Phys 331: 10.1-.2 Center of Mass & Rotation about a Fixed Axis

Fri., 11/30	**10.1-.2** Center of Mass & Rotation about a Fixed Axis	
Mon., 12/3 Tues. 12/4 Wed., 12/5 Thurs. 12/6 Fri., 12/7	**10.3-.4** Rotation about any Axis, Inertia Tensor Principle Axes **10.5-.6** Finding Principle Axes, Precession **10.7-.8** Euler's Equations	**HW10a** (10.6-.22) **HW10b** (10.36, 10.39)

10.1 Properties of the Center of Mass:

Pretty much everything can be split into CM & relative parts!

The definition of the location of the CM for a system of particles is:

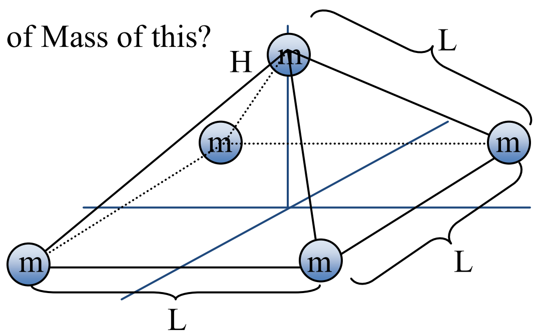
$$\vec{R} = \frac{1}{M}\sum_{\alpha} m_{\alpha}\vec{r}_{\alpha} \rightarrow \frac{1}{M}\int \left(\frac{dm}{dV}\right)\vec{r}\, dV \,,$$

where m_{α} is the mass at the position $\vec{r}_{\alpha}$, M is the total mass, and ρ is the density. Usually, we calculate <u>one</u> component of the CM at a time! For example, the *z* component is:

$$Z = \frac{1}{M}\sum_{\alpha} m_{\alpha} z_{\alpha} \rightarrow \frac{1}{M}\int \left(\frac{dm}{dV}\right) z\, dV \,.$$

Example: 10.3

Where's the Center of Mass of this?



Intuitively: Given the x-y symmetry, it will be on the z-axis; 4/5 of the mass is in the x-y plane and 1/5 is a distance H above, so the center of mass will be 1/5 of the way up from the x-y plane.

Mathematically,

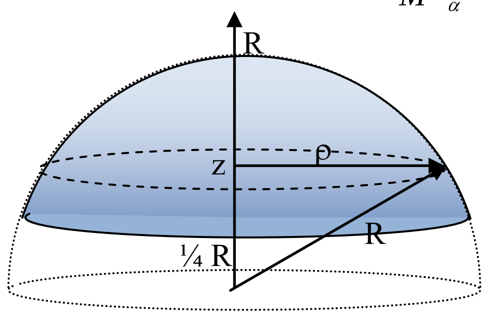
$$Z = \frac{1}{M}\sum_{\alpha} m_{\alpha} z_{\alpha} = \frac{1}{5m}\left(m0 + m0 + m0 + m0 + mH\right) = \frac{1}{5}H$$

$$X = \frac{1}{M}\sum_{\alpha} m_{\alpha} x_{\alpha} = \frac{1}{5m}\left(m\frac{L}{2} + m\frac{L}{2} + m\frac{-L}{2} + m\frac{-L}{2} + m0\right) = 0$$

Example: 10.5 (modified)

Find the Center of mass of this 1/4 of a sphere.

Given the Symmetry, it will be along the Z axis. $Z = \frac{1}{M}\sum_\alpha m_\alpha z_\alpha \rightarrow \frac{1}{M}\int \left(\frac{dm}{dV}\right) z\, dV$



Taking the differential morsel of mass to be a pancake of volume $dV = \pi\rho^2 dz = \pi\left(R^2 - z^2\right)dz$,

and the density is $\left(\frac{dm}{dV}\right) = \frac{M}{V}$. So,

$$Z = \frac{1}{M}\int\left(\frac{dm}{dV}\right) z\, dV = \frac{1}{M}\int \frac{M}{V} z\,\pi\left(R^2 - z^2\right)dz = \frac{1}{V}\int_{z=\frac{1}{4}R}^{R} z\,\pi\left(R^2 - z^2\right)dz$$

