from speaker to microphone as 344 m/s. Now say you put the speaker and microphone on an open, flat bed truck and drive 10m/s. Now the speaker and microphone are moving relative to the air. They could measure the speed of sound to be 354 m/s relative to themselves if they drove one way and 334 m/s if they drove the other.
  - **Speed of light & no medium.** Maxwell's equations predicted a value for the speed of light, c. Many figured that this must be the value measured *relative* to the medium through which it traveled. If that was the case, you'd measure different speeds depending on which way you traveled through the medium, just as for sound waves. Perhaps the most famous set of Zero-result experiments were those that set out to find this difference in speed. None was ever found. This left the question dangling, 'relative to *what* do we measure the speed of light?'
  - **Postulate 2: Constancy of the Speed of light.** Einstein came up with the startling answer: anything! No matter how fast you are moving, or in what direction, you will measure light to have the exact same speed relative to you. If you're running at someone, and shining a flashlight at them, you'd measure the speed of light as c relative to you *and* the other person would measure the speed of light as c relative to themselves!
    - **Postulate 2:** *the speed of light is the same measured relative to any inertial reference frame.*
  - This didn't come to him out of the blue. He was working on reconciling the concept of relativity with the mathematical laws of electricity and magnetism. This could be done if he threw out the old transformations of the form:
    - $\vec{v}_{ball-you} = \vec{v}_{ball-me} + \vec{v}_{me-you}$

      $\Delta\vec{x}_{ball-you} = \Delta\vec{x}_{ball-me} + \vec{v}_{me-you} \cdot \Delta t$
  - and replaced them with a new set suggested by the mathematician Lorentz, known as the Lorentz Transformations. That new set only made sense if c were frame-independent.

- This is yet another case where someone else had come-up with a mathematical patch to a perplexing problem of the day, and Einstein had the faith in logical reasoning to embrace patch and realize its profound ramifications.

Where this leads: C is a speed, distance per time. As we'll now see, the only way that c could be measured constant when you changed your own speed is if you compensatingly changed your measures of distance and time! If you change your notion of distance and time, you change your notion of every quantity measured in terms of them – age, frequency, energy, momentum…

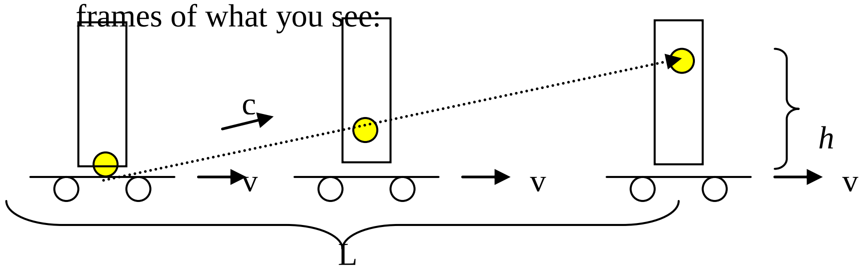
The Relativity of Time: Time Dilation

Now we're prepared to argue how time measurements differ. This is usually done by imagining a "light clock" on a skateboard. Say I've got a laser in a box; it emits a pulse of light upward. The pulse travels up to the top of the box, distance *h* above.

So, I say that it went a distance *h* and back , at speed *c* , so it took time $\Delta t_o = 2h/c$.

Now, say I hop on a skate board with my laser-in-a-box, and zip along at speed *v*. While the pulse is traveling up distance h, I'm skating horizontally a distance L.

Again, relative to me, the light is going speed *c* straight up, and it takes time $\Delta t_o = 2h/c$. But say you're standing on the sidewalk watching. Here are a few frames of what you see:



Now, you would say that the light is going at speed c up and to the right, as illustrated (note: here's where the constancy of the speed of light comes in; if we were talking about a bouncing ball, at normal speeds, we wouldn't be asserting that for you it moves diagonally just as fast as it moves vertically for me.) So, the distance traveled is the hypotenuse of this triangle up and down again: $d = 2\sqrt{L^2 + h^2}$. Going at speed *c*, this should take time $\Delta t = d/c$. For that matter, the horizontal distance is $L = v\Delta t/2$. Putting all these together, we have

$$\Delta t = \left(2\sqrt{(v\Delta t/2)^2 + h^2}\right)/c .$$

To relate this to the time measured by me, skating along with the box, $\Delta t_o = 2h/c$ or $c\Delta t_o = 2h$, so

$$\Delta t = \left(2\sqrt{\left(v\Delta t / 2\right)^2 + \left(c\Delta t_o / 2\right)^2} \right) / c$$

Solving for the time,

$$\Delta t_o = \Delta t\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

$$\gamma \equiv \frac{1}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}$$

So, since we agree about the speed of light, but not about the path, we can't agree about the time either. While this relation was motivated by considering our light pulse, it actually holds for time intervals defined by any two events observed by someone traveling with the events, and someone seeing the events travel by at speed v.

The person who saw the events taking place at the same place thought it took less time. That's the "proper" time.

- **Time Dilation:** In general, someone at home would think that *more time passed* than you do. If you traveled for 10 years, folks back home would think you were traveling for 15 years!
- **Moral:** Starting with the constancy of the speed of light, we've followed through to one bazaar implication – As measured in a reference frame moving relative to two events, the time between them seems *dilated*, or longer than when measured in the rest frame (that stationary with the events.)
- **Warning: Rest** frame means at rest *relative to the events* . When determining which frame is which in a problem, ask yourself questions like 'if I were in the space ship, would the first and second events take place the same distance and direction *from me?"* if so, you're in the 'rest frame.'

Length Contraction

Back to the light-clock on a skateboard. As I go skating by, in one tick of the clock, you say that I cover a distance $x = v\Delta t$ along the sidewalk.

Now, what do *I* think you've been doing all this time? I see you receding to the left at speed *v*. But I think time $\Delta t'$ has elapsed while you think that time Δt has elapsed. So, if *you* think *I've* gone distance $x = v\Delta t$ forward while the light pulse has bounced, then *I* think *you've* gone backwards distance

$$x' = v\Delta t' = v\Delta t\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

$$x' = x\sqrt{1 - \left(\frac{v}{c}\right)^2}$$

Along the sidewalk.

