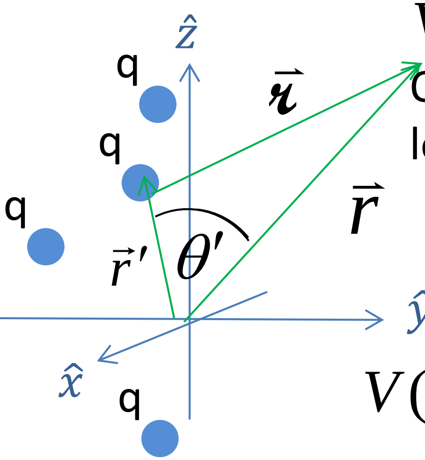
Wed. Thurs Fri.	3.4.3-.4.4 Multipole Expansion Review	HW4
Mon. Wed. Fri.	Study Day Exam 1 (Ch 2, 3) (C 14) 4.1 Polarization	

Multi-pole Expansion

Discrete charge distribution



$$V(r) = \frac{1}{4\pi\varepsilon_o} \sum_{i}^{\text{charges}} \frac{q_i}{\mathscr{r}_i} = \frac{1}{4\pi\varepsilon_o} \sum_{i}^{\text{charges}} \frac{q_i}{\sqrt{r^2 + r'^2 - 2rr'\cos\theta_{r\to r'}}}$$

$$\frac{1}{\mathscr{r}_i} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r_i'}{r}\right)^n P_n(\cos\theta_i')$$

$$P_n(\cos\theta')$$

n[th] Legendre polynomial

$$P_0 = 1$$

$$P_1(u) = u$$

$$P_2(u) = (3u^2 - 1)/2$$

$$V(r) = \frac{1}{4\pi\varepsilon_o} \sum_{\text{i}}^{\text{charges}} \left(\frac{q_i}{r} \sum_{n=0}^{\infty} \left(\left(\frac{r_i'}{r}\right)^n P_n(\cos\theta_i') \right) \right)$$

For each charge… sum terms in expansion

Flip order of summation

$$V(r) = \frac{1}{4\pi\varepsilon_o} \sum_{n=0}^{\infty} \left(r^{-(n+1)} \sum_{i}^{\text{charges}} \left(r_i'^n{}_i P_n(\cos\theta_i') q_i \right) \right)$$

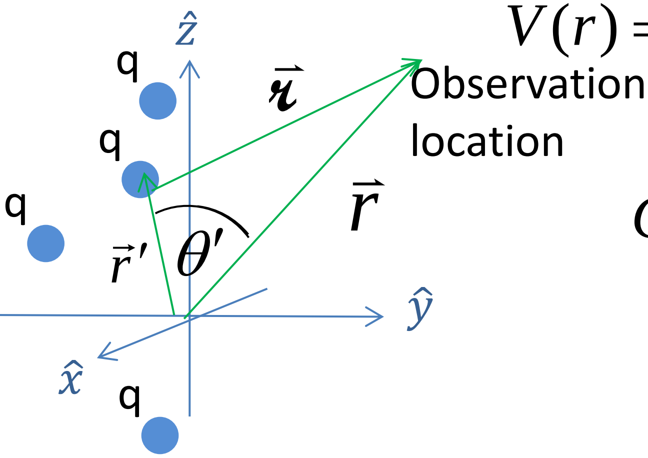
For each term in expansion… sum contribution of charges

$$V(r) = \frac{1}{4\pi\varepsilon_o} \frac{\sum_{i}^{\text{charges}} q_i}{r} + \frac{1}{4\pi\varepsilon_o} \frac{\sum_{i}^{\text{charges}} r_i' q_i \cos\theta_i'}{r^2} + \ldots$$

monopole dipole

Multi-pole Expansion

Discrete charge distribution



$$V(r) = \frac{1}{4\pi\varepsilon_o}\frac{\sum_i^{\text{charges}} q_i}{r} + \frac{1}{4\pi\varepsilon_o}\frac{\sum_i^{\text{charges}} r_i' q_i \cos\theta_i'}{r^2} + \ldots$$

