Wed. Thurs. Fri.	3.1-.2 Laplace & Images *Poster Session: Hedco7pm~9pm* **3.2** Images **T4** Relaxation Method	HW3
Mon.	**3.4.1-.4.2** Multipole Expansion	

Poisson's & Laplace's Equations

$$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$$

$$\vec{E} = -\vec{\nabla} V$$

Poisson's $$\nabla^2 V = -\rho / \varepsilon_0$$

Laplace's $$\nabla^2 V = 0$$ In "free space" where no charges are

In Cartesian

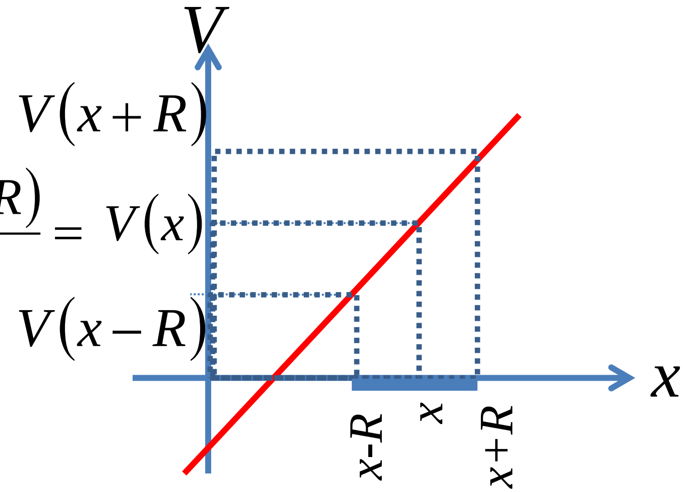
$$\nabla^2 V = \vec{\nabla} \cdot \left(\vec{\nabla} V \right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Building up: 1-D

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{d^2 V}{dx^2} = 0$$

$$\frac{V(x+R) + V(x-R)}{2} = V(x)$$



so, $\frac{dV}{dx} = \text{constant} = m$

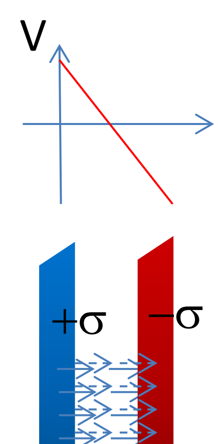
$$dV = m\,dx$$

$$\int_0^V dV = \int_{x_o}^{x} m\,dx$$

$$V = mx + x_o$$

- No local min or max
- V at mid point of range is average of V at edges

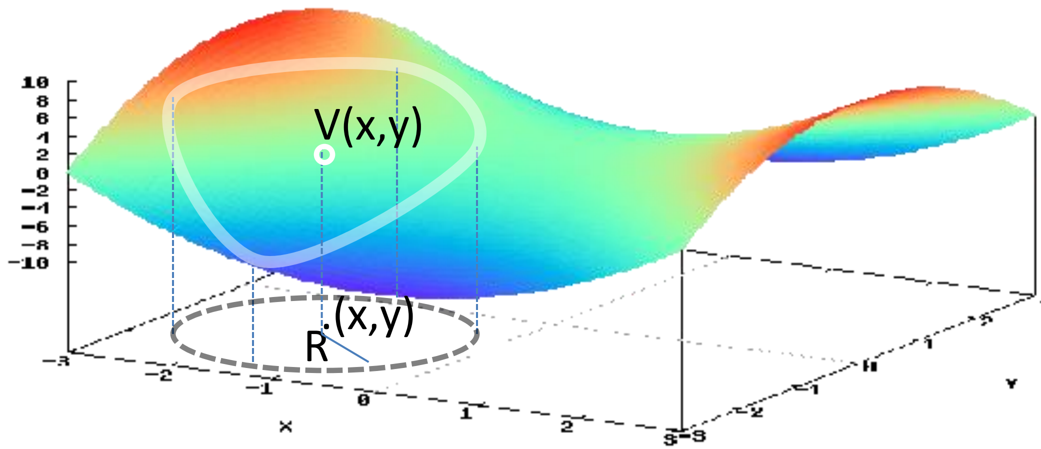
Example: voltage in parallel-plate capacitor



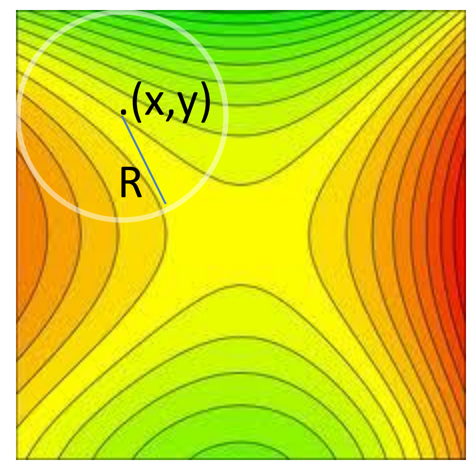
Building up: 2-D

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$\frac{\partial^2 V}{\partial x^2} = -\frac{\partial^2 V}{\partial y^2}$$



$$V(x,y) = \frac{\oint V dl_{circle}}{2\pi R}$$



- No local min or max; either flat (possible tipped plane) or saddle points

- V at mid point of range is average of V around edges (proof forthcoming)

Building up: 3-D

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Hard to visualize, but…

- No local min or max; either flat or saddle points

Consequence: can’t make a stable ‘trap’
for a charge using only electrostatic fields
– no minima for them to settle into.