To find the volume, which is in the denominator, we'd do a similar integral, but without the extra factor of z:

$$V = \int_{z=\frac{1}{4}R}^{R} \pi\left(R^2 - z^2\right)dz$$

So,

$$Z = \frac{\int_{z=\frac{1}{4}R}^{R} z\,\pi\left(R^2 - z^2\right)dz}{\int_{z=\frac{1}{4}R}^{R} \pi\left(R^2 - z^2\right)dz} = \frac{\int_{z=\frac{1}{4}R}^{R} z\left(R^2 - z^2\right)dz}{\int_{z=\frac{1}{4}R}^{R} \left(R^2 - z^2\right)dz} = \frac{\frac{1}{2}z^2R^2 - \frac{1}{4}z^4\Big|_{\frac{1}{4}R}^{R}}{zR^2 - \frac{1}{3}z^3\Big|_{\frac{1}{4}R}^{R}} = \frac{\left(\frac{1}{2}R^4 - \frac{1}{4}R^4\right) - \left(\frac{1}{2}\frac{1}{4^2}R^4 - \frac{1}{4}\frac{1}{4^4}R^4\right)}{\left(R^3 - \frac{1}{3}R^3\right) - \left(\frac{1}{4}R^3 - \frac{1}{3}\frac{1}{4^3}R^3\right)}$$

$$= R\frac{\left(\frac{1}{2} - \frac{1}{4}\right) - \left(\frac{1}{2}\frac{1}{4^2} - \frac{1}{4^5}\right)}{\left(1 - \frac{1}{3}\right) - \left(\frac{1}{4} - \frac{1}{3}\frac{1}{4^3}\right)} == R\frac{\left(1 - \frac{1}{4^4}\right)3}{4\left(\left(\frac{2}{3}\right) - \frac{1}{4}\left(1 - \frac{1}{3}\frac{1}{4^2}\right)\right)} = R\frac{\frac{253}{256}}{\left(\frac{81}{48}\right)} = R\frac{12{,}144}{20{,}736} = R\left(0.5856\right)$$

For the half-sphere, z runs from 0 to R (note, if we had some other fraction of a sphere, z would simply run over a smaller range.)

The total momentum for the system is:

$$\vec{P} = \sum_\alpha \vec{p}_\alpha = M\dot{\vec{R}},$$

and the net external force on the system is:

$$\vec{F}^{\text{ext}} = \dot{\vec{P}} = M\ddot{\vec{R}}.$$

which means the CM moves like a single particle of mass M subjected to the net external force.

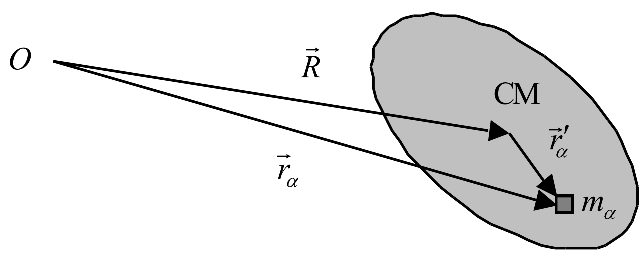
The total angular momentum about an origin O is:

$$\vec{L} = \sum_\alpha \vec{\ell}_\alpha = \sum_\alpha \vec{r}_\alpha \times \vec{p}_\alpha = \sum_\alpha \vec{r}_\alpha \times m_\alpha \dot{\vec{r}}_\alpha ,$$

where $\vec{r}_\alpha$ is the position relative to O.

We can also describe the position of each mass in the system by the position $\vec{R}$ of the CM and the position $\vec{r}'_\alpha$ relative to the CM (see the diagram below) by:

$$\vec{r}_\alpha = \vec{R} + \vec{r}'_\alpha.$$



Substituting in the relation above, we get:

$$\vec{L} = \sum_\alpha \left(\vec{R} + \vec{r}'_\alpha\right) \times m_\alpha \left(\dot{\vec{R}} + \dot{\vec{r}}'_\alpha\right),$$

$$\vec{L} = \sum_\alpha \vec{R} \times m_\alpha \dot{\vec{R}} + \sum_\alpha \vec{R} \times m_\alpha \dot{\vec{r}}'_\alpha + \sum_\alpha \vec{r}'_\alpha \times m_\alpha \dot{\vec{R}} + \sum_\alpha \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha .$$