monopole dipole

$$Q_{net} = \sum_i^{\text{charges}} q_i$$

Observe that

$$\vec{r}_i\,' \cdot \vec{r} = r_i' r \cos\theta_i'$$

or

$$\vec{r}_i\,' \cdot \hat{r} = \vec{r}_i\,' \cdot \frac{\vec{r}}{r} = r_i'\frac{r}{r}\cos\theta' = r_i'\cos\theta_i'$$

so

$$\sum_i^{\text{charges}} r_i' q_i \cos\theta_i' = \sum_i^{\text{charges}} q_i \vec{r}_i\,' \cdot \hat{r} = \left(\sum_i^{\text{charges}} q_i \vec{r}_i\,'\right) \cdot \hat{r}$$

$$= \vec{p} \cdot \hat{r}$$

Electric Dipole Moment

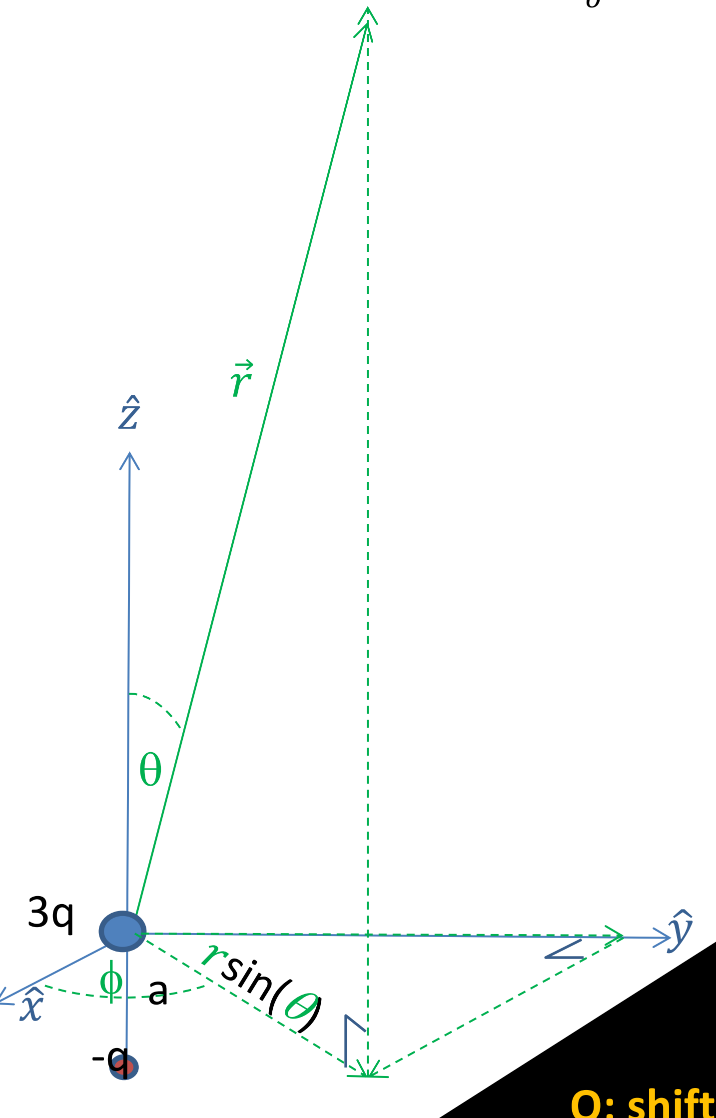
$$V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q_{net}}{r} + \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}\cdot\hat{r}}{r^2} + \ldots$$

monopole dipole

Q: explain rate each term drops off **How many terms to keep?**

Exercise: Find the first two terms in the mulitpole expansion for the figure shown below.

$$V(r)_{mono} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r} \qquad V(r)_{dipole} = \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad \vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau' = \vec{r}_1'q_1 + \vec{r}_2'q_2$$