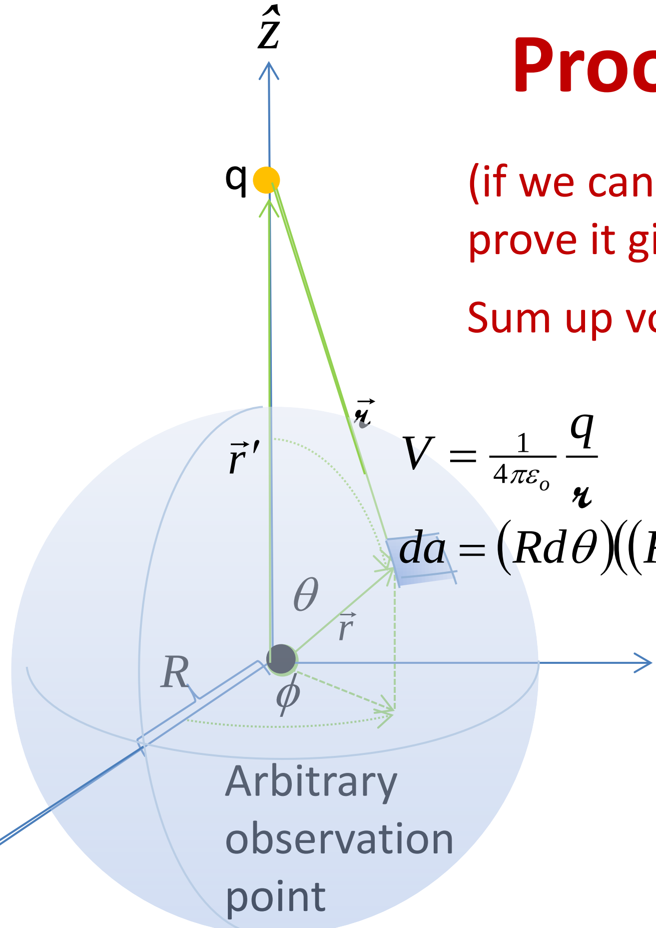
- V at mid point of range is average of V around edges
(proof, up next)

$$V(x, y, z) = \frac{\oint V da_{sphere}}{4\pi R^2}$$

Proof that $V(x,y,z)=\dfrac{\oint V da_{sphere}}{4\pi R^2}$

(if we can prove it given one point source, we can prove it given any configuration of point sources)

Sum up voltages over sphere of radius R $\oint V da$



$$V=\frac{1}{4\pi\varepsilon_o}\frac{q}{\mathscr{r}}$$

$$da=(Rd\theta)((R\sin\theta)d\phi)$$

$$\mathscr{r}=\sqrt{(\vec{r}-\vec{r}')^2}\quad=\sqrt{\vec{r}\cdot\vec{r}+\vec{r}'\cdot\vec{r}'-2\vec{r}\cdot\vec{r}}$$

$$\mathscr{r}=\sqrt{z'^2+R^2-2z'R\cos\theta}$$

$$V=\frac{1}{4\pi\varepsilon_o}\frac{q}{\sqrt{z'^2+R^2-2z'R\cos\theta}}$$

$$\oint Vda=\int_0^{\theta=\pi,}\int_0^{\phi=2\pi}\frac{1}{4\pi\varepsilon_o}\frac{q}{\sqrt{z'^2+R^2-2z'R\cos\theta}}R^2\sin\theta d\theta d\phi$$

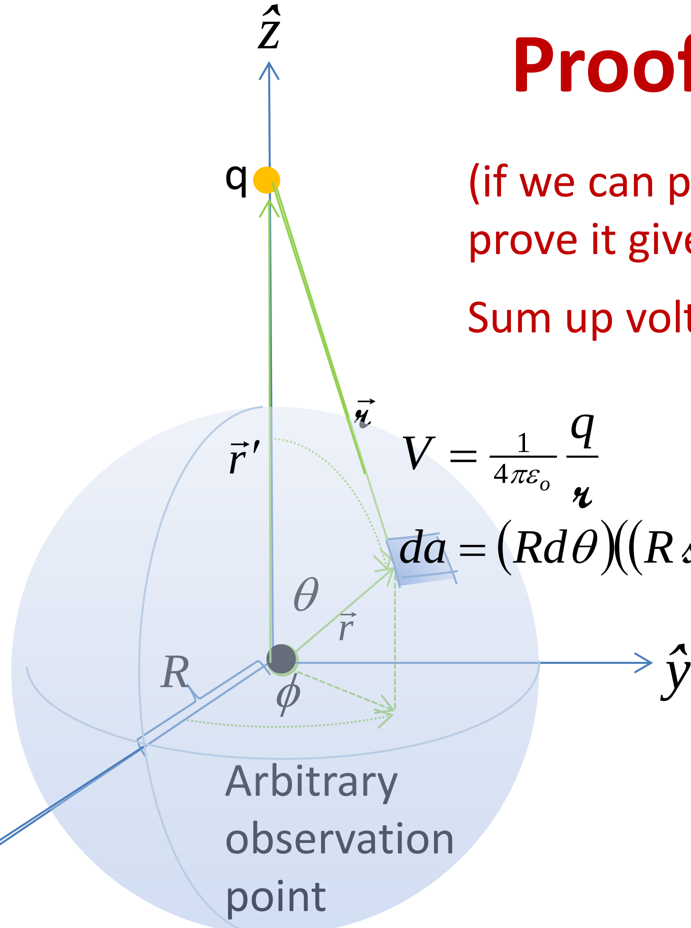
$$=\int_0^{\theta=\pi}\frac{2\pi}{4\pi\varepsilon_o}\frac{q}{\sqrt{R^2+z'^2-2z'R\cos\theta}}R^2\sin\theta d\theta$$

$$=-\frac{qR^2}{2\varepsilon_o}\int_1^{\cos\theta=-1}\frac{d(\cos\theta)}{\sqrt{R^2+z'^2-2z'R\cos\theta}}$$

Proof that $V(x,y,z) = \dfrac{\oint V da_{sphere}}{4\pi R^2}$

(if we can prove it given one point source, we can prove it given any configuration of point sources)

Sum up voltages over sphere of radius R $\oint V da$



$$V = \frac{1}{4\pi\varepsilon_o}\frac{q}{\mathscr{r}}$$

$$da = (R d\theta)((R \sin\theta) d\phi)$$

$$\oint V da = -\frac{qR^2}{2\varepsilon_o}\int_{1}^{\cos\theta=-1}\frac{d(\cos\theta)}{\sqrt{R^2+z'^2-2z'R\cos\theta}}$$

$$= \frac{qR^2}{\varepsilon_o}\frac{1}{2z'R}\sqrt{R^2+z'^2-2z'R\cos\theta}\Big|_{1}^{\cos\theta=-1}$$

$$= \frac{qR^2}{\varepsilon_o}\frac{1}{2z'R}\left(\sqrt{R^2+z'^2+2z'R}-\sqrt{R^2+z'^2-2z'R}\right)$$