The distance that *you* see me going is referred to as the "proper" length.

So, not only do we not agree about how long it's taken, we also don't agree about how far we've moved.

- **Length Contraction** Some one who travels from event to event sees the distance between them as smaller.
-

Now, space and time are kind of the backdrops against which we do physics. Rethinking them made us rethink everything else built on them – momentum, energy, mass-energy relation. How could our physics predecessors have it all so wrong? Well, if two observer's speeds are similar (compared to the speed of light), then so will be their measurements – well within most experimental uncertainties; only if their speeds are different enough will the difference in their measurements becomes noticeable. They'd not yet dealt with anything that fast.

28.7 The Relativistic Addition of Velocities

- Last time we explored the consequences of one of the two postulates of Einstein's Special Relativity, the constancy of the speed of light. The other postulate is the old fangled concept of
    - **Relativity**: The laws of physics hold in all inertial reference frames, i.e., they hold regardless of an observer's velocity.
- **Motivation Example: Moving Pool game.**
    - We return to the example of a pool game being played in a moving train car. I'm riding in the train car and you're standing in the station; we're both observing the game. Say I measure the speed of the queue ball to be $v_{ball\text{-}me}$. Since I'm standing in the car, and moving at a speed $v_{me\text{-}you}$ relative to you, you will naturally measure a different ball speed $v_{ball\text{-}you}$ relative to you.
    - *Prior* to our imposing the constancy of the speed of light, you, the ball, and I all agreed on our measures of both space and time. Then the relationship between my measure of the ball's speed and yours was fairly straightforward. To keep it simple, Say the ball is hit in the same direction that I'm moving with the train, then you would have said
        - $v_{ball\text{-}you} = v_{me\text{-}you} + v_{ball\text{-}me.}$
    - *But now,* we understand that all three of us must be in disagreement about both distance and time measurements. So, it seems doubtful that we'd have such a simple relationship between the three speeds. If the ball starts

its watch when the cue stick hits it and ends when it strikes another ball, it will measure Δt_o time to have passed while L_{ball} of table felt rolled by.

Laying out the relationships

- If I'm riding in the car, with the pool table, then I measure the distance along the table that the ball rolled to be L_o.
- The speed I would measure for the ball's motion:

$$v_{b-m} = \frac{L_o}{\Delta t_m} = \frac{L_o}{\Delta t_o \Big/ \sqrt{1-\left(\frac{v_{b-m}}{c}\right)^2}} = \frac{L_o}{\Delta t_o}\sqrt{1-\left(\frac{v_{b-m}}{c}\right)^2} \Rightarrow \frac{L_o}{\Delta t_o} = \frac{v_{b-m}}{\sqrt{1-\left(\frac{v_{b-m}}{c}\right)^2}}$$

- You, on the platform, would see the ball and table rolling by. You'd measure the ball to have rolled distance L_y along the table. The speed you would measure for the ball's motion

$$v_{b-y} = \frac{L_y}{\Delta t_y} = \frac{L_o\sqrt{1-\left(\frac{v_{b-y}}{c}\right)^2}}{\Delta t_o \Big/ \sqrt{1-\left(\frac{v_{b-y}}{c}\right)^2}} = \frac{L_o}{\Delta t_o}\sqrt{1-\left(\frac{v_{b-y}}{c}\right)^2}\sqrt{1-\left(\frac{v_{b-y}}{c}\right)^2} = \frac{L_o}{\Delta t_o}\left(1-\left(\frac{v_{b-y}}{c}\right)^2\right)$$

$$v_{b-y} = \frac{v_{b-m}}{\sqrt{1-\left(\frac{v_{b-m}}{c}\right)^2}}\left(1-\left(\frac{v_{b-y}}{c}\right)^2\right)$$

- This can be rearranged into a simpler form
  - $v_{b-y} = \dfrac{v_{b-m} + v_{m-y}}{1+\frac{v_{b-m}v_{m-y}}{c^2}}$. Note that the numerator is exactly what we'd expect classically, the difference is the denominator isn't just 1.
- In general
  - $v_{A-B} = \dfrac{v_{A-C} + v_{C-B}}{1+\frac{v_{A-C}v_{C-B}}{c^2}}$ relates relative speeds of three parties, A, B, and C. (ball, you, and me).

- Of course, $v_{A-B} \approx v_{A-C} + v_{C-B}$ has been a perfectly good approximation for anything moving at a normal speed. So, our new relationship between relative speeds should agree very well with our old one for slow speeds.

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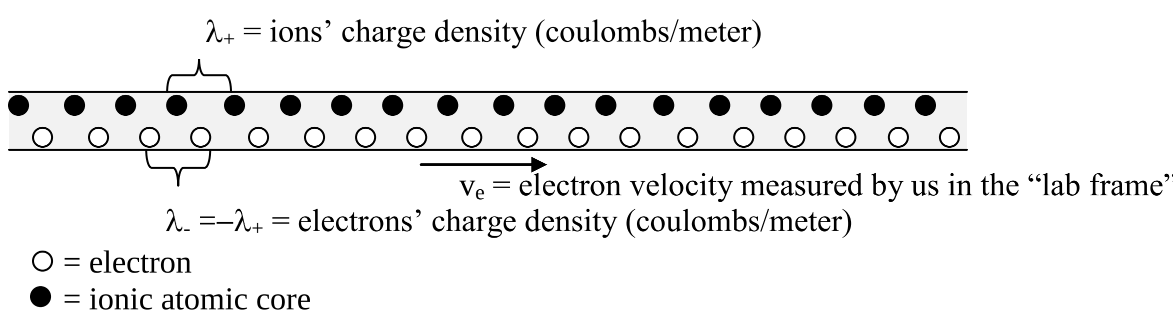
  - For speeds of everyday experience, the classical and special relativistic relationships differ imperceptibly.