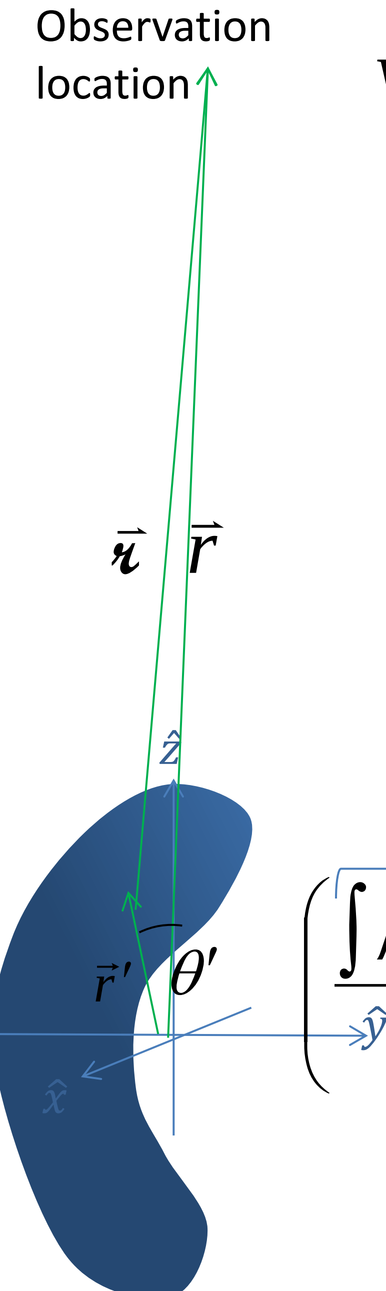
Q: shift origin effect

coordinate-free representation of dipole field: Please do problem 3.36.

Electric field of dipole in other coordinates: Is it possible to show how the electric field of a dipole would look like in Cartesian or cylindrical coordinates?

Multi-pole Expansion

Continuous charge distribution



$$V(r) = \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{\mathscr{r}}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

$$V(r) = \frac{1}{4\pi\varepsilon_o}\int\left(\left(\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')\right)\rho(\vec{r}\,')d\tau'\right)$$

Re-ordering sums

$$V(r) = \frac{1}{4\pi\varepsilon_o}\sum_{n=0}^{\infty}\left(\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\rho(\vec{r}\,')d\tau'\right)$$

$$\left(\frac{\int\rho(\vec{r}\,')d\tau'}{r} + \frac{\int r'\cos\theta'\rho(\vec{r}\,')d\tau'}{r^2} + \frac{\int r'^2\left(3(\cos\theta')^2-1\right)\rho(\vec{r}\,')d\tau'}{2r^3} + \ldots\right.$$

n^th Legendre polynomial

$$P_n(\cos\theta')$$

$$P_0 = 1$$

$$P_1(u) = u$$

$$P_2(u) = \frac{(3u^2-1)}{2}$$

Multi-pole Expansion

$$V(r) = \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{\mathscr{r}}$$

$$V(r) = \frac{1}{4\pi\varepsilon_o}\left(\frac{\int \rho(\vec{r}\,')d\tau'}{r} + \frac{\int r'\cos\theta'\rho(\vec{r}\,')d\tau'}{r^2} + \frac{\int r'^2\left(3(\cos\theta')^2 - 1\right)\rho(\vec{r}\,')d\tau'}{2r^3} + \dots \right.$$

