Phys 331: Ch 9, .1-.3 Noninertial Frames: Acceleration, Ties, and Angular Velocity.1

Mon., 11/19 Tues.,11/20	**9.3-.5** Noninertial Frames: L, d/dt, Newton's 2nd.	**HW9a** (9.2, 9.8)
Mon., 11/26 Tues. 11/27	**9.6-.7** Fictional Forces: Centrifugal & Coriolis	**HW9b** (9.14, 9.24)

Equipment

- Globe
- Mass-spring rod (just one mass)
- Tinker toys
- Styerfoam balls and knitting needles?

Non-inertial Frames

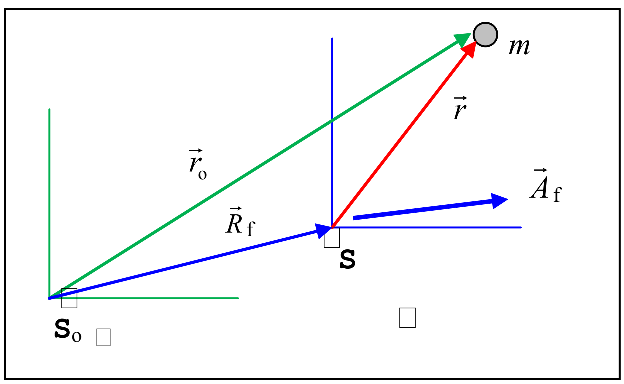
As we've seen a few times now, the position, velocity, and acceleration measurements made relative to two different reference frames depends on how the two frames are positioned, moving, and accelerating relative to each other.

$$\vec{r} = \vec{r}_o - \vec{R}_f$$

$$\dot{\vec{r}} = \dot{\vec{r}}_o - \vec{V}_f$$

And

$$\ddot{\vec{r}} = \ddot{\vec{r}}_o - \vec{A}_f$$



If S_o is an inertial frame, then Newton's 2nd Law relates the net force and acceleration measured relative to it:

$$\vec{F}_{net} = m\ddot{\vec{r}}_o$$

Or

$$\ddot{\vec{r}}_o = \frac{\vec{F}_{net}}{m}$$

So,

$$\ddot{\vec{r}} = \frac{\vec{F}_{net}}{m} - \vec{A}_f$$

Fictitious Inertial / Frame Force

$m\ddot{\vec{r}} = \vec{F}_{net} + \vec{F}_{frame}$

$\vec{F}_{frame} \equiv -m\vec{A}_f$

Last time we applied this ideal for *translating* non-inertial frames to a few situations and got familiar with how this fictitious frame force is equivalent to a gravitational force – it scales with an object's mass; indeed, that recognition is was a key step in the development of Einstein's General Relativity.

This Time

Now it's time for us to consider how things look when frames are *rotating* relative to each other. It's not terribly different.

As you're familiar, riding on a spinning merry-go-round means you're accelerating even if your *speed* isn't changing because the *direction* of your motion is always changing.

Harkening back to our discussion of position, velocity, and acceleration in polar coordinates, you may recall that

$$\vec{a}_o = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{r} + \left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right)\hat{\phi}$$

So, it's pretty clear that someone say standing on a merry-go-round, and simply riding along in a uniformly rotating frame, will be seen to have the frame's acceleration of

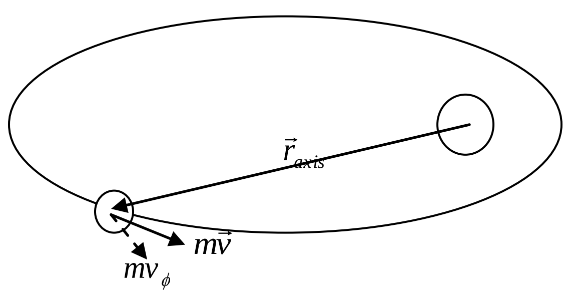
$$\vec{A}_f = \left(-r\dot{\phi}^2\right)\hat{r}$$

But what *isn't* so obvious is how someone *moving* in that frame is seen by an inertial observer to accelerate, or what fictitious forces such a someone would perceive when judging things' motions relative to this rotating frame.

What we're interested in doing today is relating the person's acceleration relative to a rotating frame, **a** and acceleration relative to an inertial observer, $\mathbf{a_o}$. We'll start by simply considering how we describe rotation.

9.3 Angular Velocity Vector:

Recall that an object's angular momentum about a point is a vector. For example, if the Earth is going around the sun, as illustrated,



$\vec{L}_{axi} = \vec{r}_{axis} \times m\vec{v}$

The cross-product gives a direction for this vector which is easy to remember by the right-hand-rule: start with your right hand aligned with the first vector (r), and then curl your fingers in the direction of the second (v), then your thumb points in the direction of the angular momentum vector. In this case, it's out of the board.