Relativistic Electric Interaction: Magnetization

- Now, all of the examples that the reading and we have discussed are pretty exotic. You might get the impression that Relativistic effects were essentially the domain of theory, exotic experiments, and astrophysics – not everyday life. You'd be mostly right. However, there's one relativistic effect that is quite common and perceptible. In fact the effect was observed and theoretically described (though not explained) before special relativity was dreamt of. That's magnetism. As good physicists point out, it's an oversimplification to say that you can derive the rules of magnetism from simply the electric Coulomb interaction and special relativity; to do a general and rigorous derivation, you need to make a few technical assumptions. Only to the extent that you find these assumptions "reasonable" or "self-evident" can you say your derivation isn't a little circular. That disclaimer issued the derivation that I'm about to sketch, which is limited to charged particles with constant velocities, is quite sturdy. For the details, see Chapter 5 of Purcell's Electricity and Magnetism.

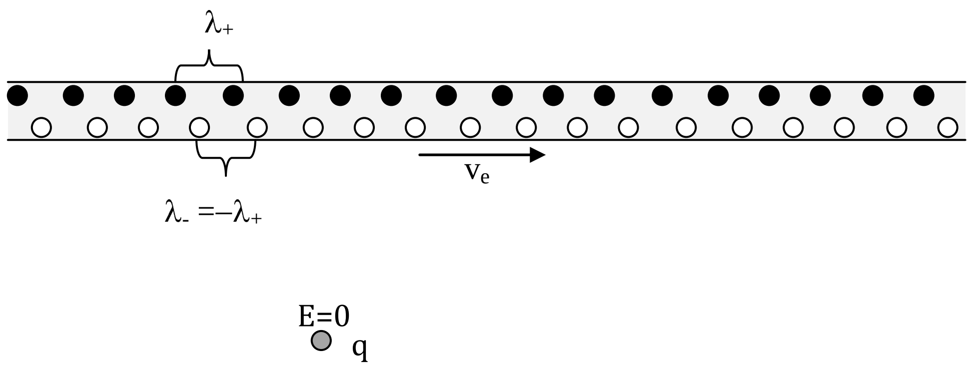
- We'll consider the interaction of a current carrying wire and a point charge moving along side it.

- In a metal, one electron from each atom is free to move about from atom to atom. Thus, it's convenient to think about the metal as two coexisting populations: the free conduction electrons and the ionic atomic cores (the atoms minus their free electrons). In a schematic cartoon, we can try to represent that as follows

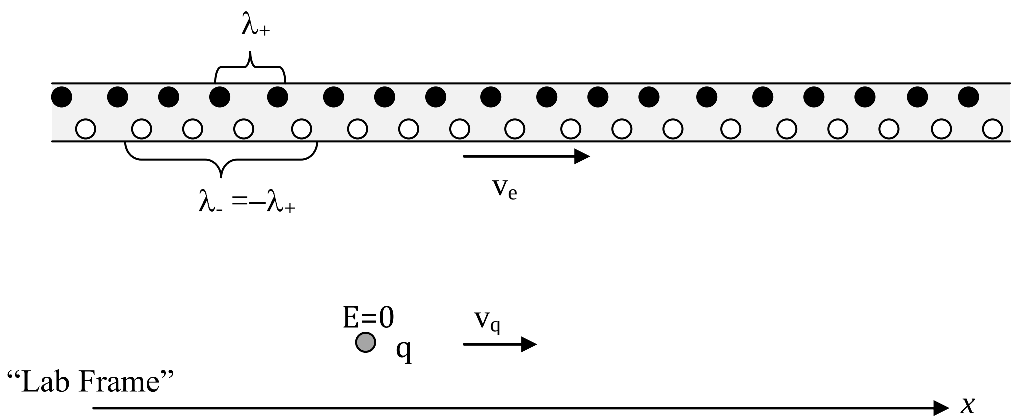


- Here, the charge densities are simply $\lambda_+ = \dfrac{e}{\Delta x_{atoms}}$ and $\lambda_- = -\dfrac{e}{\Delta x_e}$

- Now, let's think about how this interacts with a charged particle. First, let's say the particle is just sitting, some distance away from the wire.



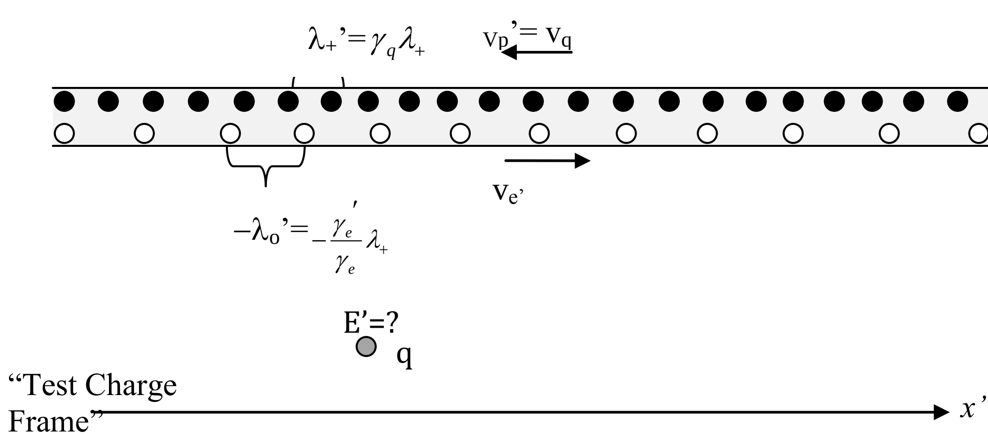
- From its perspective, as from ours, the wire is net neutral, and there's no electric field, no interaction.
  - $E = E_+ + E_- = \frac{1}{4\pi\varepsilon_o}\frac{2\lambda_+}{r} + \frac{1}{4\pi\varepsilon_o}\frac{2\lambda_-}{r} = \frac{1}{4\pi\varepsilon_o}\frac{2\lambda_+}{r} + \frac{1}{4\pi\varepsilon_o}\frac{-2\lambda_+}{r} = 0$
- Now let's say that it's moving in the same direction as the electrons in the wire.