$$\int \rho(\vec{r}\,')d\tau' = Q$$

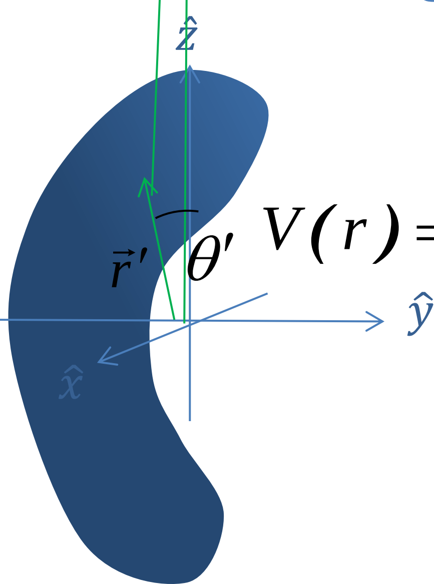
$$\int r'\cos\theta'\rho(\vec{r}\,')d\tau'$$

$$\int \hat{r}\cdot\vec{r}\,'\rho(\vec{r}\,')d\tau'$$

$$\hat{r}\cdot\left(\int \vec{r}\,'\rho(\vec{r}\,')d\tau'\right)$$

$$\hat{r}\cdot\vec{p}$$

Continuous charge distribution



$$V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r} + \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}\cdot\hat{r}}{r^2} + \dots$$

Monopole   Dipole   Quadrapole

Dipole Moment

$$\vec{p} \equiv \int \vec{r}\,'\rho(\vec{r}\,')d\tau'$$

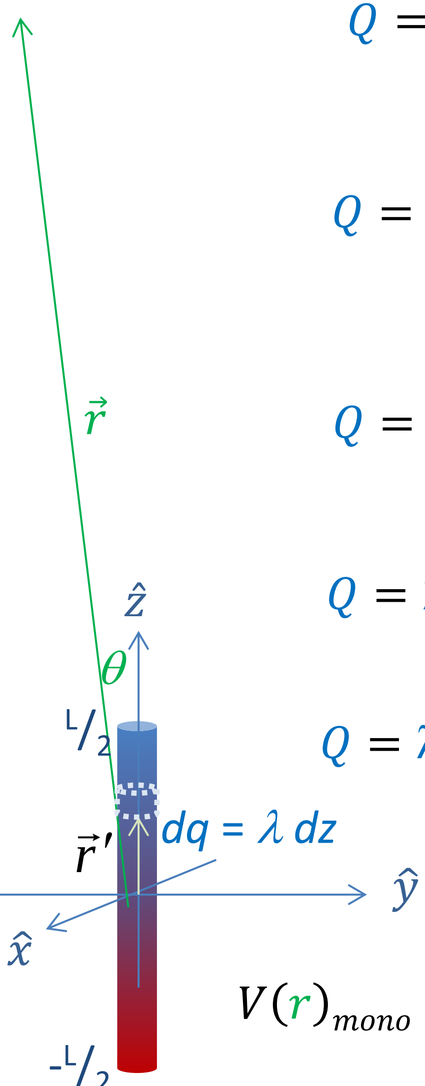
Analogy:
If *ρ(r')* where *mass* density, then

$$\frac{\int \vec{r}\,'\rho(\vec{r}\,')d\tau'}{\int \rho(\vec{r}\,')d\tau'} = \frac{\int \vec{r}\,'\rho(\vec{r}\,')d\tau'}{Mass}$$

Would be the center of *mass* or mass-averaged location.

Example: Find the first two terms in the mulitpole expansion for the figure shown below if $\lambda(\vec{r}) = \lambda_o(2z'/L)^3$.

$$V(r)_{mono} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r} \qquad V(r)_{dipole} = \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad \vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau'$$



$$Q = \int_{z'=-\frac{L}{2}}^{z'=\frac{L}{2}} \lambda(\vec{r})dz'$$

$$Q = \int_{z'=-\frac{L}{2}}^{z'=\frac{L}{2}} \lambda_o\left(\frac{2z'}{L}\right)^3 dz'$$

$$Q = \lambda_o\frac{L}{2}\int_{\left(\frac{2z'}{L}\right)=-1}^{\left(\frac{2z'}{L}\right)=1}\left(\frac{2z'}{L}\right)^3 d\left(\frac{2z'}{L}\right)$$

