Fri., 11/2	**8.1-.2** 2-Body Central Forces, Relative Coordinates.	
Mon. 11/5 Wed., 11/7 Thurs. 11/8 Fri., 11/9	**Review** **Exam 2** (Ch 5-7) **8.3-.4** Equations of Motion for 2-Body Central Force	

From the get-go, the book considers just a *two* body system; I want to put this in context of things we've done before in this class and 231 with *multi*-body systems. So I'll start general, and then take it down a notch to just two bodies.

Let's recall how we can dived up the energy of a system.

When you've got a multi-particle system, the total energy of the system is the sum of the kinetic energies of each particle and the potential energies of their shared interactions.

$$E_{system} = \sum_i \left(T_i + \sum_{j<i} U_{i,j} \right)$$

(the j<i ensures that we don't double count the shared potential energies)

Now, if you recall,

$$\vec{F}_{net.ext} = \frac{d\vec{P}_{total}}{dt} \text{ where } \vec{P}_{total} = \sum m_i \dot{\vec{r}}_i \, .$$

Which means that, if you could stick bright red dot at the "center of mass" of the whole system, $\vec{R}_{cm} \equiv \frac{\sum m_i \vec{r}_i}{\sum m_i}$, that red dot would respond to the external forces just like a single particle with the whole system's mass: $\vec{P}_{total} = M_{total} \dot{\vec{R}}_{cm}$.

For that reason, when you're considering the total energy in a compound system, it's often convenient to break it up into the center of mass's kinetic energy, and the rest, the "internal" energy, or the energy 'relative to the center of mass.'

$$E_{system} = \sum_i \left(T_i + \sum_{j<i} U_{i,j} \right) = T_{CM} + \sum_i \left(T_{i-cm} + \sum_{j<i} U_{i,j} \right) = \tfrac{1}{2} M_{total} \dot{R}_{cm}^2 + \sum_i \left(\tfrac{1}{2} m_i \dot{r}_{i-cm}^2 + \sum_{j<i} U_{i,j} \right)$$

where $\vec{r}_{i-cm} = \vec{r}_i - \vec{r}_{cm}$ (if you derive this relationship, it's not immediately apparent, but the cross-terms do indeed end up summing to 0.)

For example, when a ball goes flying through the air, you can track the center-of-mass motion, but, on the atomic scale, there's also a lot of jiggling and bonds that hold it together, or when you look at a distant star system, may be all that you can resolve is the 'net' motion of the system, the kinetic energy of the center of mass, but maybe there are a few stars and planets that make up that system and there's energy associated with their gravitational interactions and their motions relative to the center of mass.

Okay, now down to n=2

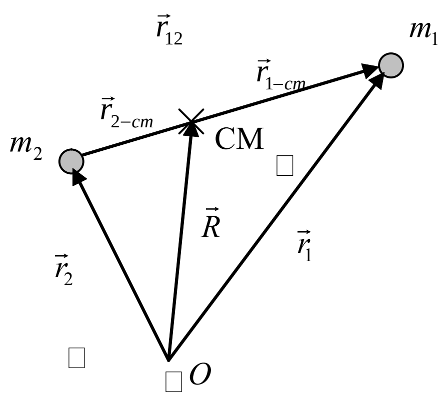
Of course, the simplest 'compound' system is one of just two particles. In that case, our sums reduce to

$$E_{system} = \tfrac{1}{2} m_1 \dot{r}_1^2 + \tfrac{1}{2} m_2 \dot{r}_2^2 + U_{1,2}$$

Or, in terms of center of mass, and relative motions,

$$E_{system} = \tfrac{1}{2} (m_1 + m_2) \dot{R}_{cm}^2 + \tfrac{1}{2} m_1 \dot{r}_{1-cm}^2 + \tfrac{1}{2} m_2 \dot{r}_{2-cm}^2 + U_{1,2}$$

Where $\vec{R}_{cm} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2}$ and $\vec{r}_{1-cm} = \vec{r}_1 - \vec{R}_{cm} = \vec{r}_1 - \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \dfrac{m_2 (\vec{r}_1 - \vec{r}_2)}{m_1 + m_2}$ and $\vec{r}_{2-cm} = \vec{r}_2 - \vec{R}_{cm} = \dfrac{m_1 (\vec{r}_2 - \vec{r}_1)}{m_1 + m_2}$