- To *us,* the wire still looks neutral, so there shouldn't be any interaction. But the charge q is the best judge of whether or not it's interacting; what does the situation look like to *it?*
  - **Different Velocities.** From its perspective, the positive atoms appear to be moving backwards at v_q, and the electrons appear to be moving forwards a little slower. By the velocity addition rule, they should be moving forward at

- $$v_{A-B} = \frac{v_{A-C} + v_{C-B}}{1 + \frac{v_{A-C}v_{C-B}}{c^2}}$$
  $$v_e' = \frac{v_e - v_q}{1 - \frac{v_e v_q}{c^2}}$$

- **Different Charge Densities.** But that's not all.
  - **Ion Density.** According to the charge, the length between two ions is contracted by a factor of $\frac{1}{\gamma_q} = \sqrt{1-\left(\frac{v_q}{c}\right)^2}$: $\Delta x_{atom}' = \frac{\Delta x_{atom}}{\gamma_q}$.
    That means that the *density* of ions appears to be *increased* by a factor of gamma. $\lambda_+' = \frac{e}{\Delta x_{atoms}'} = \frac{e}{\frac{\Delta x_{atoms}}{\gamma_q}} = \gamma_q \lambda_+$
  - **Electron density.** Similarly, the electrons are *less* dense. It's a two step process to figure out the new value.
    - First, given the distance *we* measure between electrons in the lab frame, Δx_e, we can transform to find the Proper length of their separation, Δx_e^o.
      - $\Delta x_e^{o} = \gamma_e \Delta x_e$ where $\gamma_e = \frac{1}{\sqrt{1-\left(\frac{v_e}{c}\right)^2}}$
    - Now we can transform from their rest frame back to the charge's frame in which their moving at v_e'.
      - $\Delta x_e' = \frac{1}{\gamma_e'}\Delta x_e^o = \frac{\gamma_e}{\gamma_e'}\Delta x_e$ where $\gamma_e' = \frac{1}{\sqrt{1-\left(\frac{v_e'}{c}\right)^2}}$
    - So the density of negative charge is
      $$\lambda_-' = -\frac{e}{\Delta x_e'} = -\frac{e}{\frac{\gamma_e \Delta x_e}{\gamma_e'}} = \frac{\gamma_e'}{\gamma_e}\lambda_- = -\frac{\gamma_e'}{\gamma_e}\lambda_+$$



- Putting all this together
  - $$E' = E_+' + E_-' = \tfrac{1}{4\pi\varepsilon_o}\frac{2\lambda_+'}{r} + \tfrac{1}{4\pi\varepsilon_o}\frac{2\lambda_-'}{r} = \tfrac{1}{4\pi\varepsilon_o}\frac{2\lambda_+}{r}\left[\gamma_q - \frac{\gamma_e'}{\gamma_e}\right] = \tfrac{1}{4\pi\varepsilon_o}\frac{2\lambda_+\gamma_q \frac{v_q}{c}\frac{v_e}{c}}{r}$$
    $$F' = \tfrac{1}{4\pi\varepsilon_o}\frac{2q\lambda_+\gamma_q \frac{v_q}{c}\frac{v_e}{c}}{r}$$
  - This force points radially out, i.e., straight down.
- Finally, we transform back to our lab frame to see what force we observe acting on the charge.
  - Force is change in momentum per time. The momentum component perpendicular to the transformation is the same, where as the time in the frame with the charge moving is longer by a factor of gamma.
    - $$F' = \frac{\Delta p_y'}{\Delta t'} = \frac{\Delta p_y}{(\Delta t/\gamma_p)} = \gamma_p F$$
      $$F = \frac{1}{\gamma_p}F'$$
  - $$F = F'/\gamma_q = \tfrac{1}{4\pi\varepsilon_o}\frac{2q\lambda_+\frac{v_q}{c}\frac{v_e}{c}}{r} = \tfrac{1}{4\pi\varepsilon_o c^2}\frac{2\lambda_+ v_e}{r}qv_q$$
    - $$\mu_o \equiv \tfrac{1}{\varepsilon_o c^2}$$
      $$I = \lambda_+ v_e$$
  - $$F_y = -\tfrac{\mu_o}{4\pi}\frac{2I_x}{r}qv_{p.x}$$
- In the next chapter, we will see that this is *exactly* the "magnetic" force that the wire exerts on the charge.

  - **Stationary Charge & Current => no interaction.** Last time (Monday), we reasoned that if we had a current carrying wire and a charged particle just *sat* beside it, the charged particle would see the wire as net neutral, and wouldn't feel a thing.
  - **Moving Charge & Current => Interaction.** However, if the charge moved alongside the wire, while the wire still looks neutral to *us*, it *doesn't* look neutral to the moving charge (thank you Einstein.) That means that, while *we* wouldn't have expected the charge to feel a force, *it does*. We call it the magnetic force.
  - **E&M Unification.** Really, as we can conclude from the wire & charge example, the "Magnetic Interaction" and the "Electric Interaction" are two special cases of something more general: the Electro-Magnetic Interaction. Though this course won't get to it, these two have further been unified

with the "Weak Interaction", making the "Electro(magnetic)-Weak Interaction."

Magnetic Interact as Distinct from Electric. While it is *conceptually* satisfying to understand the unification of electric and magnetic interactions, much of the time, it is *practically* convenient to treat them individually. That is what we'll do the vast majority of the time. So let's start thinking about the Magnetic Interaction in its own right.

Demo: Current carrying wires interacting

- **Intro. A new way to think about magnetism.** We're going to consider the interaction of two current carrying wires. Now, I expect that you already have some practical experience with magnetism – who among us *hasn't* used a magnet to post something to a refrigerator, or used a compass when hiking, or clipped together the cars of a child's train? Some of you may even have had some quantitative experience with magnetism in high-school physics. I urge you to put all that from your mind and get ready to start afresh. The essence of magnetism is found not by looking at fridge magnets, but at currents.