$$Q = \lambda_o\frac{L}{2}\int_{\beth=-1}^{\beth=1}\beth^3\, d\beth$$

$$Q = \lambda_o\frac{L}{8}(1^4-(-1)^4) = 0$$

$$V(r)_{mono} = 0$$

$$\vec{p} = \int_{z'=-\frac{L}{2}}^{z'=\frac{L}{2}} \vec{r}'\lambda(\vec{r}')dz'$$

$$\vec{p} = \int_{z'=-\frac{L}{2}}^{z'=\frac{L}{2}} z'\hat{z}\lambda_o\left(\frac{2z'}{L}\right)^3 dz'$$

$$\vec{p} = \lambda_o\frac{L^2}{4}\int_{\left(\frac{2z'}{L}\right)=-1}^{\left(\frac{2z'}{L}\right)=1}\left(\frac{2z'}{L}\right)^4 d\left(\frac{2z'}{L}\right)\hat{z}$$

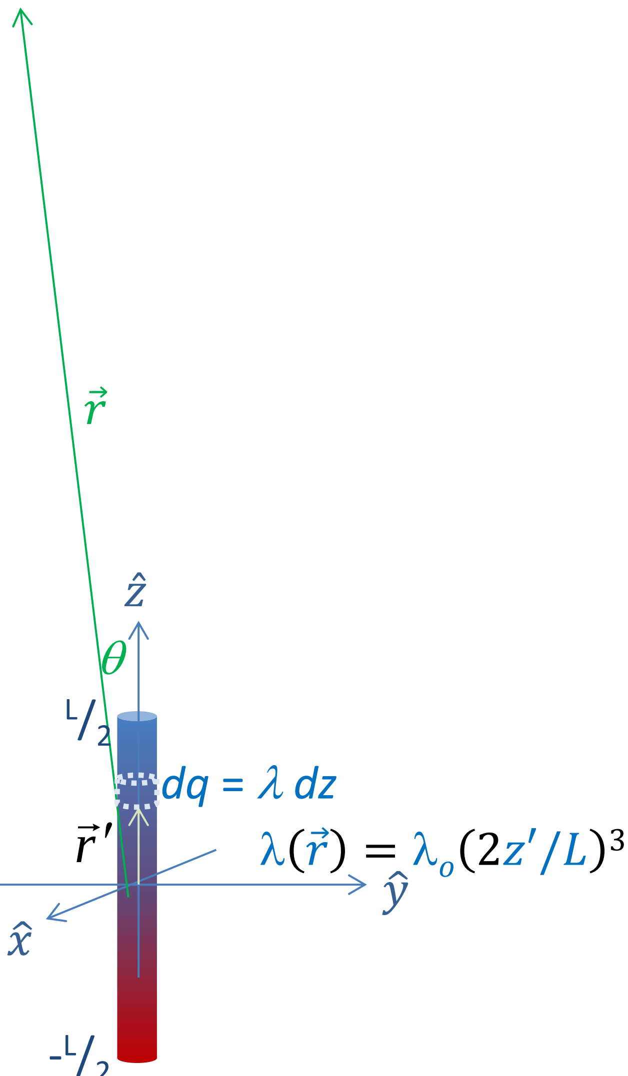
$$\vec{p} = \lambda_o\frac{L^2}{20}(1^5-(-1)^5)\hat{z} = \lambda_o\frac{L^2}{10}\hat{z}$$

$$V(r)_{dipole} = \frac{1}{4\pi\varepsilon_o}\lambda_o\frac{L^2}{10}\frac{\hat{z}\cdot\hat{r}}{r^2}$$

$$V(r)_{dipole} = \frac{\lambda_o L^2}{40\pi\varepsilon_o}\frac{\cos(\theta)}{r^2}$$

Example: Find the corresponding approximate field expression.