If we define $\vec{r}_{12} \equiv \vec{r}_1 - \vec{r}_2$, then we can write these two relative positions as simply

$$\vec{r}_{1-cm} = \frac{m_2}{m_1 + m_2} \vec{r}_{12} \Rightarrow \dot{r}^2{}_{1-cm} = \left(\frac{m_2}{m_1 + m_2} \right)^2 \dot{r}^2{}_{12}$$

$$\vec{r}_{2-cm} = -\frac{m_1}{m_1 + m_2} \vec{r}_{12} \Rightarrow \dot{r}^2{}_{2-cm} = \left(\frac{m_1}{m_1 + m_2} \right)^2 \dot{r}^2{}_{12}$$

So,

$$E_{system} = \tfrac{1}{2}(m_1 + m_2)\dot{R}_{cm}^2 + \tfrac{1}{2}m_1\left(\frac{m_2}{m_1+m_2}\right)^2 \dot{r}^2_{12} + \tfrac{1}{2}m_2\left(\frac{m_1}{m_1+m_2}\right)^2 \dot{r}^2_{12} + U_{1,2}$$

$$E_{system} = \tfrac{1}{2}(m_1 + m_2)\dot{R}_{cm}^2 + \tfrac{1}{2}\frac{(m_2 + m_1)m_1 m_2}{(m_1+m_2)^2}\dot{r}^2_{12} + U_{1,2}$$

$$E_{system} = \tfrac{1}{2}(m_1 + m_2)\dot{R}_{cm}^2 + \tfrac{1}{2}\frac{m_1 m_2}{(m_1+m_2)}\dot{r}^2_{12} + U_{1,2}$$

So, if we define

$$M \equiv (m_1 + m_2)$$

$$\mu \equiv \frac{m_1 m_2}{(m_1+m_2)}$$

We can write the system's energy in short-hand as

$$E_{system} = \tfrac{1}{2}M\dot{R}_{cm}^2 + \tfrac{1}{2}\mu\dot{r}^2_{12} + U_{1,2}$$

Better yet, the simplest, fundamental potential energies for two interacting particles are gravitational

$$U_{grav}(r_{12}) = \frac{-Gm_1 m_2}{r_{12}} \text{ and the corresponding force is } \vec{F}_{grav1\leftarrow 2}(r_{12}) = \frac{-Gm_1 m_2}{r_{12}^{\;2}}\hat{r}_{12}$$

The textbook (at least the 1st printing) does <u>not</u> seem to give the gravitational constant, $G = 6.67\times10^{-11}\ \mathrm{m^3/kg\cdot s^2}$. It is not needed for some calculations, but it is useful or necessary for others.

And the electric potential

$$U_{elect}(\vec{r}_1, \vec{r}_2) = \frac{1}{4\pi\varepsilon_o}\frac{q_1 q_2}{r_{12}} \text{ and the corresponding force is } \vec{F}_{elect1\leftarrow 2}(r_{12}) = \frac{1}{4\pi\varepsilon_o}\frac{q_1 q_2}{r_{12}^{\;2}}\hat{r}_{12}$$

$$\frac{1}{4\pi\varepsilon_o} = 9\times10^9\ Jm/C^2$$

Notice that both only depend upon the separation, not the absolute positions of the two interacting particles.

We can describe a two-body system with either the Lagrangian approach or Newton's second law. We will use a little bit of each.

Of course, the Lagrangian is

$$\mathcal{L} = T - U = \tfrac{1}{2}m_1\dot{\vec{r}}_1^{\,2} + \tfrac{1}{2}m_2\dot{\vec{r}}_2^{\,2} - U(r_{12}).$$

…

$$\mathcal{L}_{system} = \tfrac{1}{2} M\dot{R}_{cm}^2 + \tfrac{1}{2}\mu\dot{r}^2{}_{12} - U(r_{1,2})$$

Notice that there are only two degrees of freedom: the position of the center of mass and the separation of the two objects from each other.

One way of conceptually subdividing this is

$$\mathcal{L} = T - U = \tfrac{1}{2} M\dot{\vec{R}}^2 + \left(\tfrac{1}{2}\mu\dot{\vec{r}}^2 - U\left(r\right)\right) = \mathcal{L}_{cm} + \mathcal{L}_{rel}.$$

There are no mixed terms, so there will be independent equations of motion for $\vec{R}$ and $\vec{r}$ just as if they described two non-interacting systems.