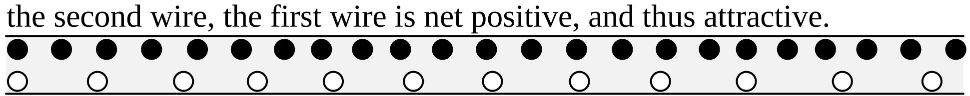
Q: I'm going to run current through these two wires. *What do you expect to see?*

Leading Questions

- What did we discuss Monday?
- If *we* see both wires as neutral, what does an electron flowing through one wire see in the other wire?
- So what force does it feel?
- Now, if *all* the moving electrons in the wire feel that, what should the wire do?
- **Do the demo:** Currents run parallel.

Q: *What's happening here?*

- **Charge flow:** In these two wires, charged particles are moving parallel. Mind you, they're moving over a backdrop of atomic ion cores, so the wires are neutral to us (except for a very miniscule charge build up).
- **Attraction of parallel:** They attract each other!
    - **Q:** *What does the charge distribution in the first wire look like from the perspective of the electrons moving in the second wire?*
        - **From the perspective of the moving charges:** This is exactly what we reasoned out on Monday, but instead of a wire and a moving charge, we have a wire and another wire's worth of moving charges. From the perspective of the electrons moving in the second wire, the first wire is net positive, and thus attractive.



- **Repulsion of Anti-parallel:** Guess what happens if I switch the direction of charge flow so their anti-parallel? Apparently charges moving in opposite directions repel each other.
    - **Q:** *What does the charge distribution in the first wire look like from the perspective of the electrons moving in the second wire?*
        - **From the perspective of the moving charges:** Apparently there's a net negative charge. That's because the positive atomic cores appear to be moving back yeay fast (so yeahy compressed separation) while the electrons appear to be moving back even faster (so even more compressed separation.)
- **Perpendicular force for Perpendicular wires:** It's not easy for me to show, but if I had one wire running up and the other coplanar but running perpendicularly across it, that second wire would be pushed as to spin into alignment.
- **Zero Force for encircling.** Finally, if the second wire encircled the first and the charges ran around it – there would be *no* push at all!

This is a most bazaar interaction!!

- **Magnetic Force Expression**
    - **Intro.** Monday, we actually derived the force equation for a charge interacting with a wire. The conceptually (if not mathematically) simpler case is the interaction of just two moving point charges.
    - **Electric.** Recall that Coulomb's law describes the electric force between two charged particles: $\vec{F}_{E.1\leftarrow 2} = \frac{1}{4\pi\varepsilon_o}\frac{q_1 q_2}{r_{1-2}^2}\hat{r}_{1-2}$
    - **Magnetic.** Here's the Magnetic force for two, constant velocity charged particles:
        - $\vec{F}_{M.1\leftarrow 2} = \frac{\mu_o}{4\pi}\frac{q_1\vec{v}_1 \times \left(q_2\vec{v}_2 \times \hat{r}_{1-2}\right)}{r_{1-2}^2}\left\{\frac{1-\left(\frac{v_2}{c}\right)^2}{\left(1-\hat{r}_{1-2}\cdot\frac{\vec{v}_2}{c}\right)^3}\right\}$ Ack!

            - Of course, all particle-2 quantities would be evaluated at the retarded time (where was, and how fast was q_2 at the time the radiation was emitted that now reaches q_1, at time $t_r = t - cr_{1-2}$. (This expression follows from Griffiths' 10.66, or 10.67; since Griffiths makes the point that the "Biot-Savart law for a particle" is not exact, so, for the sake of consistency, it's worth noting this now).

- **v<<c.** Fortunately, the term in squiggly brackets is negligible unless v is on order of c. So, for most cases we can approximate this as
  - $\vec{F}_{M.1\leftarrow 2} \approx \frac{\mu_o}{4\pi}\frac{q_1\vec{v}_1 \times \left(q_2\vec{v}_1 \times \hat{r}_{1-2}\right)}{r_{1-2}^2}$
    - Yes, these are cross-products, and yes, this is still kind of ugly.
- **There's still an Electric interaction.** Mind you, just because the charges are moving doesn't mean that their *electric* interaction goes away –that's there too (though we have to deal with retarded times), but now there's an additional magnetic interaction.

- **Basis of *all* Magnetism.**
  - Whether we're talking two moving charges, currents in wires, or even two magnets or the Earth and a compass – this interaction lies at the heart of *all* magnetism. (one might quibble about magnets and electron orbital angular momentum and spin, but at least at the semi-classical level, this is still the picture.)

- **Magnetic Field**
  - **Electric Field.** Now, it was convenient to define the Electric field as essentially the Electric force without the particle 1 specific factor:
    - $\vec{E}_{1\leftarrow 2} = \frac{1}{4\pi\varepsilon_o}\frac{q_2}{r_{1-2}^2}\hat{r}_{1-2}$, so $\vec{F}_{E.1\leftarrow 2} = q_1\vec{E}_{E.1\leftarrow 2}$
  - **Magnetic Field.** Similarly, it will be convenient to define the *Magnetic* field as essentially the Magnetic force without the particle 1 specific factor:
    - $\vec{B}_{1\leftarrow 2} \approx \frac{\mu_o}{4\pi}\frac{\left(q_2\vec{v}_1 \times \hat{r}_{1-2}\right)}{r_{1-2}^2}$, so $\vec{F}_{M.1\leftarrow 2} = q_1\vec{v}_1 \times \vec{B}_{1\leftarrow 2}$.
      - **Units (T) Tesla = N/(C m/s)**
      - $\frac{\mu_0}{4\pi} = 1\times 10^{-7}\,\frac{\mathrm{T}\cdot\mathrm{m}^2}{\mathrm{C}\cdot\mathrm{m/s}}$
    - **Mathematically simpler.** Mathematically speaking, the magnetic field is at least one step nicer to look at than is the force (one fewer cross products). So, we will spend a lot of time getting familiar with this field.
    - **Conceptually Abstract.** One thing to notice is that, since the force is the *cross* product of the velocity and the field, unlike the

Electric field, the Magnetic field does *not* point in the direction of the force. That is part of the price we pay for defining it so simply.