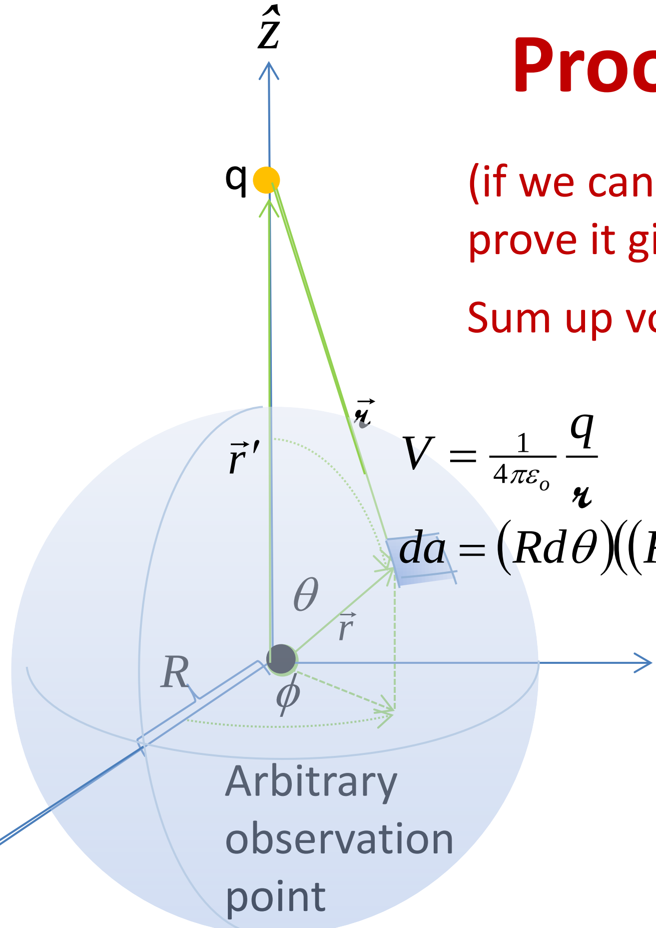
$$= \frac{qR^2}{\varepsilon_o}\frac{1}{2z'R}\left(\sqrt{(R+z')^2}-\sqrt{(R-z')^2}\right)$$

$$= \frac{qR^2}{\varepsilon_o}\frac{1}{2z'R}\left(|R+z'|-|R-z'|\right)$$

Proof that $V(x,y,z)=\frac{\oint Vda_{sphere}}{4\pi R^2}$

(if we can prove it given one point source, we can prove it given any configuration of point sources)

Sum up voltages over sphere of radius R $\oint Vda$



$$V=\frac{1}{4\pi\varepsilon_o}\frac{q}{\mathcal{r}}$$

$$da=(Rd\theta)((R\sin\theta)d\phi)$$

$$\oint Vda=\frac{qR^2}{\varepsilon_o}\frac{1}{2z'R}\left(|R+z'|-|R-z'|\right)$$

$$R<z'\Rightarrow|R-z'|=z'-R$$

$$=\frac{qR^2}{\varepsilon_o}\frac{1}{2z'R}(R+z'-z'+R)\quad=\frac{qR^2}{\varepsilon_o}\frac{1}{z'}$$

$$V(x,y,z)=\frac{\oint Vda_{sphere}}{4\pi R^2}=\frac{\left(\frac{qR^2}{\varepsilon_o}\frac{1}{z'}\right)}{4\pi R^2}=\frac{1}{4\pi\varepsilon_o}\frac{q}{z'}$$

Exactly what we knew it was all a long!

Why bother then?

Means, you can determine the voltage in the interior from that on the perimeter (even if you don't know the source charge configuration).

Example: General Solution if V depends only on distance from z-axis

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}V\right) + \frac{1}{s^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

If V depends only on *s*,

$$\nabla^2 V = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial}{\partial s}V\right) + \frac{1}{s^2}0 + 0 = 0$$

$$\frac{1}{s}\frac{d}{ds}\left(s\frac{d}{ds}V\right) = 0$$

$$\frac{d}{ds}\left(s\frac{d}{ds}V\right) = 0$$

$$s\frac{d}{ds}V = c$$

$$\frac{d}{ds}V = \frac{c}{s}$$

$$dV = \frac{c}{s}ds$$

$$\int_{V(a)}^{V(b)} dV = \int_a^b \frac{c}{s}ds$$

$$V(b) - V(a) = c\,\ln\left(\frac{b}{a}\right)$$

Exercise: General Solution if V depends only on distance from origin

Uniqueness Theorems (by hook or crook)

Voltages: (as with any differential equation) Regardless of *how* you've found it, if you've found one solution to Laplace / Poisson's equation that satisfies the boundary conditions, you've found the *only* solution.

Proof: (prove the opposite to be false)

If both V_1 and V_2 are solutions, that is $\nabla^2 V_1 = \nabla^2 V_2 = -\frac{1}{\varepsilon_o}\rho$

Since it's a *linear* differential equation, $V_3 = V_1 + V_2$ must also be a solution, that is, $\nabla^2 V_3 = -\frac{1}{\varepsilon_o}\rho$

$$\nabla^2 (V_1 + V_2) = -\frac{1}{\varepsilon_o}\rho$$

$$\nabla^2 V_1 + \nabla^2 V_2 = -\frac{1}{\varepsilon_o}\rho$$

$$\left(-\frac{1}{\varepsilon_o}\rho\right) + \left(-\frac{1}{\varepsilon_o}\rho\right) \neq -\frac{1}{\varepsilon_o}\rho$$

Mustn't be true after all – apparently there's only one solution.

Uniqueness Theorems (by hook or crook)

Fields: Given a charge density in a cavity within a charged conductor, the field within the conductor is uniquely determined by the *inner* charge distribution and the conductor's charge amount.