On the other hand, the magnitude of the angular momentum is simply

$$L = r_{axis} m v \sin\theta_{r-v}$$

where one way of looking at the factor of sine is that it picks off just the component of the velocity which is perpendicular to the radius, i.e., the $\hat{\phi}$ component which we can rephrase as

$$v_\phi = r_{axis}\dot{\phi},$$

so,

$$L = r_a m r_a \dot{\phi} = m r_a^{\,2} \dot{\phi}$$

also written as

$$L = m r_a^{\,2} \omega$$

where

$$\omega \equiv \dot{\phi}.$$

Which is an identification that we've used a lot in the last two chapters.

So, one could say that the angular momentum vector is

$$\vec{L} = m r_a^{\,2} \omega \hat{L}$$

For convenience, it is conventional to define the "angular velocity vector" to be

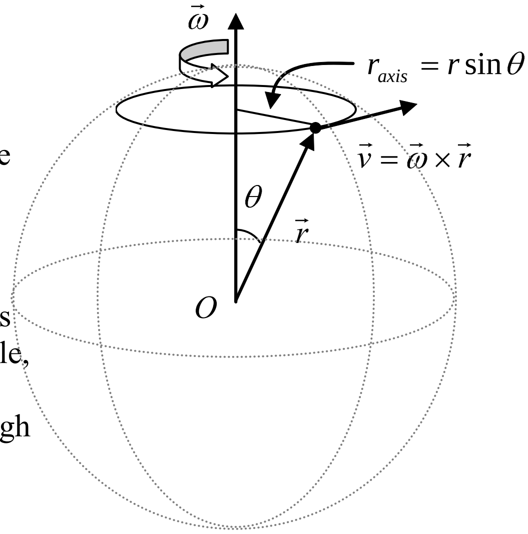
$\boxed{\vec{\omega} \equiv \dot{\phi}\hat{L}}$, that is, to assign it the direction of the associated angular momentum (found by the right-hand rule).

(note, this is *not* the direction you'd get if you simply took $\dfrac{d(\phi\hat{\phi})}{dt}$ or the direction of $\dfrac{d(\hat{\phi})}{dt}$ alone, so this really is a new definition).

One way to think about the direction is imagine laying your right hand so your fingers arc along a segment of the object's trajectory, then your thumb points in the direction of the angular velocity. Or, for that matter, if the object were something like a tether ball going around a pole, than your thumb would be pointing parallel to the pole / axis about which it's moving.

Rigid Body angular – linear velocity.

Demo: Globe



In the preceding development, we used that $v_\phi = r_{axis}\dot{\phi}$.

Now, if we are interested in *rigid body* rotation (like the Earth's), then the angular component of the velocity is the *only* component:

$$\vec{v} = v_\phi\hat{\phi}$$

And if it's convenient for us to select an origin which lies on the axis of rotation that we're interested in (for example, rotation about the Earth's N-S axis with the origin at the center of the Earth.), then about an axis which goes through the origin,

$$\vec{v} = r_{axis}\omega\hat{\phi} = (r\sin\theta)\omega\hat{\phi} = \vec{\omega}\times\vec{r}$$

$$\frac{d}{dt}\vec{r} = \vec{\omega}\times\vec{r}$$

We can generalize the relation above to any vector that rotates with the body. Most importantly, if we embed a coordinate system in the rotating object, then the rate with which a direction unit vector itself spins around is:

$$\frac{d\hat{x}}{dt} = \vec{\omega}\times\hat{x}\text{ . (ditto for y and z)}$$

This is the time derivative of the unit vector according to an observer who is <u>not</u> rotating, for example, the rate with which $\hat{r}$ spins around.

Rotating Coordinate Systems

The above will be useful when dealing with rotating coordinate systems

We will use the following notation:

- $\vec{\omega}$ - will refer to the angular velocity of a body (object) as measured relative to some frame
- $\vec{\Omega}$ - will be the angular velocity of a noninertial, rotating reference frame as observed by an inertial / non-rotating frame just as we've used V and A for a frame's velocity and acceleration.

For example, the earth rotates once every 24 hours, so when we measure things here on Earth, we're measuring relative to a frame rotating with

$$\Omega = \frac{2\pi\text{ rad}}{24\times 3600\text{ s}} \approx 7.3\times 10^{-5}\text{ rad/s},$$

and $\vec{\Omega}$ points toward the North Pole (sun "sets" in the west, so we rotate toward the east).