Factor out the terms that are <u>not</u> summed over to get:

$$\vec{L} = \left(\sum_\alpha m_\alpha\right)\vec{R} \times \dot{\vec{R}} + \vec{R} \times \left(\sum_\alpha m_\alpha \dot{\vec{r}}'_\alpha\right) + \left(\sum_\alpha m_\alpha \vec{r}'_\alpha\right) \times \dot{\vec{R}} + \sum_\alpha \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha$$

The "weighted" sum of the positions relative to the center of mass is zero, because $\vec{r}'_\alpha = \vec{r}_\alpha - \vec{R}$ and (the final two terms are equal by definition):

$$\sum_\alpha m_\alpha \vec{r}'_\alpha = \sum_\alpha m_\alpha \left(\vec{r}_\alpha - \vec{R}\right) = \sum_\alpha m_\alpha \vec{r}_\alpha - \left(\sum_\alpha m_\alpha\right)\vec{R} = \sum_\alpha m_\alpha \vec{r}_\alpha - M\vec{R} = 0.$$

The summation in the second term is zero because:

$$\sum_\alpha m_\alpha \dot{\vec{r}}'_\alpha = \frac{d}{dt}\left(\sum_\alpha m_\alpha \vec{r}'_\alpha\right) = \frac{d}{dt}(0) = 0.$$

This leaves:

$$\vec{L} = \left(\vec{R} \times M\dot{\vec{R}}\right) + \sum_\alpha \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha = \vec{R} \times \vec{P} + \sum_\alpha \vec{r}'_\alpha \times m_\alpha \dot{\vec{r}}'_\alpha ,$$

which means that the angular momentum can be broken into two parts:

$$\boxed{\vec{L} = \vec{L}(\text{motion of CM}) + \vec{L}(\text{motion relative to CM})}.$$

By analogy to the earth's motion, we can label these as *orbital* and *spin* parts:

$$\vec{L} = \vec{L}_{\text{orb}} + \vec{L}_{\text{spin}},$$

where $\vec{L}_{\text{orb}} = \vec{R} \times \vec{P}$. The rate of change of the orbital angular momentum is:

$$\dot{\vec{L}}_{\text{orb}} = \frac{d}{dt}\left(\vec{R} \times \vec{P}\right) = \dot{\vec{R}} \times \vec{P} + \vec{R} \times \dot{\vec{P}}.$$

The first term is zero because $\dot{\vec{R}} \,||\, \vec{P}$, so using Newton's second law, $\vec{F}^{\text{ext}} = \dot{\vec{P}}$, gives:

$$\boxed{\dot{\vec{L}}_{\text{orb}} = \vec{R} \times \vec{F}^{\text{ext}}},$$

so once again the CM acts like a particle of mass M subjected to the net external force.

The rate of change of the total angular momentum is:

$$\dot{\vec{L}} = \sum \vec{r}_\alpha \times \vec{F}_{\alpha.net}$$

However, the terms involving *internal* forces all vanish assuming that they obey Newton's 3rd and are central. For example, consider the two terms that involve the force of particle 1 on particle 2 and that of particle 2 on particle 1:

$$\vec{r}_1 \times \vec{F}_{1\leftarrow 2} + \vec{r}_2 \times \vec{F}_{2\leftarrow 1} = \vec{r}_1 \times \vec{F}_{1\leftarrow 2} - \vec{r}_2 \times \vec{F}_{1\leftarrow 2} = \left(\vec{r}_1 - \vec{r}_2\right) \times \vec{F}_{1\leftarrow 2} = \vec{r}_{1\leftarrow 2} \times \vec{F}_{1\leftarrow 2} = 0$$