Proof: (prove the opposite to be false)

Assume both E_1 and E_2 are solutions. Their difference is $\vec{E}_3 \equiv \vec{E}_2 - \vec{E}_1$

$$\vec{E}_3 \cdot \vec{\nabla} V_3 + V_3 \left(\vec{\nabla} \cdot \vec{E}_3 \right) = \vec{\nabla} \cdot \left(V_3 \vec{E}_3 \right)$$ Product rule

$$-\vec{E}_3 = \vec{\nabla} V_3 \qquad \nabla \cdot \vec{E}_3 \equiv \nabla \cdot \vec{E}_2 - \nabla \cdot \vec{E}_1$$

$$-E_3^2 + 0 = \vec{\nabla} \cdot \left(V_3 \vec{E}_3 \right)$$

The area is the surface of the enclosing conductor, i.e., an equipotential, so V is constant over the integral.

$$-\int_{vol} E_3^2 d\tau = \int_{vol} \vec{\nabla} \cdot \left(V_3 \vec{E}_3 \right) d\tau$$

Gauss's Law

The integrand is clearly never negative, so no way for it to sum to zero by some contributions canceling others; must *always* be 0

$$E_3^2 = 0 \qquad \vec{E}_3 \equiv \vec{E}_2 - \vec{E}_1 = 0 \qquad \vec{E}_2 = \vec{E}_1$$

Charge distribution uniquely determines field

Hooks and Crook: Interesting ways of finding V and E

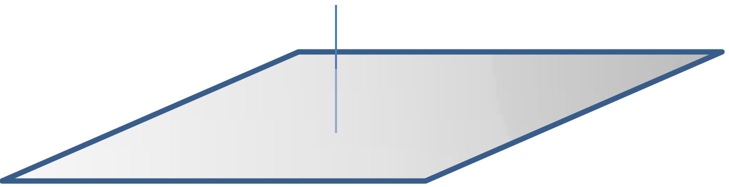
Images: replace a problem with a simpler equivalent one (based on corollary of the first uniqueness theorem)

Relaxation: a computational method based on the potential at a point being the average of the values at the same distance (more about Next Time).

Multipole Expansion: a method for getting approximate answers for V far from a charge distribution (section 3.4)

Charge Images Reflected in Conductors

Images: replace a problem with a simpler equivalent one (based on corollary of the first uniqueness theorem)



You know the flat surface is an equipotential / the electric field goes perpendicularly into it.
Given your charge above, where could you put another charge to get these V and E properties in the plane?

$$\frac{q_o}{r_o} + \frac{q_i}{r_i} = V_{surface} = const$$

Voltage on conductor surface is constant

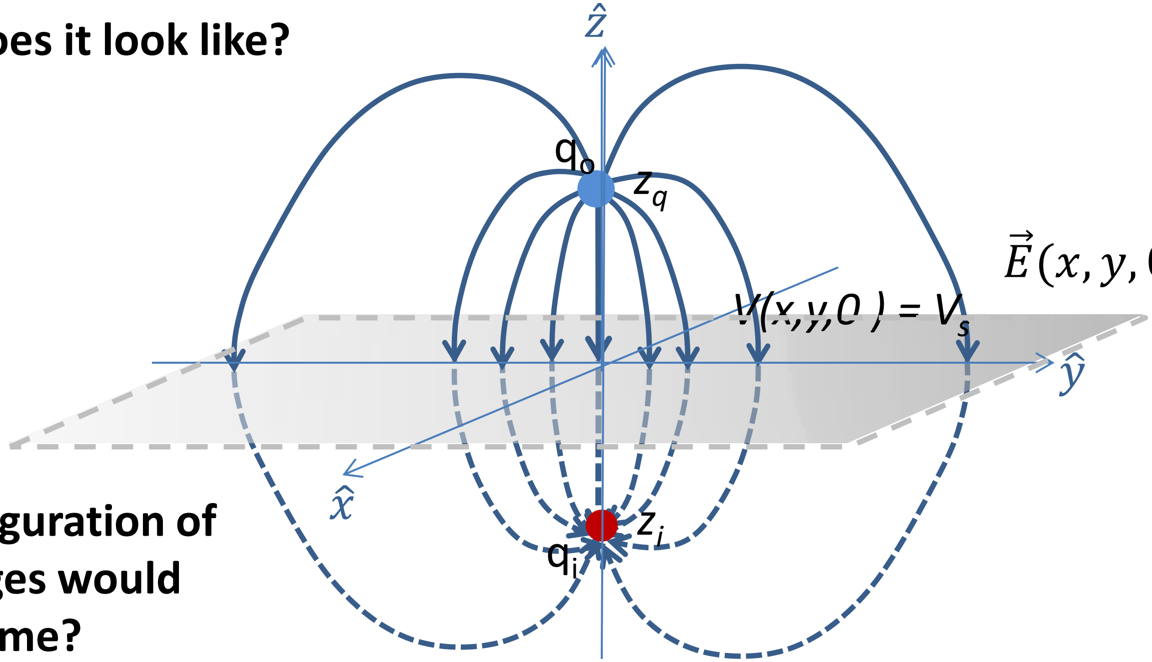
Much more next time

Charge Images Reflected in Conductors

Images: replace a problem with a simpler equivalent one (based on corollary of the first uniqueness theorem – if your solution works on the boundary, it works everywhere)

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

What does it look like?



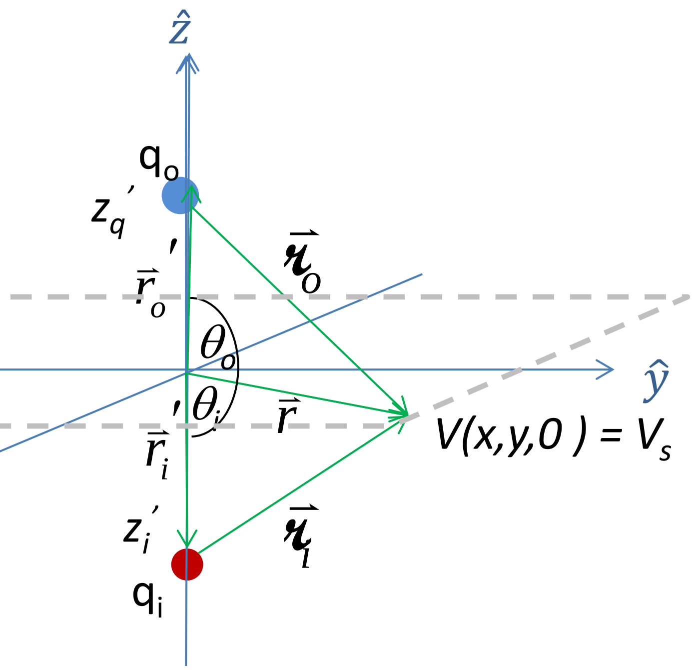
$\vec{E}(x,y,0)$ points straight down.

What configuration of point charges would look the same?

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

In gory detail (for the experience), determine image's charge and location



For observation location in plane of the conductor

$\theta_o = \theta_i = 90°$

$Cos\ \theta_o = Cos\,\theta_i = 0$

$$\frac{q_o}{\sqrt{r^2 + z_o'^2}} + \frac{q_i}{\sqrt{r^2 + z_i'^2}} = 4\pi\varepsilon_o V_s$$

Must be true for *all* location in plane, so can choose easy-to-evaluate locations to determine values of q_i and z_i.