Far from the sources, $|r| >> |r'|$, $V(\vec{r}) \approx V(\vec{r})_{mono} + V(\vec{r})_{dipole}$



$$V(\vec{r}) \approx 0 \quad + \frac{\lambda_o L^2}{40\pi\varepsilon_o}\frac{\cos(\theta)}{r^2}$$

$$\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$$

$$\vec{E}(\vec{r}) = -\frac{\partial}{\partial r}V(\vec{r})\hat{r} - \frac{1}{r}\frac{\partial}{\partial\theta}V(\vec{r})\hat{\theta} - \frac{1}{r sin(\theta)}\frac{\partial}{\partial\varphi}V(\vec{r})\hat{\varphi}$$

$$\vec{E}(\vec{r}) \approx -\frac{\partial}{\partial r}\left(\frac{\lambda_o L^2}{40\pi\varepsilon_o}\frac{\cos(\theta)}{r^2}\right)\hat{r} - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\lambda_o L^2}{40\pi\varepsilon_o}\frac{\cos(\theta)}{r^2}\right)\hat{\theta}$$

$$\vec{E}(\vec{r}) \approx -\frac{\lambda_o L^2}{40\pi\varepsilon_o}\left(\cos(\theta)\frac{\partial}{\partial r}\left(\frac{1}{r^2}\right)\hat{r} + \frac{1}{r}\frac{1}{r^2}\frac{\partial}{\partial\theta}(\cos(\theta))\hat{\theta}\right)$$

$$\vec{E}(\vec{r}) \approx -\frac{\lambda_o L^2}{40\pi\varepsilon_o}\left(-2\cos(\theta)\left(\frac{1}{r^3}\right)\hat{r} + \frac{-1}{r^3}\sin(\theta)\hat{\theta}\right)$$

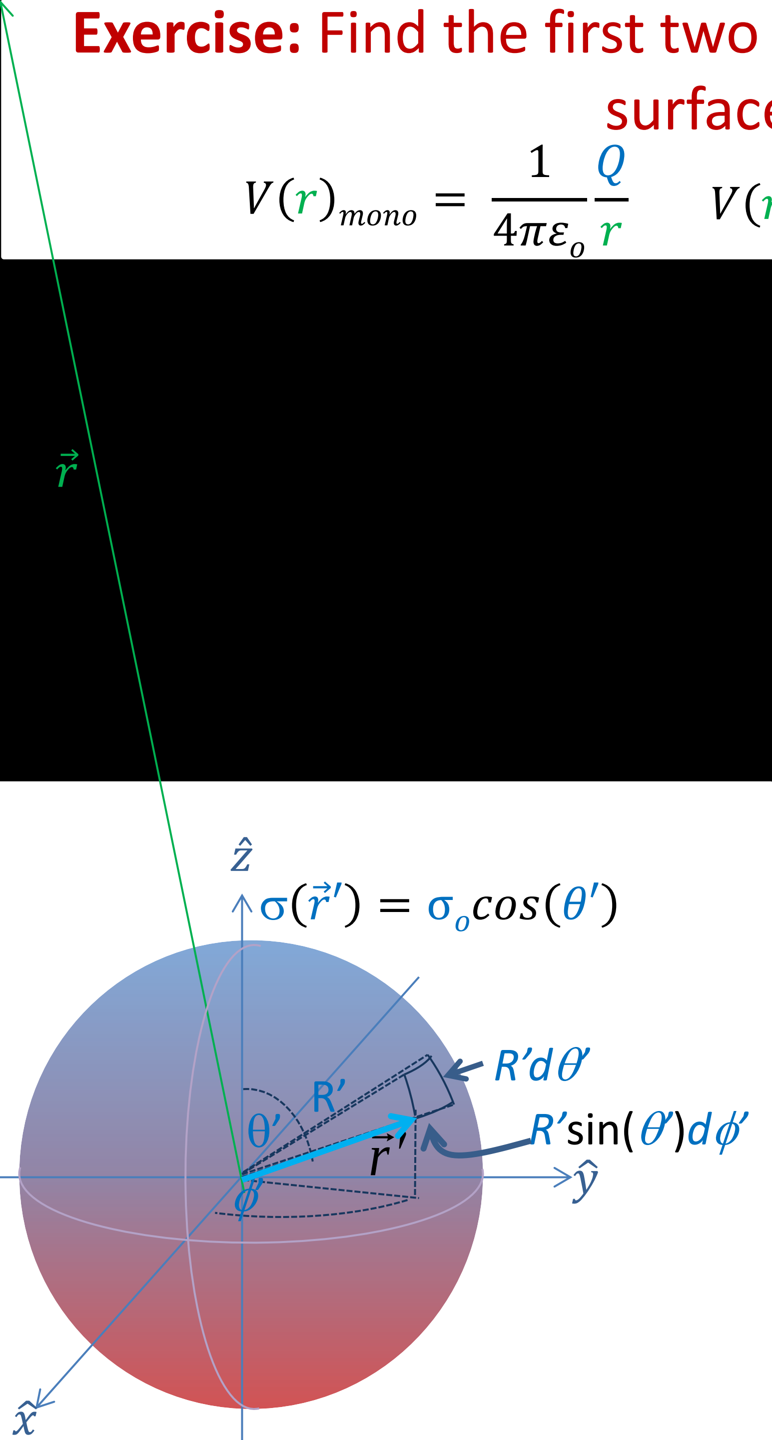
$$\vec{E}(\vec{r}) \approx \frac{\lambda_o L^2}{40\pi\varepsilon_o}\frac{1}{r^3}(2\cos(\theta)\,\hat{r} + \sin(\theta)\hat{\theta})$$