Newton's 3rd   Central Force

In this way, each internal force term disappears, so what're we're left with are just the *external* forces:

$$\dot{\vec{L}} = \vec{\Gamma}^{\text{ext}} = \sum \vec{r}_\alpha \times \vec{F}_\alpha^{\text{ext}} = \sum \left(\vec{r}\,'_\alpha + \vec{R}\right) \times \vec{F}_\alpha^{\text{ext}} = \sum \vec{r}\,'_\alpha \times \vec{F}_\alpha^{\text{ext}} + \vec{R} \times \sum \vec{F}_\alpha^{\text{ext}},$$

$$\dot{\vec{L}} = \sum \vec{r}\,'_\alpha \times \vec{F}_\alpha^{\text{ext}} + \vec{R} \times \vec{F}^{\text{ext}},$$

so the rate of change of the spin angular momentum is ($\vec{F}_\alpha^{\text{ext}}$ is the external force on m_α):

$$\dot{\vec{L}}_{\text{spin}} = \dot{\vec{L}} - \dot{\vec{L}}_{\text{orb}} = \left(\sum \vec{r}\,'_\alpha \times \vec{F}_\alpha^{\text{ext}} + \vec{R} \times \vec{F}^{\text{ext}}\right) - \left(\vec{R} \times \vec{F}^{\text{ext}}\right),$$

$$\boxed{\dot{\vec{L}}_{\text{spin}} = \sum \vec{r}\,'_\alpha \times \vec{F}_\alpha^{\text{ext}} = \vec{\Gamma}^{\text{ext}}(\text{about CM})}.$$

The total kinetic energy for a system is:

$$T = \sum_\alpha \tfrac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^{\,2}.$$

The speed squared can be expressed in terms of the velocity of the CM and the velocity relative to the CM:

$$\dot{\vec{r}}_\alpha^{\,2} = \left(\dot{\vec{R}} + \dot{\vec{r}}\,'_\alpha\right)^2 = \dot{\vec{R}}^2 + 2\dot{\vec{R}} \cdot \dot{\vec{r}}\,'_\alpha + \left(\dot{\vec{r}}\,'_\alpha\right)^2,$$

so:

$$T = \tfrac{1}{2}\left(\sum m_\alpha\right)\dot{\vec{R}}^2 + \dot{\vec{R}}\cdot\left(\sum_\alpha \tfrac{1}{2} m_\alpha \dot{\vec{r}}'\right) + \sum_\alpha \tfrac{1}{2} m_\alpha \dot{\vec{r}}_\alpha'^2 .$$

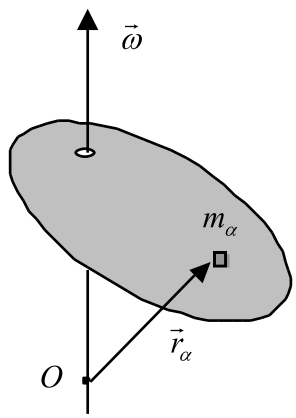
We have already shown that the middle term is zero, so the kinetic energy can be split into two parts:

$$T = \tfrac{1}{2} M\dot{\vec{R}}^2 + \sum_\alpha \tfrac{1}{2} m_\alpha \dot{\vec{r}}_\alpha'^2 ,$$

$$\boxed{T = T(\text{motion of CM}) + T(\text{motion relative to CM})}.$$

10.2 Rotation about a Fixed Axis:

Suppose a body is rotating about a fixed axis, which we will call the z axis, so $\vec{\omega} = (0,0,\omega)$. The origin O lies somewhere along this axis of rotation. Imagine the body divided into several small masses m_α with positions $\vec{r}_\alpha$ (see the diagram below).



The angular momentum relative to the origin (or any point on the axis of rotation) is:

$$\vec{L} = \sum \vec{\ell}_\alpha = \sum m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha .$$

The velocity of the mass at $\vec{r}_\alpha = (x_\alpha, y_\alpha, z_\alpha)$ is:

$$\vec{v}_\alpha = \vec{\omega}\times\vec{r}_\alpha = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \omega \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = (-\omega y_\alpha, \omega x_\alpha, 0),$$

so:

$$\vec{\ell}_\alpha = m_\alpha \vec{r}_\alpha \times \vec{v}_\alpha = m_\alpha \cdot \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ x_\alpha & y_\alpha & z_\alpha \\ -\omega y_\alpha & \omega x_\alpha & 0 \end{vmatrix} = m_\alpha \omega\left(-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2\right).$$