$r \to \infty$

$$\frac{q_o}{\sqrt{\infty^2 + z_o'^2}} + \frac{q_i}{\sqrt{\infty^2 + z_i'^2}} = 4\pi\varepsilon_o V_s$$

$$0 + 0 = 4\pi\varepsilon_o V_s$$

Apparently works at all only if $V_s = 0$.

$r \to 0$

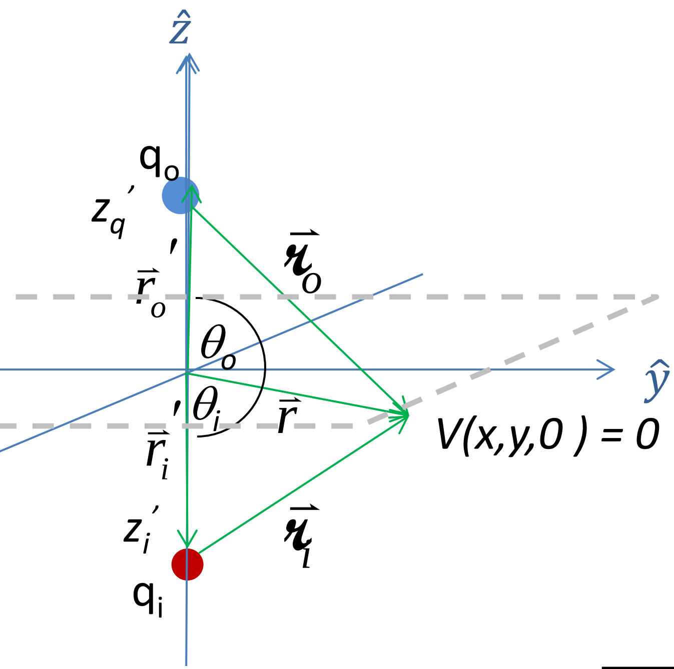
$$\frac{q_o}{\sqrt{z_o'^2}} + \frac{q_i}{\sqrt{z_i'^2}} = 0$$

$$\frac{q_i}{|z_i'|} = -\frac{q_o}{|z_o'|}$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

In gory detail (for the experience), determine image's charge and location



Return to $$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{q_i}{\sqrt{r^2+z_i'^2}}=4\pi\varepsilon_o V_s$$

Plug in what we've learned: $V_s = 0 \qquad q_i = -q_o\left|\frac{z_i'}{z_o'}\right|$

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{-q_0\left|\frac{z_i'}{z_o'}\right|}{\sqrt{r^2+z_i'^2}}=0$$

$$\frac{1}{\sqrt{r^2+z_o'^2}}=\left|\frac{z_i'}{z_o'}\right|\frac{1}{\sqrt{r^2+z_i'^2}}$$

$$\frac{1}{|z_o'|\sqrt{\left(\frac{r}{z_o'}\right)^2+1}}=\left|\frac{z_i'}{z_o'}\right|\frac{1}{|z_i'|\sqrt{\left(\frac{r}{z_i'}\right)^2+1}}$$

$$\frac{1}{\sqrt{\left(\frac{r}{z_o'}\right)^2+1}}=\frac{1}{\sqrt{\left(\frac{r}{z_i'}\right)^2+1}}$$

$$|z_o'|=|z_i'|$$

$$z_i'=-z_o'$$

$$q_i=-q_o\left|\frac{z_i'}{z_o'}\right|$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

Digression: Force between q_o and Surface

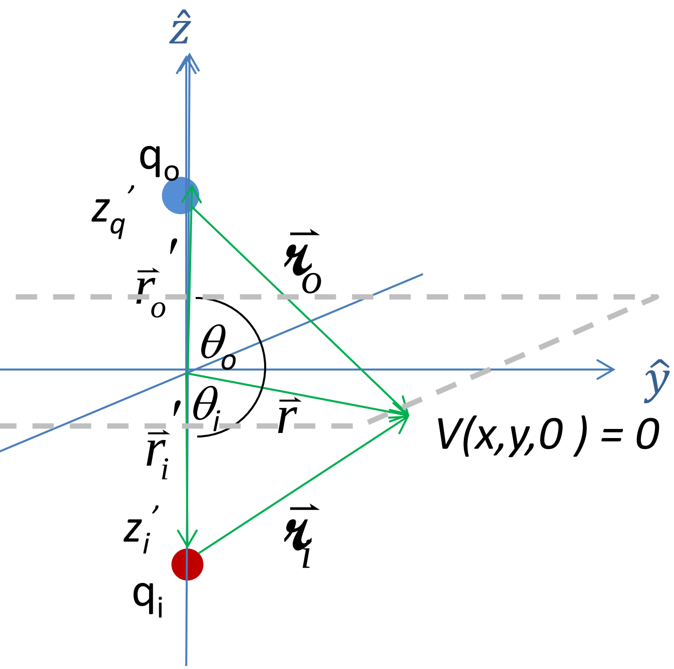


Image charge distance and magnitude

$$z_i' = -z_o' \qquad q_i = -q_o$$

Field everywhere above the plane is as if there were these two charges

$$\vec{F}_{i\to o} = \frac{1}{4\pi\varepsilon_o}\frac{q_i q_o}{(z_o - z_i)^2}\hat{z}$$

$$\vec{F}_{i\to o} = -\frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z_o)^2}\hat{z}$$

So,

$$\vec{F}_{o\to surface} = \frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z_o)^2}\hat{z}$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

Digression: Work of moving q_o into place (and arranging surface charge)