- **Theoretical: Biot-Savart Law**
  - Now let's see how our observation of the magnetic field due to a current relates to our mathematical parameterization of it. The Biot-Savart Law that I've introduced speaks about the field due to a moving point charge, but what's a current other than a string of moving point charges (against a neutralizing backdrop of stationary ions). We'll derive the expression for a full current's field, but for now, let's return to the mathematical expression we have for the magnetic field due to a moving point charge.
    - $\vec{B}_{1\leftarrow 2} \approx \frac{\mu_o}{4\pi}\frac{\left(q_2\vec{v}_1 \times \hat{r}_{1-2}\right)}{r_{1-2}^2}$
      - Okay, its strength depends on the charge and its velocity as well as the distance from the charge. One might have guessed as much. But what do we mean by crossing the velocity into the unit vector?
      - **Cross Product**

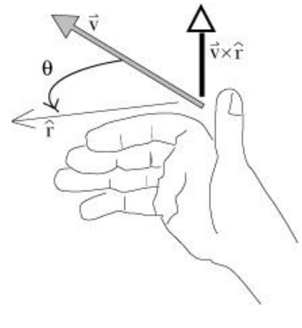
$$\vec{A}\times\vec{B} \equiv \det\begin{vmatrix}\hat{x} & \hat{y} & \hat{z}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix} = \left\langle \left(A_yB_z - A_zB_y\right), \left(A_zB_x - A_xB_z\right), \left(A_xB_y - A_yB_x\right)\right\rangle$$

magnitude: $\left|\vec{A}\times\vec{B}\right| = AB\sin\theta$

- where θ is the angle between $\vec{A}$ and $\vec{B}$

direction: determined with the "right-hand rule" (RHR)

- point fingers of right hand in direction of first vector
- rotate wrist so you can curl the fingers to the second vector's direction
- your thumb will point in the direction of the cross product



The direction of the cross product is also perpendicular to the two vectors being multiplied. If the vectors are parallel, the cross product is zero.

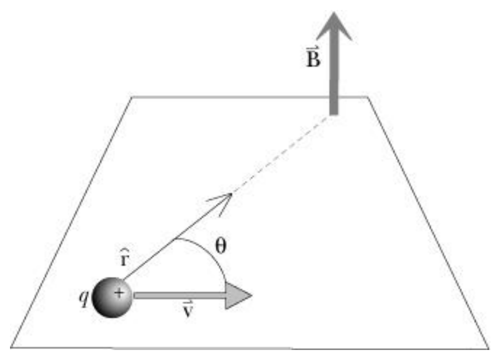
Demo: 17_Crossproduct.py

Here's a visual to go along with the math. See how the product's direction and magnitude varies with the magnitudes and relative angles of the multiplied vectors.

Clicker Questions 17.3a-c

- **Back to Biot-Savart**
  - Now that we 'get' the cross-product, let's return to looking at the Biot-Savart expression for a moving charge's magnetic field.
  - The magnetic field at a location $\vec{r}$ relative to a charge q moving a velocity $\vec{v}$ is:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{q\,\vec{v}\times\hat{r}}{r^2},$$



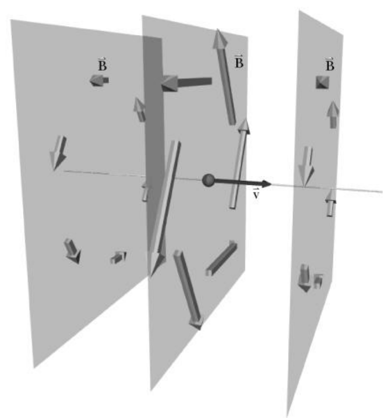
Clicker Questions 17.3d-g

Demo: **17_Bparticle_1loc_PRIVATE.py**

See how the direction and magnitude vary according to the $1/r^2$ and the v×r.

Demo: 17_Bproton_PRIVATE.py

In 3-D, the magnetic field looks like the following (use VPython to demonstrate)



For a whole string of charges, i.e., a current, it looks then like this

Demo: 17_long_wire.py

Lorentz Force Law

We've already talked about source charges ($q_1, q_2, \ldots$) producing an electric field $\vec{E}$, which results in an electric force on a test charge (Q) of $\vec{F}_{elec} = \vec{F}_E = Q\vec{E}$. Moving charges (or currents) produce a magnetic field $\vec{B}$. The calculation of the magnetic field is more complicated, so we start by considering the resulting magnetic force on a test charge of $\vec{F}_{mag} = \vec{F}_B = Q\vec{v} \times \vec{B}$. Often, the two forces are summarized in the Lorentz force law as $\vec{F} = Q\left[\vec{E} + \left(\vec{v} \times \vec{B}\right)\right]$.

- There is no magnetic force on a test charge if it is not moving (v = 0).
- There is no magnetic force if the particle moves parallel to the magnetic field.
- The magnetic force is perpendicular to both the velocity and magnetic field, so problems are "inherently 3-D." You don't need to make perspective drawings! Use ⊗ to represent a vector pointing "into the page" and ⊙ for a vector pointing "out of the page."
- Use the right-hand rule (RHR) to determine the direction of $\vec{v} \times \vec{B}$. The magnetic force is in the opposite direction if the test charge is negative.
- Remember that for a standard (right-handed) coordinate system $\hat{x} \times \hat{y} = \hat{z}$. If you get coordinate axes wrong, you'll get directions wrong.
- Magnetic forces do no work. The motion during an infinitesimally small time dt is $d\vec{\ell} = \vec{v}\,dt$. The magnetic force is perpendicular to the velocity, so the work done by the magnetic force in this time is $dW_{mag} = \vec{F}_{mag} \cdot d\vec{\ell} = 0$. This also means that a magnetic force cannot change the speed of a particle (because of the work-KE theorem). The velocity (direction) can change.

Cyclotron Motion in a Uniform Magnetic Field

A "uniform" magnetic field means the same size and direction in a region. Suppose a charged particle is initially moving with a speed v perpendicular to a magnetic field.