The total angular momentum is:

$$L_x = -\sum m_\alpha x_\alpha z_\alpha \omega,$$

$$L_y = -\sum m_\alpha y_\alpha z_\alpha \omega,$$

$$L_z = \sum m_\alpha \left(x_\alpha^2 + y_\alpha^2\right)\omega.$$

In Chapter 3, there was a brief mention that the angular momentum is not necessarily in the same direction as the angular velocity, but we ignored the components that were perpendicular to the angular velocity. The *z* component can be written as:

$$L_z = \sum m_\alpha \rho_\alpha^2 \omega = I_z \omega,$$

where $\rho_\alpha = \sqrt{x_\alpha^2 + y_\alpha^2}$ is the distance from the axis of rotation and the *moment of inertia* about the *z* axis is:

$$I_z = \sum m_\alpha \rho_\alpha^2 .$$

The total kinetic energy of the rotating body is:

$$T = \tfrac{1}{2}\sum m_\alpha v_\alpha^2 = \tfrac{1}{2}\sum m_\alpha \left(\rho_\alpha \omega\right)^2 = \tfrac{1}{2} I_z \omega^2,$$

is related to the moment of inertia for rotation about a fixed axis.

The other two components can be written as:

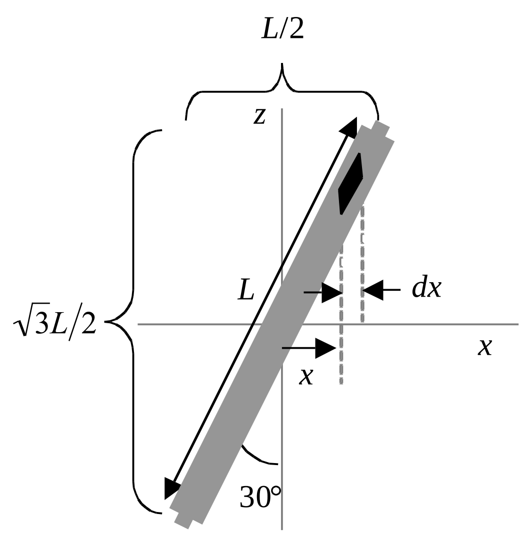
$$L_x = I_{xz}\omega \quad \text{and} \quad L_y = I_{yz}\omega,$$

where the *products of inertia* are:

$$I_{xz} = -\sum m_\alpha x_\alpha z_\alpha \quad \text{and} \quad I_{yz} = -\sum m_\alpha y_\alpha z_\alpha .$$

DEMO: Bars perpendicular to the axis and at an angle. When spun the one at an angle "wants" to wobble because its angular momentum is not along the axis.

Example: Find the moment and products of inertia for rod of mass *M* and length *L* in the *xz* plane at an angle of 30° with the *z* axis which passes through the middle.



The rod extends a length $L\sin 30^\circ = L/2$ in the horizontal direction and $L\cos 30^\circ = \sqrt{3}L/2$ in the vertical direction.

Divide the rod into slices. A representative one at *x* of width *dx* is shown above. The moment of inertia about the *z* axis is (since *y* = 0 for each piece):

$$I_z = \sum m_\alpha \rho_\alpha^2 = \sum m_\alpha x_\alpha^2 = \sum\left(\left(\frac{M}{L/2}\right)dx\right)x^2 = \sum x^2\left(M\,\frac{dx}{L/2}\right) \to \frac{2M}{L}\int_{-L/4}^{+L/4} x^2 dx,$$

$$I_z = \frac{4M}{L}\int_0^{+L/4} x^2 dx = \frac{4M}{L}\left(\frac{x^3}{3}\right)\Bigg|_0^{L/4} = \frac{ML^2}{48}.$$

The products of are inertia are (since *y* = 0 for each piece):

$$I_{yz} = -\sum m_\alpha y_\alpha z_\alpha = 0,$$

and (since $\frac{x_\alpha}{z_\alpha} = \tan 30° \Rightarrow z_\alpha = \sqrt{3}x_\alpha$):

$$I_{xz} = -\sum m_\alpha x_\alpha z_\alpha = -\sqrt{3}\sum m_\alpha x_\alpha^2 = -\sqrt{3}\left(\frac{ML^2}{48}\right)$$