Addition of Angular velocities

By the way, angular velocities (with respect to the same origin) add just as regular velocities do. For example, if you watch two air planes arcing across the sky, if you see one going through 10° per minute and another following it and going 4° per minute, then you see their separation growing by 6° per minute.

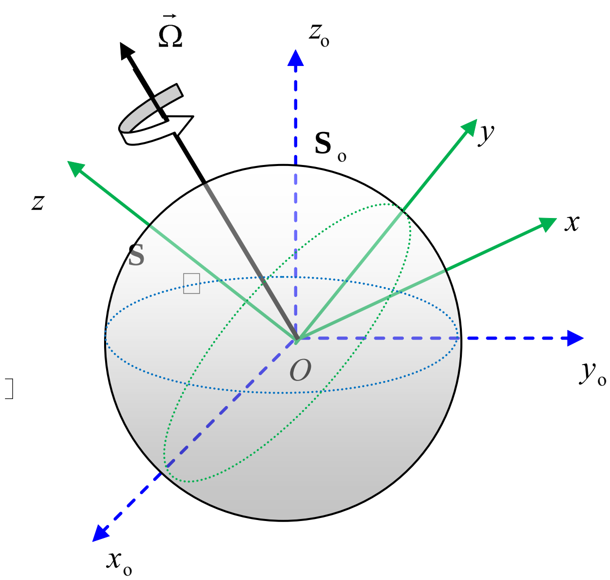
$$\vec{\omega}_{1,2} = \vec{\omega}_{2,you} - \vec{\omega}_{2,you}$$

Similarly,

$$\vec{\omega} = \vec{\omega}_o - \vec{\Omega}$$

<u>Time Derivatives in a Rotating Frame</u>:

Okay, now we're ready to consider the two perspectives of someone in a rotating frame and someone not; for example, someone on the spinning Earth and someone hanging out in space. Suppose an Earth-anchored noninertial frame **S** is rotating with a constant angular velocity $\vec{\Omega}$ relative to an Heavens-anchored inertial frame $\mathbf{S}_o$ as shown below.



If $\vec{Q}$ is *some* vector, say a position or velocity of bug walking across the surface of a globe, we would like to be able to relate its time derivatives in the two frames (that of the spinning globe and that of the relatively-stationary room.) Since the derivatives won't be the same, we will adopt the following notation to distinguish them:

$$\left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}_o} = \text{rate of change of vector } \vec{Q} \text{ relative to inertial frame } \mathbf{S}_o,$$

$$\left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}} = \text{rate of change of vector } \vec{Q} \text{ relative to rotating frame } \mathbf{S}.$$

Rotating Frame. Suppose the unit vector in the rotating frame **S** are $\hat{x}$, $\hat{y}$, and $\hat{z}$. The vector can be written as:

$$\vec{Q} = Q_x\hat{x} + Q_y\hat{y} + Q_z\hat{z}. \text{ (measured in rotating frame)}$$

In the rotating frame **S**, the unit vectors do not appear to change, so the derivative in that frame is simply:

$$\left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}} = \frac{dQ_x}{dt}\hat{x} + \frac{dQ_y}{dt}\hat{y} + \frac{dQ_z}{dt}\hat{z}. \text{ (measured in rotating frame)}$$

Inertial Frame. Relative to the inertial frame $\mathbf{S}_o$'s axes, the (rotating) unit vectors are seen to rotate / change. So when taking the derivative in *this* frame, we must use the product rule:

$$\left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}_o} = \frac{dQ_x}{dt}\hat{x} + \frac{dQ_y}{dt}\hat{y} + \frac{dQ_z}{dt}\hat{z} + Q_x\frac{d\hat{x}}{dt} + Q_y\frac{d\hat{y}}{dt} + Q_z\frac{d\hat{z}}{dt}.$$

Now, the first three terms are simply the vector's rate of change as seen in the rotating frame, and for the latter three, since the unit vectors are anchored in the rotating frame, we can invoke:

$$\left(\frac{d\hat{x}}{dt}\right)_{\mathbf{S}_o} = \vec{\Omega}\times\hat{x}, \text{ and similarly for the y and z directions.}$$

This gives:

$$\left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}_o} = \left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}} + Q_x\vec{\Omega}\times\hat{x} + Q_y\vec{\Omega}\times\hat{y} + Q_z\vec{\Omega}\times\hat{z} = \left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}} + \vec{\Omega}\times\left(Q_x\hat{x} + Q_y\hat{y} + Q_z\hat{z}\right)$$