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

The speed will remain constant and the force is always perpendicular to the velocity. Also, the force will remain the same size: $F_B = QvB$ (the velocity stays perpendicular to the magnetic field). Therefore, there will be uniform circular motion and the acceleration is $a = v^2/R$, where R is the radius. Applying Newtons's second law gives

$$QvB = m\left(v^2/R\right) \quad \text{or} \quad p = mv = QBR \tag{5.3}$$

Working this through a little more rigorously,

$$\frac{d\vec{p}}{dt} = \vec{F}_{mag}$$

$$m\frac{d\vec{v}}{dt} = q\vec{v} \times \vec{B}$$

$$m\begin{Bmatrix} dv_x/dt \\ dv_y/dt \\ dv_z/dt \end{Bmatrix} = q\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ B_x & B_y & B_z \end{vmatrix} = q\begin{Bmatrix} v_y B_z - v_z B_y \\ v_z B_x - v_x B_z \\ v_x B_y - v_y B_x \end{Bmatrix}$$

In the simple case that the field points in the z-direction, we get

$$\frac{dv_x}{dt} = \frac{qB_z}{m} v_y$$

$$\frac{dv_y}{dt} = -\frac{qB_z}{m} v_x$$

Taking another derivative of the second relation gives

$\dfrac{d^2 v_y}{dt^2} = -\dfrac{qB_z}{m}\dfrac{dv_x}{dt}$; plugging in from the first expression yields

$\dfrac{d^2 v_y}{dt^2} = -\left(\dfrac{qB_z}{m}\right)^2 v_y$; similarly, $\dfrac{d^2 v_x}{dt^2} = -\left(\dfrac{qB_z}{m}\right)^2 v_x$

These are solved by

$$v_y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$
$$v_x(t) = C_3 \cos(\omega t) + C_4 \sin(\omega t)$$

Plugging them back into

$$\frac{dv_x}{dt} = \frac{qB_z}{m} v_y$$

Gives

$$\omega\left(-C_3 \sin(\omega t) + C_4 \cos(\omega t)\right) = \frac{qB_z}{m}\left(C_1 \cos(\omega t) + C_2 \sin(\omega t)\right)$$

Which tells us that

$$\omega = \frac{qB_z}{m}$$
$$-C_3 = C_2$$
$$C_4 = C_1$$

So,

$$v_y(t) = C_1 \cos(\omega t) + C_2 \sin(\omega t)$$
$$v_x(t) = C_1 \sin(\omega t) - C_2 \cos(\omega t)$$

Appealing to boundary conditions, if we say $\vec{v} = \langle 0, v_{yo}, 0 \rangle$ at *t=0*, then

$$v_y(t) = v_{yo} \cos(\omega t)$$
$$v_x(t) = v_{yo} \sin(\omega t)$$

Hmm… looks like a circle of radius v_{yo}.

There we have proven what we only argued in Phys 232 – that the charged particle will go around in circles.

If the particle has a component of velocity in the direction of the magnetic field, that component will remain unchanged because there is no magnetic force in the same direction as the magnetic field. The resulting trajectory will be helical.

Motion in Parallel Electric and Magnetic Fields.

What if there's an electric field parallel to the magnetic field?

What do you expect?

then there's a force pushing up in the z direction, increasing the charge's velocity, *but only in the z-direction.* So, that doesn't at all effect the x-y components, thus it does not at all effect the magnetic force which makes them spin around and around. You'd just have

$$v_y(t) = v_{yo}\cos(\omega t)$$
$$v_x(t) = v_{yo}\sin(\omega t)$$
$$v_z(t) = v_{z_o} + qE_z t$$

Motion in "Crossed" Electric and Magnetic Fields

Now consider the more complicated situation when there are uniform electric and magnetic fields at right angles. Suppose $\vec{B}$ is in the *x* direction and $\vec{E}$ is in the *z* direction. Suppose a charged particle is initially moving in the *yz* plane. There will not be any force in the *z* direction to take it out of the plane.

We'll use "dot notation" for time derivatives: $v_x = dx/dt = \dot{x}$, $a_x = d^2x/dt^2 = \ddot{x}$, etc. The cross product in the Lorentz force law is

$$\vec{v}\times\vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ B & 0 & 0 \end{vmatrix} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dot{x} & \dot{y} & \dot{z} \\ B & 0 & 0 \end{vmatrix} = B\dot{z}\hat{y} - B\dot{y}\hat{z},$$

so Newton's second law in this situation is

$$\vec{F} = Q\left[\vec{E} + \left(\vec{v}\times\vec{B}\right)\right] = Q\left[E\hat{z} + \left(B\dot{z}\hat{y} - B\dot{y}\hat{z}\right)\right] = \left(QB\dot{z}\right)\hat{y} + \left(QE - QB\dot{y}\right)\hat{z} = m\vec{a} = m\left(\ddot{y}\hat{y} + \ddot{z}\hat{z}\right).$$

The differential equations for the *y* and *z* components are

$$QB\dot{z} = m\ddot{y} \quad \text{and} \quad QE - QB\dot{y} = m\ddot{z}.$$

If we define the *cyclotron frequency*, $\omega \equiv QB/m$, the differential equations can be written as

$$\ddot{y} = \omega\dot{z} \quad \text{and} \quad \ddot{z} = \omega\left(\frac{E}{B} - \dot{y}\right).$$

The solution (you can check them by plugging them back in, methods of solution taught in PHYS 331) is

$$y(t) = C_1\cos\omega t + C_2\sin\omega t + (E/B)t + C_3,$$
$$z(t) = C_2\cos\omega t - C_1\sin\omega t + C_4.$$

The four constants (C's) must be determined by "initial conditions" – position and velocity.

