Physics 331 – Advanced Mechanics
Second Approximation of the Range with Linear Drag

We can extend the method that the book employed in approximating the range in order to refine that approximation. In a nut shell, here's the method. Ultimately, we want to express the range as an expansion of the form

$$R = R_0 + C_1\left(\frac{v_{oy}}{v_{ter,l}}\right) + C_2\left(\frac{v_{oy}}{v_{ter,l}}\right)^2 + C_3\left(\frac{v_{oy}}{v_{ter,l}}\right)^3 + \ldots$$

We begin with the expression

$$0 = \left(\frac{v_{yo} + v_{\text{ter},l}}{v_{xo}}\right)R + v_{\text{ter},l}\tau_l \ln\left(1 - \frac{R}{v_{xo}\tau_l}\right),$$

apply a Taylor series expansion on the log term and rearrange to get

$$\left(\frac{v_{yo} + v_{\text{ter},l}}{v_{xo}}\right)R - v_{\text{ter},l}\tau_l\left[\frac{R}{v_{xo}\tau_l} + \frac{1}{2}\left(\frac{R}{v_{xo}\tau_l}\right)^2 + \frac{1}{3}\left(\frac{R}{v_{xo}\tau_l}\right)^3 + \frac{1}{4}\left(\frac{R}{v_{xo}\tau_l}\right)^4 + \ldots\right] = 0.$$

Here, I've held just one more term in the expansion than the book did for its Eq'n 2.41.

As they did to get Eq'n 2.42, we eliminate a common factor of R and then rearrange the expression to have

$$R = \left(\frac{2v_{xo}v_{yo}}{g}\right) - R\left[\frac{2}{3}\left(\frac{R}{\tau_l v_{xo}}\right) + \frac{1}{2}\left(\frac{R}{\tau_l v_{xo}}\right)^2 + \ldots\right]$$

I've written it this way so that the terms are still in terms $\varepsilon \equiv \left(\frac{R}{v_{x0}\tau}\right)$ which we're supposing to be rather small.

Clearly, when it's *quite* small, you can just drop everything in the square brackets to get

$$R \approx R_o = R_{vac} \equiv \left(\frac{2v_{xo}v_{yo}}{g}\right)$$

But what if it's not *that* small, what if you the first term in the square brackets is still significant? Then here's the next order approximation – take your 0^{th}-order estimate, plug it back into the expression and just keep the lowest-order terms:

$$R = \left(\frac{2v_{xo}v_{yo}}{g}\right) - \left(\frac{2v_{xo}v_{yo}}{g}\right)\left[\frac{2}{3}\left(\frac{1}{\tau_l v_{xo}}\frac{2v_{xo}v_{yo}}{g}\right) + \frac{1}{2}\left(\frac{R}{\tau_l v_{xo}}\right)^2 + \ldots\right]$$

Too small to keep

Thanks to remembering $v_{\text{ter},l} = \frac{mg}{b}$ and $\tau_l = \frac{m}{b}$, that simplifies to

$$R \approx R_1 = R_{\text{vac}}\left(1 - \frac{4}{3}\left(\frac{v_{yo}}{v_{\text{ter},l}}\right)\right).$$

Now, what if $\varepsilon \equiv \left(\frac{R}{v_{x0}\tau}\right)$ is too large for us to ignore the squared term, how do we refine our expression? Take our *1st-order* approximation, R_1, and plug *it* back in. This looks like

$$R = \left(\frac{2v_{xo}v_{yo}}{g}\right) - R_{\text{vac}}\left(1-\frac{4}{3}\left(\frac{v_{yo}}{v_{\text{ter},l}}\right)\right)\left[\frac{2}{3}\left(\frac{1}{\tau_l\, v_{xo}}R_{\text{vac}}\left(1-\frac{4}{3}\left(\frac{v_{yo}}{v_{\text{ter},l}}\right)\right)\right)+\frac{1}{2}\left(\frac{1}{\tau_l\, v_{xo}}R_{\text{vac}}\left(1-\frac{4}{3}\left(\frac{v_{yo}}{v_{\text{ter},l}}\right)\right)\right)^2+\ldots\right]$$

which, admittedly, isn't too pretty, but it cleans up to become

$$R = R_{\text{vac}}\left[1-\frac{4v_{yo}}{3v_{\text{ter},l}}\left(1-\frac{4v_{yo}}{3v_{\text{ter},l}}\right)^2-\frac{2v_{yo}^2}{v_{\text{ter},l}^2}\left(1-\frac{4v_{yo}}{3v_{\text{ter},l}}\right)^3\right]+\ldots.$$

Squaring out and cubing out (ugly) and then just keeping terms of order $\left(v_{yo}/v_{\text{ter},l}\right)^2$:

$$R \approx R_{\text{vac}}\left[1-\frac{4v_{yo}}{3v_{\text{ter},l}}+\frac{4v_{yo}}{3v_{\text{ter},l}}\left(\frac{8v_{yo}}{3v_{\text{ter},l}}\right)-\frac{2v_{yo}^2}{v_{\text{ter},l}^2}\right].$$

Combine the last two terms to get the second correction the range in vacuum:

$$R \approx R_2 = R_{\text{vac}}\left[1-\frac{4}{3}\left(\frac{v_{yo}}{v_{\text{ter},l}}\right)+\frac{14}{9}\left(\frac{v_{yo}}{v_{\text{ter},l}}\right)^2\right].$$

So, in a roundabout way, we're slowly building a power-series expansion (like a Taylor series) for the range, *R,* in powers of $\left(\frac{v_{yo}}{v_{\text{ter},l}}\right)$.