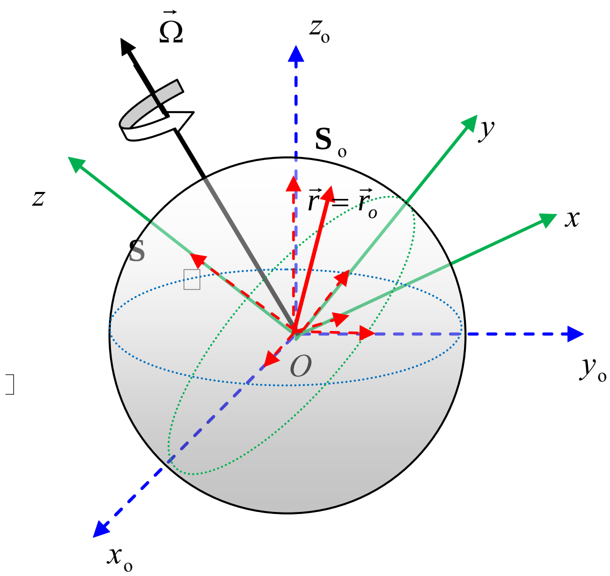
$$\boxed{\left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}_o} = \left(\frac{d\vec{Q}}{dt}\right)_{\mathbf{S}} + \vec{\Omega}\times\vec{Q}}$$

And that relates how *any* vector Q is seen to evolve from the perspective of the rotating frame and from that of the inertial frame.

Newton's Second Law in a Rotating Frame:

Obviously then, we want to apply this to the case of a position vector, $\vec{r}$. For simplicity, we'll give both frames the same reference point / origin, so

$\vec{r}_o = \vec{r}$ (the position vector is drawn as the same arrow, though it projects differently onto the different sets of axes)



Yet, by virtue of the two frames rotating relative to each other

$$\left(\frac{d\vec{r}}{dt}\right)_{\mathbf{S}_o} = \left(\frac{d\vec{r}}{dt}\right)_{\mathbf{S}} + \vec{\Omega}\times\vec{r}$$
$$\dot{\vec{r}}_o = \dot{\vec{r}} + \vec{\Omega}\times\vec{r} \qquad .$$
$$\dot{\vec{r}} = \dot{\vec{r}}_o - \vec{\Omega}\times\vec{r}$$

Which is essentially what we had when frames were just *translating* relative to each other if you think of the second term as the velocity with which the inertial observe sees the point of the frame moving, $\vec{V}_\phi$.

The second derivative in the inertial frame $\mathbf{S}_o$ is:

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{\mathbf{S}_o} = \left(\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)_{\mathbf{S}_o}\right)_{\mathbf{S}_o} = \left(\frac{d}{dt}\left[\left(\frac{d\vec{r}}{dt}\right)_{\mathbf{S}} + \vec{\Omega}\times\vec{r}\right]\right)_{\mathbf{S}_o} .$$

Substitute in the first derivative to get:

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{\mathbf{S}_o} = \left(\frac{d}{dt}\left[\left(\frac{d\vec{r}}{dt}\right)_{\mathbf{S}} + \vec{\Omega}\times\vec{r}\right]\right)_{\mathbf{S}} + \vec{\Omega}\times\left[\left(\frac{d\vec{r}}{dt}\right)_{\mathbf{S}} + \vec{\Omega}\times\vec{r}\right]$$

Since we're imagining that the frame is spinning at a constant rate, $\vec{\Omega}$, the result can be written as:

$$\left(\frac{d^2\vec{r}}{dt^2}\right)_{\mathbf{S}_o} = \left(\frac{d^2\vec{r}}{dt^2}\right)_{\mathbf{S}} + 2\vec{\Omega}\times\left(\frac{d\vec{r}}{dt}\right)_{\mathbf{S}} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right),$$

$$\ddot{\vec{r}}_o = \ddot{\vec{r}} + 2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)$$

Or we could rewrite this solving for the acceleration measured relative to the rotating frame,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_o - \left(2\vec{\Omega}\times\dot{\vec{r}} + \vec{\Omega}\times\left(\vec{\Omega}\times\vec{r}\right)\right)$$

Which is quite nearly of the same form as what we had when the frames were translating relative to each other since the terms in brackets look a lot like the acceleration of the point in the rotating frame (except that it depends on the actual object's velocity relative to the frame).

We can rewrite it as

$$\ddot{\vec{r}} = \ddot{\vec{r}}_o + 2\dot{\vec{r}}\times\vec{\Omega} + \left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega}$$

(note that the sign flip comes with reversing the order of the cross product. This is generally true for cross-products; for example, crossing 'forward' with 'left' leaves your thump point in 'up', but crossing 'left' with 'forward' leaves your thumb pointing in the opposite direction – down.)