Examples/Exercises:

Problem 5.2 (a) & (c)

For both parts, the particle starts from the origin, $y(0)=0$ and $z(0)=0$, so we know that $C_3=-C_1$ and $C_4=-C_2$. Taking the time derivative of the general solution gives

$$\dot y(t) = -C_1\omega \sin \omega t + C_2\omega \cos \omega t + E/B\,,$$
$$\dot z(t) = -C_2\omega \sin \omega t - C_1\omega \cos \omega t.$$

a. We are also given $\vec v(0)=(E/B)\hat y$, so $\dot y(0)=E/B$ and $\dot z(0)=0$. The first condition implies that $C_2=0$, which means that $C_4=0$. The second condition implies that $C_1=0$, so $C_3=0$.

The solution that satisfies the initial conditions is

$$y(t)=(E/B)t,$$
$$z(t)=0.$$

In other words, the particle moves with a constant velocity $\vec v(t)=(E/B)\hat y$. The electric and magnetic forces balance at this velocity (perpendicular to *E* and *B* at the right speed).

c. We are also given $\vec v(0)=(E/B)(\hat y+\hat z)$, so $\dot y(0)=E/B$ and $\dot z(0)=E/B$. The first condition implies that $C_2=0$, which means that $C_4=0$. The second condition implies that $C_1=-E/\omega B$, so $C_3=E/\omega B$.

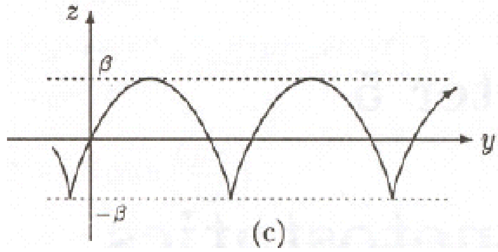
The solution that satisfies the initial conditions is

$$y(t)=-(E/\omega B)\cos\omega t+(E/B)t+(E/\omega B)=(E/\omega B)[1+\omega t-\cos\omega t],$$
$$z(t)=(E/\omega B)\sin\omega t.$$

With just the cosine in $y(t)$, this would be clockwise motion around a circle of radius $E/\omega B$ with a angular frequency ω. The particle is moving clockwise around a circle whose center is moving along the *y* axis. Another way to see this is to isolate the sine and cosine, square and add them, which gives one. That expression can be rearranged to give

$$\left[y(t)-\frac{E}{\omega B}(1+\omega t)\right]^2+[z(t)]^2=\left(\frac{E}{\omega B}\right)^2.$$

This is the equation for a circle of radius $E/\omega B$ with the center at $y=E/\omega B(1+\omega t)$ and z = 0. The graph is shown below.

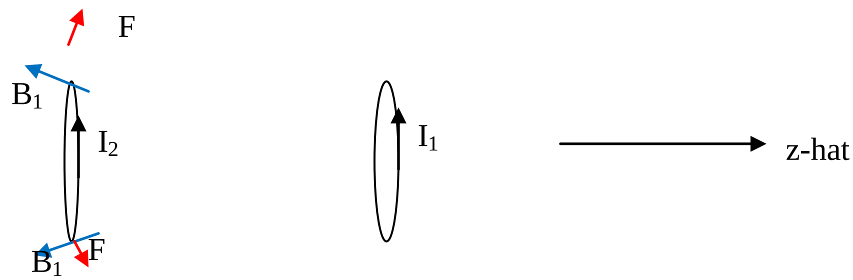


No Work by F_{mag}.

The book points out that the magnetic force can't possibly do any work since the force is inherently perpendicular to the motion.

It's important to remember that work results in a change in kinetic energy, and thus *speed,* but it takes no work to change *direction.*

Superconducting Current Loops. With that in mind, think of one superconducting electromagnet dragging another electromagnet to itself. I'm considering superconductors so we don't need to worry about any magnets –once we've got the current started circulating, it'll keep going. So, how is this not work?



Notice that the magnetic field at loop 2 due to current 1 diverges; that means that there's a magnetic force with an up and out component.

To the extent that the wire's inter-atomic bonds don't stretch much, they prevent the wire from stretching outward; to the extent that the ion cores strongly attract the circulating electrons, they hold the electrons from moving out radially, thus cancel the local outward components of the force, leaving only the small horizontal component. Thus the force accelerates the electrons sideways, and, thanks to their strong attraction to the ion cores, they bring the ring with them.

But doesn't acceleration mean change in kinetic energy and thus work is getting done? Not if the increase in motion *along* the z-direction is offset by a *decrease in motion around* the z-axis. That is, if the electron's slow their rotation to compensate for the increase in their axial motion.

Does this happen? Absolutely. Remember lenses law? If an external source *increases* the magnetic flux through a ring (which will happen as ring 2 approaches ring 1), then a change in the ring's current gets induced so that its gets induced in that ring so *its* field tries to oppose the change. There you have it – the current in ring 2 *slows* its circulation, thus allowing for the speeding up translation. And NO NET WORK is done.

Regular Current Loops (with batteries)

If, instead of having a superconducting current loop we had a regular one, then we'd need a battery to drive the current. I still think the current's circulation would slow down; we can see that if we think about this as a faraday's effect with a non-coulombic emf induced along the current loop and thus partly combating the battery's emf. So even with the battery, I don't think any additional work (above that already associated with resistively heating the wire) gets done.

Preview

For Wednesday, you'll read about currents and magnetic forces on them.

Second attempt on HW #4 is due tomorrow. Any questions?

"I really like the conceptual scenarios such as example 5.2, where you're given electric/magnetic fields, or initial v, charge, etc., and you have to figure out what your particle will do. Do you think we could do a few of these in class?"
Rachael Hach Post a response
Admin

Flag as inappropriate

"Griffiths says that magnetic forces do no work, but doesn't explain why. Can we discuss why this is."
Jessica Hide response Post a response
Admin

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I also found this a bit odd. So does that mean the magnetic force is technically not applying an acceleration to a particle? Because if a particle is following a curved trajectory, then it is accelerating. I understand mathematically that the force is perpendicular to the path so there is no work done, but conceptually it seems strange.

Casey McGrath

"How important will it be for us to understand and know how to use the relativity material, and can we maybe see an example where relativity comes into play in an electro/magnetostatics problem?"
Sam Post a response
Admin

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"Can we briefly go over the derivation of equation 5.10?"
Spencer Post a response
Admin