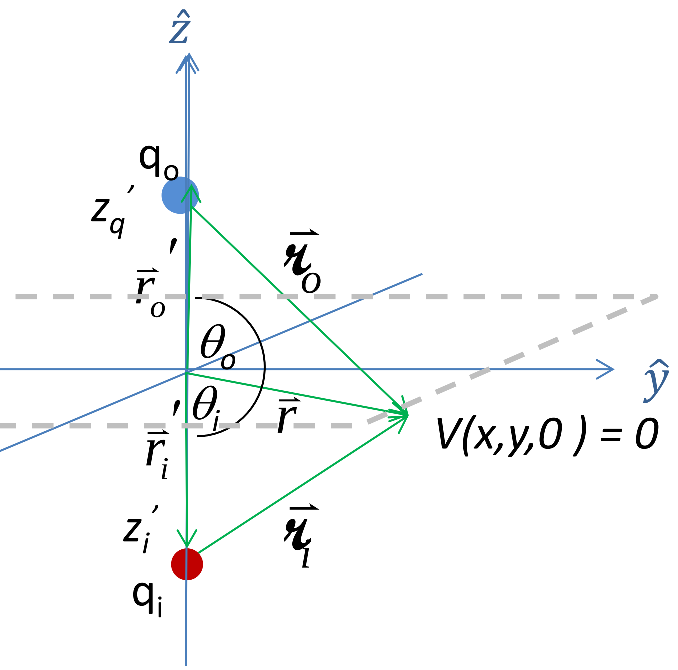


Image charge distance and magnitude

$$z_i' = -z_o' \qquad q_i = -q_o$$

Field everywhere above the plane is as if there were these two charges

$$\vec{F}_{surface\to o} = -\frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z_o)^2}\hat{z}$$

$$W = \int_\infty^{z_o} \vec{F}_{you\to o}\cdot d\vec{l} = -\int_\infty^{z_o} \vec{F}_{surface\to o}\cdot d\vec{l} = \int_{z_o}^{\infty} \vec{F}_{surface\to o}\cdot d\vec{l}$$

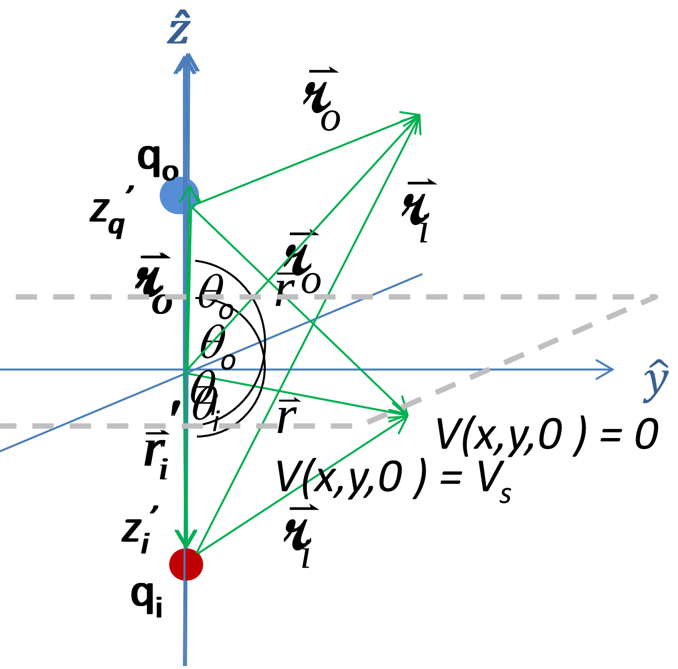
$$W = \int_{z_o}^{\infty} -\frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z)^2}\hat{z}\cdot d\vec{l} = \int_{z_o}^{\infty} -\frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z)^2}dz$$

$$W = -\frac{q_o^2}{16\pi\varepsilon_o}\int_{z_o}^{\infty}\frac{1}{z^2}dz = \frac{q_o^2}{16\pi\varepsilon_o}\frac{1}{z}\bigg|_{z_o}^{\infty} = -\frac{q_o^2}{16\pi\varepsilon_o}\frac{1}{z_o}$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

Return to determine image's charge and location



Return to

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{q_i}{\sqrt{r^2+z_i'^2}}=4\pi\varepsilon_o V_s$$

Plug in what we've learned: $z_i' = -z_o'$ $\quad q_i = -q_o$

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{-q_o}{\sqrt{r^2+z_o'^2}}=4\pi\varepsilon_o V_s$$

Works on the surface:

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{-q_o}{\sqrt{r^2+z_o'^2}}=0$$

Must work *everywhere* above the plane:

$$\frac{q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos\theta_o}}+\frac{q_i}{\sqrt{r^2+z_i'^2-2rz_i'\cos\theta_i}}=4\pi\varepsilon_o V(\vec{r})$$

becomes:

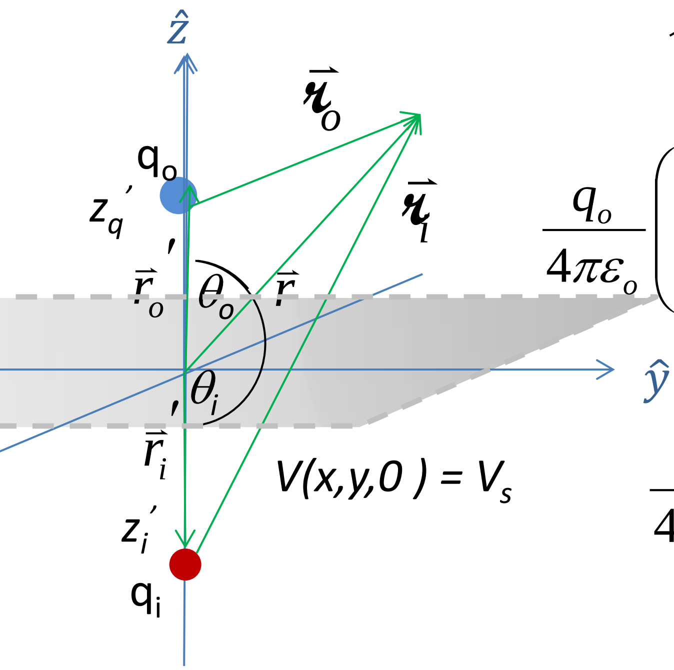
$$\frac{q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos\theta_o}}+\frac{-q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos(\pi-\theta_o)}}=4\pi\varepsilon_o V(\vec{r})$$

$$\frac{q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos\theta_o}}+\frac{-q_o}{\sqrt{r^2+z_o'^2+2rz_o'\cos(\theta_o)}}=4\pi\varepsilon_o V(\vec{r})$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

Return to determine image's charge and location



Must work *everywhere* above the plane:

$$\frac{q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos\theta_o}}+\frac{-q_o}{\sqrt{r^2+z_o'^2+2rz_o'\cos(\theta_o)}}=4\pi\varepsilon_o V(\vec{r})$$

