Wed. 10/6 Thurs. 10/7 Fri. 10/8	**4.9** Energy of 2 Particle Interaction **5.1-.3 (2.6)** Hooke's Law, Simple Harmonic (Complex Sol'ns)	**HW4**

Announcement: SPS Comet Observing trip with CSUSB

Handouts: Intro to Making Plots with Python

Plotting

- Over the last few days we've come up with some interesting equations. Often you want to be able to plot them. VPython *can* do it, but it's sometimes a little clunky. The Enthought version of Python is better. The handout walks you through using the Enthought version to do this.
    - Note: at the beginning of the semester I'd asked you to install VPython and Enthought's python on your computer (the latter *is* free, but you have to get through to a screen where you can indicate "educational use".) So you've got *two* versions of Phython's Idle on your machine. To use Enthought's plotting package, you've got to make sure you open the right one: go to "EPD" folder on your start menu and open Idle from there. In that say 'open new window', and then type your code in that new window.

Remind them of:

Example 2: A 2-kg particle moves in one dimension under a force:

$$F(x) = -bx + 2c\sin(ax),$$

where a = 1 m^{-1}, b = 1 N/m, and c = 1 N. The argument of the sine is in radians. (a) Find the potential energy with the reference point at the origin so that $U(0) = 0$. Sketch the potential and show the classically allowed and forbidden regions if the total energy is E = -0.5 J. (b) Identify the three points of equilibrium and determine if each is stable or unstable.

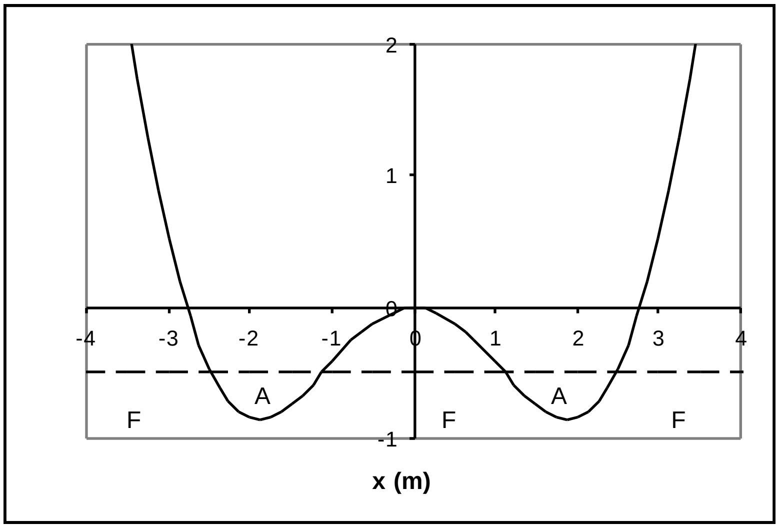
(a) The potential is found by integrating the force (with a minus sign!):

$$U(x) = -\int_0^x F(x')\,dx' = -\int_0^x \left[-bx' + 2c\sin(ax')\right]dx',$$

$$U(x) = \left[\frac{bx'^2}{2} + \frac{2c}{a}\cos(ax')\right]_0^x = \left(\frac{bx^2}{2} + \frac{2c}{a}\cos(ax) - \frac{2c}{a}\right).$$

The constants a, b, and c are all one and each term is in Joules when x is in meters. The graph below shows $U(x)$ vs. x. For large x, the x^2 dominates the oscillating term. The dashed line is E = -0.5 J and the allowed (A) and forbidden (F) regions are labeled. The particle is only allowed to be where $E > U(x)$ because $E = T + U(x)$ and T must be positive.

Then use this as an example to plot:



(b) The points of equilibrium are where $F(x) = -dU/dx = 0$. This gives the transcendental equation:

$$x = \frac{2c}{b}\sin(ax)$$

The solution $x = 0$ corresponds to an unstable equilibrium because $d^2U/dx^2 < 0$. The other two solutions can be found approximately by making successive guesses to get $x \approx \pm 1.896$, which are stable because $d^2U/dx^2 > 0$.

General 1-D & Time dependence

If energy is conserved, $E = T + U(s)$, then:

$$T = \tfrac{1}{2} m\dot{s}^2 = E - U(s),$$

which can be used to find the velocity as a function of position:

$$\dot{s}(s) = \pm\sqrt{\frac{2}{m}}\sqrt{E - U(s)}.$$

The velocity is $\dot{x} = dx/dt$, so $dt = dx/\dot{x}$. This can be integrated to find the time for motion between two points:

$$t = \int_{s_0}^{s} \frac{ds}{\dot{s}(s')} = \sqrt{\frac{m}{2}} \int_{x_0}^{x} \frac{ds'}{\sqrt{E - U(s')}}.$$

In practice, this can be difficult to calculate because the integrand goes to infinity as it approaches the turning point where $\dot{x} = 0$. Even for the simple pendulum, there is no analytical solution (see Prob. 4.38). Energy conservation is typically <u>not</u> a good way to get information about time.