Newton's second law applies in the inertial reference frame $\mathbf{S}_o$, so:

$$m\ddot{\vec{r}}_o = \vec{F}_{net}.$$

Inserting that in the previous expression gives our expression for the acceleration of an object relative to the rotating from when subject to a net force:

$$\boxed{m\ddot{\vec{r}} = \vec{F}_{net} + 2m\dot{\vec{r}}\times\vec{\Omega} + m\left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega}},$$

which is Newton's second law in the rotating frame. To make this concrete, think again of the bug walking across the surface of a spinning globe.

It has two extra terms called the *Coriolis force*:

$$\vec{F}_{\mathrm{cor}} = 2m\dot{\vec{r}}\times\vec{\Omega},$$

and the *centrifugal force*:

$$\vec{F}_{\mathrm{cf}} = m\left(\vec{\Omega}\times\vec{r}\right)\times\vec{\Omega}.$$

This might look a little more familiar if we write it as $\vec{F}_{\mathrm{cf}} = mr_{axis}\Omega^2\hat{r}_{axis}$

Then, from the perspective of someone riding along in the rotating frame, when you see something accelerate relative to you, it does so as if the real forces *plus* these two fictitious ones were being applied to it

$$\boxed{m\ddot{\vec{r}} = \vec{F}_{net} + \vec{F}_{cor} + \vec{F}_{cf}}$$

Right-Hand Rule examples

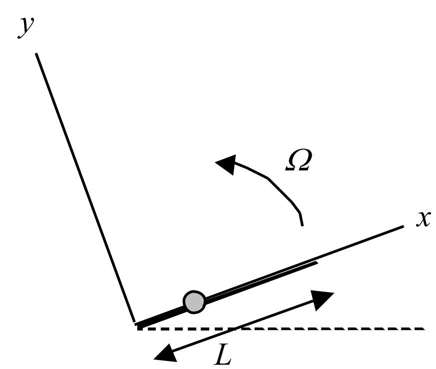
What if someone were walking North toward / near the North pole

What if someone were walking East near the North pole?

Walking South across the Equator?

Walking West along the Equator?

Example: (Ex. 5.8 F&C) A rod of length L is rotating at a constant rate of Ω in a horizontal plane. A bead starts just off the axis of rotation sliding out with a speed of $v(0) = \Omega L_o$. Ignore friction. How long will it take the bead to reach the other end of the rod?



Use coordinates that rotate with the rod so that the x axis follows the rod, so the equation of motion is:

$$m\ddot{\vec{r}} = \vec{F} + 2m\dot{\vec{r}} \times \vec{\Omega} + m\left(\vec{\Omega} \times \vec{r}\right) \times \vec{\Omega}.$$

The angular momentum is $\vec{\Omega} = \Omega\hat{z}$. The bead is constrained to move along the x axis and the rod can only exert a normal force in the y direction since there is no friction, so:

$$m\ddot{x}\hat{x} = N_y\hat{y} + N_z\hat{z} - mg\hat{z} + 2m\dot{x}\hat{x} \times \left(\Omega\hat{z}\right) + m\left(\Omega\hat{z} \times x\hat{x}\right) \times \Omega\hat{z},$$

$$m\ddot{x}\hat{x} = N_y\hat{y} + N_z\hat{z} - mg\hat{z} - 2m\dot{x}\Omega\hat{y} + m\Omega^2 x\hat{x}.$$

The three component equations are:

$$m\ddot{x} = m\Omega^2 x, \qquad N_y = 2m\dot{x}\Omega, \qquad N_z = mg.$$

The second equation relates the size of the horizontal component of the normal force to the Coriolis force. The solution to the first equation is:

$$x(t) = Ae^{\Omega t} + Be^{-\Omega t},$$

so:

$$\dot{x}(t) = A\Omega e^{\Omega t} - \Omega B e^{-\Omega t}.$$

The initial conditions are $x = L_o$ and $\dot{x}(0) = \Omega L_o$, so:

$$A + B = L_o \quad \text{and} \quad (A - B)\Omega = \Omega L_o \ .$$

The second is $A - B = L_o$, so adding the conditions gives $A = L_o$ and subtracting them gives $B = 0$. The position is:

$$x(t) = L_o e^{\Omega t}.$$

Let T be the time when the bead reaches the other end, so:

$$x(T) = L = L_o e^{\Omega T},$$

$$T = \frac{1}{\Omega}\ln\left(\frac{L}{L_o}\right),$$

Note that the units are right because the angular velocity has units of radians (unitless) over time.