If $V(x,y,0)=V_s\neq 0$ **simply add the constant offset.**

$$\frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{r^2+z_o'^2-2rz_o'\cos\theta_o}}-\frac{1}{\sqrt{r^2+z_o'^2+2rz_o'\cos(\theta_o)}}\right)+V_s=V(\vec{r})$$

or

$$\frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{x^2+y^2+(z-z_o')^2}}-\frac{1}{\sqrt{x^2+y^2+(z+z_o')^2}}\right)+V_s=V(\vec{r})$$

Even for the *real* system

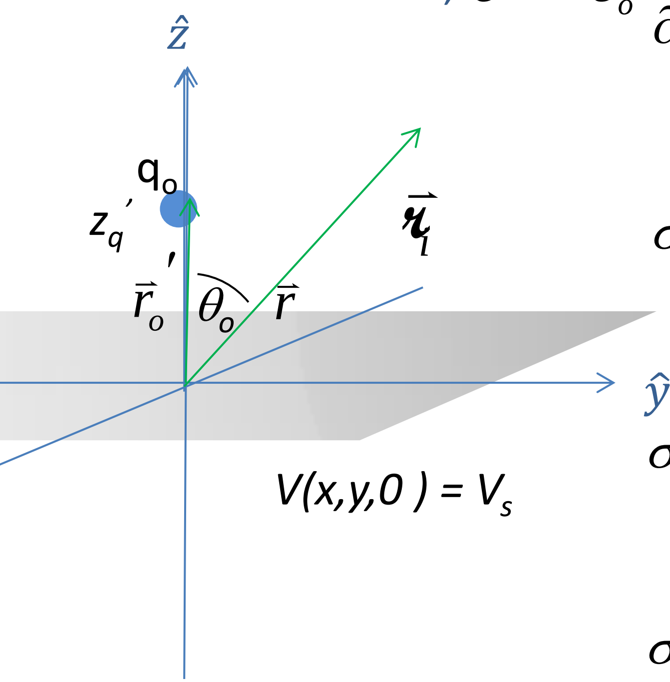
Surface Charge Density

$$\frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{x^2+y^2+(z-z_o')^2}}-\frac{1}{\sqrt{x^2+y^2+(z+z_o')^2}}\right)+V_s=V(\vec{r})$$

From a Gaussian pill-box: $E_n=\dfrac{\sigma}{\varepsilon_o}$ *n* denotes component perpendicular to surface

In terms of V: $-\dfrac{\partial V}{\partial n}=E_n=\dfrac{\sigma}{\varepsilon_o}$ So, $\sigma=-\varepsilon_o\left.\dfrac{\partial V}{\partial n}\right|_{surface}$

In this case, $\sigma=-\varepsilon_o\left.\dfrac{\partial V}{\partial z}\right|_{z=0}=-\dfrac{q_o}{4\pi}\dfrac{\partial}{\partial z}\left.\left(\dfrac{1}{\sqrt{x^2+y^2+(z-z_o')^2}}-\dfrac{1}{\sqrt{x^2+y^2+(z+z_o')^2}}\right)\right|_{z=0}$



$$\sigma=-\frac{q_o}{4\pi}\left.\left(\frac{-1}{2}\frac{2(z-z_o')}{\left(x^2+y^2+(z-z_o')^2\right)^{\frac{3}{2}}}-\frac{-1}{2}\frac{2(z+z_o')}{\left(x^2+y^2+(z+z_o')^2\right)^{\frac{3}{2}}}\right)\right|_{z=0}$$

$$\sigma=-\frac{q_o}{4\pi}\left(\frac{(z_o')}{\left(x^2+y^2+(z_o')^2\right)^{\frac{3}{2}}}+\frac{(z_o')}{\left(x^2+y^2+(z_o')^2\right)^{\frac{3}{2}}}\right)$$

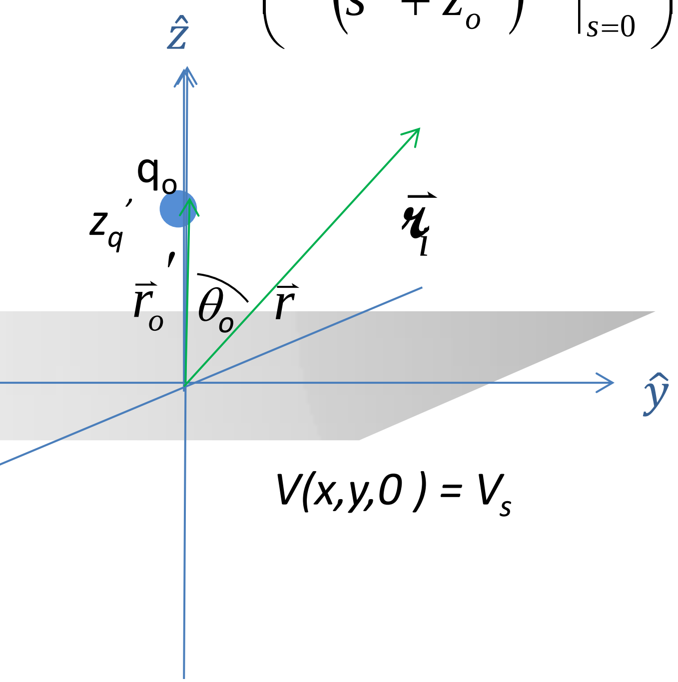
$$\sigma=-\frac{q_o}{2\pi}\frac{z_o'}{\left(x^2+y^2+z_o'^2\right)^{\frac{3}{2}}}$$

Surface Charge

$$q_{surf} = \int \sigma da$$

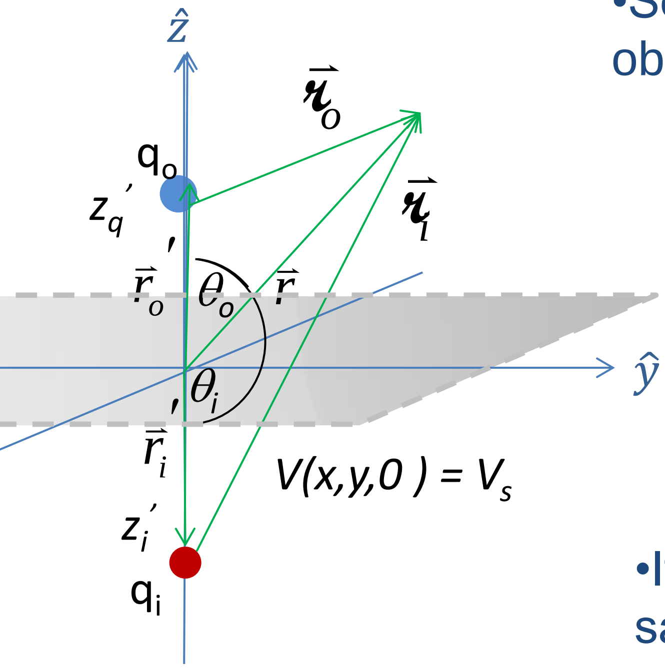
$$q_{surf} = \int -\tfrac{q_o}{4\pi}\frac{2z_o'}{\left(x^2+y^2+z_o'^2\right)^{3/2}}dxdy = -\tfrac{q_o}{4\pi}\int\frac{2z_o'}{\left(s^2+z_o'^2\right)^{3/2}}sd\phi ds = -q_o\int\frac{z_o'}{\left(s^2+z_o'^2\right)^{3/2}}\tfrac{1}{2}ds^2$$