Or since $\vec{p} = \lambda_o \dfrac{L^2}{10}\hat{z}$,

$$\vec{E}(\vec{r}) \approx \frac{1}{4\pi\varepsilon_o}\frac{p}{r^3}(2\cos(\theta)\,\hat{r} + \sin(\theta)\hat{\theta})$$

Exercise: Find the first two terms in the multi-pole expansion for this surface charge distribution.

$$V(r)_{mono} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r} \qquad V(r)_{dipole} = \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad \vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau'$$



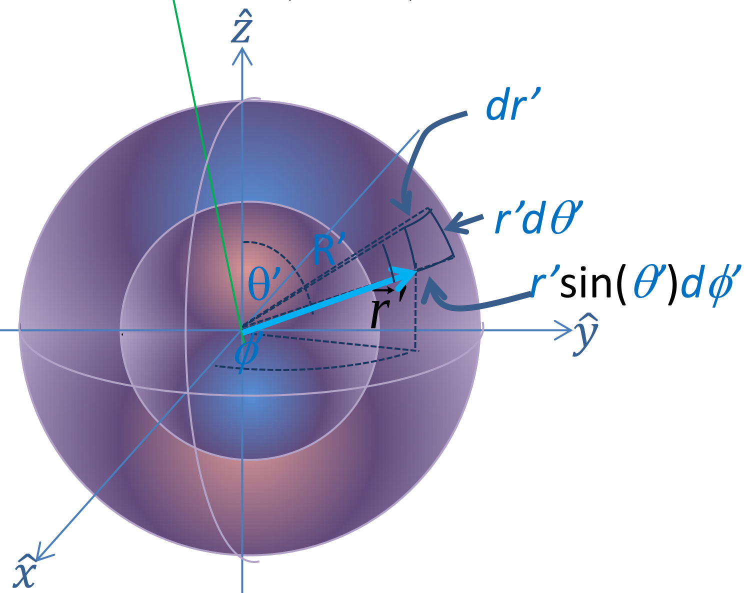
Exercise: Find the first two terms in the multi-pole expansion for this volume charge distribution.

$$V(r)_{mono} = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r} \qquad V(r)_{dipole} = \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad \vec{p} = \int \vec{r}'\rho(\vec{r}')d\tau'$$

[redacted]

$\vec{r}$

$$\rho(\vec{r}') = \rho_o\left(2 - \frac{R}{r'}\right)\sin(2\theta')$$



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