The Lagrangian for the CM is like that of a free particle (no force or potential) of mass *M*. In the first term,

$$\dot{\vec{R}}^2 = \dot{R}_x^2 + \dot{R}_y^2 + \dot{R}_z^2 .$$

The coordinates R_x, R_y, and R_z are ignorable (not present in $\mathcal{L}$), so the equations for the center of mass motion are:

$$M\dot{R}_x = \text{constant} \quad M\dot{R}_y = \text{constant} \quad \text{and} \quad M\dot{R}_z = \text{constant},$$

or the total momentum of the system is constant:

$$\vec{P} = M\dot{\vec{R}} = \text{constant(vector)}.$$

The Lagrangian for the relative motion is like that of a particle of mass μ moving in a potential $U(r)$.

Example: 8.2

What if there *is* an external agent acting upon this system, for example, say we have two masses interacting with each other (say joined by a spring) near the surface of the earth (i.e., where a particle-earth potential energy is mgy.) Then everything plays out as said, but with an addition to the potential energy.

$$U = U(r_{1,2}) + U_{1,E}(y_1) + U_{2,E}(y_2) = U(r_{1,2}) + m_1 g(y_1) + m_2 g(y_2) = U(r_{1,2}) + g\left(m_1 y_1 + m_2 y_2\right)$$

$$U = U(r_{1,2}) + \left(m_1 + m_2\right) g \frac{\left(m_1 y_1 + m_2 y_2\right)}{m_1 + m_2}$$

$$U = U(r_{1,2}) + MgY_{cm}$$

For this reason, the "center of Mass" is also known as the "center of gravity" since a body responds to a uniform gravitational field like a single particle with all the mass at this point.

Then the lagrangian would be

$$\mathcal{L}_{system} = \tfrac{1}{2} M\dot{R}_{cm}^2 + \tfrac{1}{2}\mu\dot{r}^2{}_{12} - \left(U(r_{1,2}) + MgY_{cm}\right)$$

Since there are no cross terms, one can easily group the center-of-mass and the separation terms and refer to them separately as the center-of-masse's Lagrangian and the separation' lagrangian.

$$\mathcal{L}_{system} = \mathcal{L}_{CM} + \mathcal{L}_{separation}$$
$$\left(\tfrac{1}{2}M\dot{R}^2_{cm} - MgY_{cm}\right) + \left(\tfrac{1}{2}\mu\dot{r}^2{}_{12} - U(r_{1,2})\right)$$

Solve Center of Mass problem

It's really easy to get the equations of motion of the center of mass from subjecting $\mathcal{L}_{CM}$ to

$$\frac{\partial}{\partial X_{cm}} = \frac{d}{dt}\frac{\partial}{\partial \dot{X}_{cm}} \qquad \frac{\partial}{\partial Y_{cm}} = \frac{d}{dt}\frac{\partial}{\partial \dot{Y}_{cm}} \qquad \frac{\partial}{\partial Z_{cm}} = \frac{d}{dt}\frac{\partial}{\partial \dot{Z}_{cm}}$$

yielding

$$X_{cm} = X_{cm.o} + \dot{X}_{cm}\Delta t$$
$$Y_{cm} = Y_{cm.o} + \dot{Y}_{cm.o}\Delta t - \tfrac{1}{2}g(\Delta t)^2$$
$$Z_{cm} = Z_{cm.o} + \dot{Z}_{cm}\Delta t$$

Solve the separation problem

Let's say that what we've got are two masses joined by a spring.

So, what if the potential were a spring potential? What would the lagrangian's look like and what would be the equations of motion for the separation?

$$\mathcal{L}_{system} = \mathcal{L}_{CM} + \mathcal{L}_{separation}$$
$$\left(\tfrac{1}{2}M\dot{R}^2_{cm} - MgY_{cm}\right) + \left(\tfrac{1}{2}\mu\dot{\vec{r}}^2{}_{12} - \tfrac{1}{2}k(r_{1,2} - r_o)^2\right)$$

Note that $\frac{1}{2}\mu\dot{\vec{r}}^2$ isn't as simple as it looks – there are three coordinates:

In Cartesian: $\frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)$

In Polar/Cylindrical: $\frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\left(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2\right)$

In Spherical: $\frac{1}{2}\mu\dot{\vec{r}}^2 = \frac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)$

Since the spring potential only depends on the spring's stretch, Spherical coordinates seem the natural ones to choose.

$$\mathcal{L}_{separation} = \left(\tfrac{1}{2}\mu\dot{\vec{r}}^2{}_{12} - \tfrac{1}{2}k(r_{1,2} - r_o)^2\right) = \tfrac{1}{2}\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - \tfrac{1}{2}k(r - r_o)^2$$

$$\frac{\partial \mathcal{L}_{separation}}{\partial \theta} = \frac{d}{dt}\frac{\partial \mathcal{L}_{separation}}{\partial \dot{\theta}} = \mu r^2 2\sin\theta\cos\theta\dot{\phi}^2 = \frac{d}{dt}\mu r^2\dot{\theta} = \mu 2r\dot{r}\dot{\theta} + \mu r^2\ddot{\theta}$$
$$\frac{\partial \mathcal{L}_{separation}}{\partial \phi} = \frac{d}{dt}\frac{\partial \mathcal{L}_{separation}}{\partial \dot{\phi}} = 0 = \frac{d}{dt}\mu r^2\sin^2\theta\dot{\phi}$$
$$\frac{\partial \mathcal{L}_{separation}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}_{separation}}{\partial \dot{r}} = \mu r\dot{\theta}^2 + \mu r\sin^2\theta\dot{\phi}^2 - k(r - r_o) = \mu\ddot{r}$$

Now, let's define the direction that the spring is initially pointing to be z, that is θo=0, ϕ_o=90°,. Then we have that

$$\mu r^2\sin^2\theta\dot{\phi} = 0$$

And the only way for that to be true as r and q may evolve is if

$\dot{\phi} = 0$ always.

So,

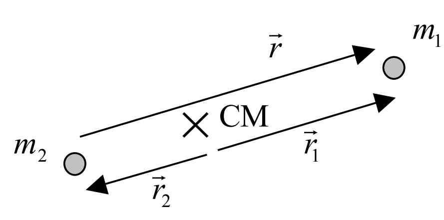
$\mu r\dot{\theta}^2 - k(r - r_o) = \mu\ddot{r}$ and $\mu r^2\ddot{\theta} = -\mu 2r\dot{r}\dot{\theta}$ or backing up, $\mu r^2\dot{\theta} = const$.

Nest time, we'll consider systems that actually are rotating, for now, if we've got the simple case that it wasn't *originally* rotating, then we see that it won't start rotating spontaneously, and we've got simply

$$-k(r - r_o) = \mu\ddot{r}$$

We can get the same result starting from Newton's laws. We already know that if there is no net external force on a system, its center of mass moves with a constant velocity. To simplify the analysis, we can choose the center of mass as the origin of an inertial reference frame so that (see the diagram below):

$$\vec{R} = 0 \quad \text{and} \quad m_1\vec{r}_1 + m_2\vec{r}_2 = 0.$$



The force on particle 1 will depend on its position relative to particle 2 (see the diagram above) which can be written as (only if CM is the origin, use the second equation above):

$$\vec{r} = \vec{r}_1 - \vec{r}_2 = \vec{r}_1 - \left(-\frac{m_1}{m_2}\vec{r}_1 \right) = \left(\frac{m_1 + m_2}{m_2} \right)\vec{r}_1 .$$

Suppose that we want to describe the motion of particle 1. The size of the force depends on the size of the separation and is along $\vec{r}$, so its equation of motion is:

$$m_1 \frac{d^2\vec{r}_1}{dt^2} = -f(r)\frac{\vec{r}}{r},$$

where $f(r)$ is the magnitude of the force. Substitute in $\vec{r}_1 = \frac{m_2}{m_1 + m_2}\vec{r}$ to get a description of the motion of particle 1 relative to particle 2:

$$m_1\left(\frac{m_2}{m_1 + m_2} \right)\frac{d^2\vec{r}}{dt^2} = \mu\frac{d^2\vec{r}}{dt^2} = -f(r)\frac{\vec{r}}{r}.$$

In other words, we can solve for the position of particle 1 *relative* to particle 2 *as if* particle 2 is not moving if we use the reduced mass for particle 1.