$$q_{surf} = -q_o\left(-\frac{z_o'}{\left(s^2+z_o'^2\right)^{1/2}}\Bigg|_{s=0}^{s=\infty}\right) = q_o\left(0-\frac{z_o'}{\left(z_o'^2\right)^{1/2}}\right) = -q_o$$



General Approach

- Draw picture
- Appeal to symmetry (and intuition about mirrors)
- Apply the condition $\sum \frac{q_o}{\mathcal{r}_o} + \sum \frac{q_i}{\mathcal{r}_i} = const$ on conductor
- See what you've got to do to remove dependence on the observation location on conductor.

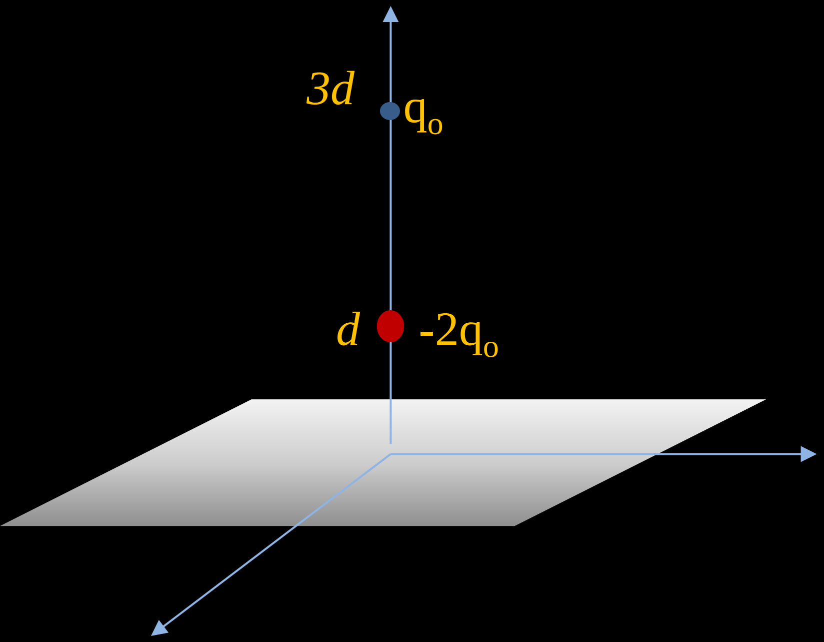


1)Mathematically, you've got 3 free parameters: the constant, the image charge's value, and the image charge's location.

2) Since the relation should be true for *all* observation locations on the conducting surface, choose easy ones to help you determine the three parameter's values.

- If you've got a solution that works for the boundary and satisfies Poisson's equation, then you've got *the* solution.

Exercise: where and what are the image charges?

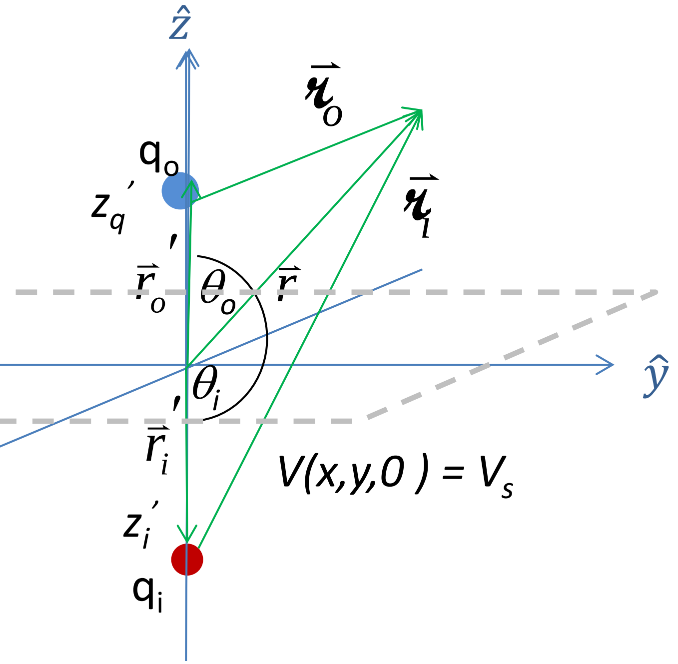


Wed. Thurs. Fri.	3.1-.2 Laplace & Images *Poster Session: Hedco7pm~9pm* **3.2** Images **T4** Relaxation Method	 HW3
Mon.	**3.4.1-.4.2** Multipole Expansion	

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

In gory detail (for the experience), determine image's charge and location



For observation location anywhere (above the conductor)

$$\frac{q_o}{\mathscr{r}_o} + \frac{q_i}{\mathscr{r}_i} = 4\pi\varepsilon_o V(\vec{r})$$

$$\frac{q_o}{\sqrt{(\vec{r} - \vec{r}_o')^2}} + \frac{q_i}{\sqrt{(\vec{r} - \vec{r}_i')^2}} = 4\pi\varepsilon_o V(\vec{r})$$

$$\frac{q_o}{\sqrt{r^2 + r_o'^2 - 2rr_o'\cos\theta_o}} + \frac{q_i}{\sqrt{r^2 + r_i'^2 - 2rr_i'\cos\theta_i}} = 4\pi\varepsilon_o V(\vec{r})$$

$$\frac{q_o}{\sqrt{r^2 + z_o'^2 - 2rz_o'\cos\theta_o}} + \frac{q_i}{\sqrt{r^2 + z_i'^2 - 2rz_i'\cos\theta_i}} = 4\pi\varepsilon_o V(\vec{r})$$