Example 3: (2.10 of Fowles & Cassiday 5th ed.) A particle of mass m is released from rest at $s = b$ and its potential energy is $U(s) = -k/s$. (a) Find its velocity as a function of position. (b) How long does it take the particle to reach the origin?

(a) At $s = b$, the kinetic energy is $T = 0$ so the total energy is $E = U(b) = -k/b$. Since energy is conserved:

$$E = -k/b = T + U(s) = \tfrac{1}{2} m\dot{s}^2 - k/x,$$

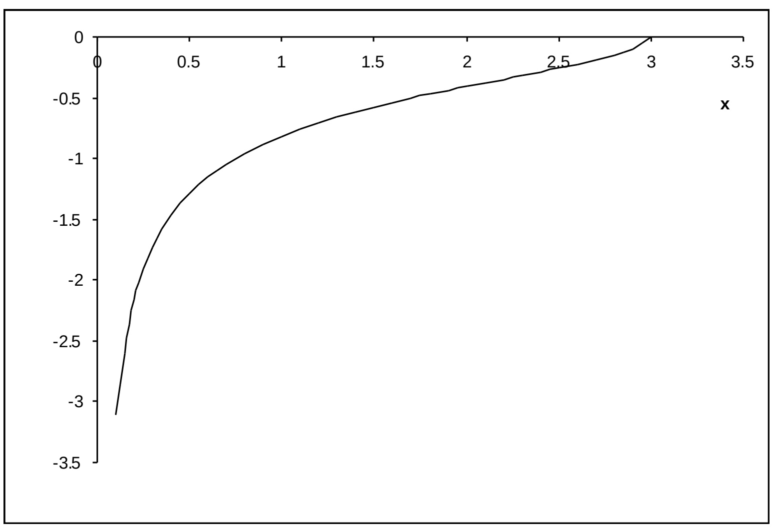
so taking the negative root because the potential attracts the particle toward the origin:

$$\dot{s}(s) = -\sqrt{\frac{2k}{m}\left(\frac{1}{s} - \frac{1}{b}\right)}.$$

Say k = 3N/m, m = 2kg, b = 3m.

Exercise: Try your hand at getting Enthought's Python to plot this.

An example of this is shown below.



(b) Since $\dot{s} = ds/dt$, the time required to move from b to 0 is:

$$\int_0^t dt = t = \int_b^0 \frac{ds}{\dot{s}(s)} = -\sqrt{\frac{m}{2k}} \int_b^0 \frac{ds}{\sqrt{1/s - 1/b}} = +\sqrt{\frac{mb}{2k}} \int_0^b \frac{\sqrt{s}\, ds}{\sqrt{b - s}}$$

Use the integral (from the front cover of the text):

$$\int \frac{\sqrt{y}\, dy}{\sqrt{1 - y}} = \sin^{-1}\left(\sqrt{y}\right) - \sqrt{y(1 - y)}$$

with the change of variables $s = by$ and $ds = b\, dy$. The integral for the time becomes:

$$t=\sqrt{\frac{mb}{2k}}\int_0^1 \frac{\sqrt{by}\,b\,dy}{\sqrt{b-by}}=\sqrt{\frac{mb^3}{2k}}\int_0^1 \frac{\sqrt{y}\,dy}{\sqrt{1-y}}=\sqrt{\frac{mb^3}{2k}}\left[\sin^{-1}\left(\sqrt{y}\right)-\sqrt{y(1-y)}\right]_0$$

$$t=\sqrt{\frac{mb^3}{2k}}\sin^{-1}(1)=\sqrt{\frac{mb^3}{2k}}\left(\frac{\pi}{2}\right)$$

$$t=\pi\sqrt{\frac{mb^3}{8k}}$$

Rigid Bodies: (Systems of multi particles) The book discusses multi-particle systems. One common situation is if they define a rigid object. Then the total kinetic energy can be rephrased as the kinetic energy of the center of mass + the kinetic energy of all the parts *about* the center of mass:

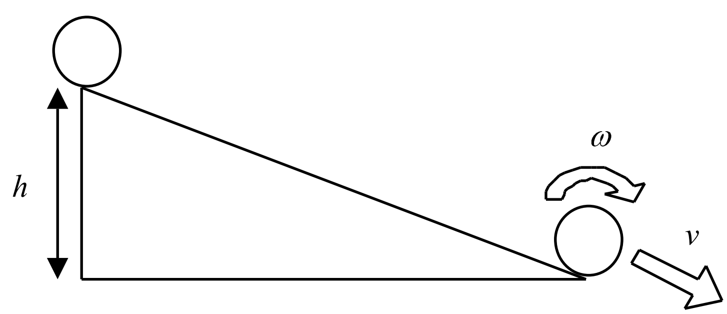
$$\sum \tfrac{1}{2} m_i v_{i.cm}^2 = \tfrac{1}{2}\sum m_i \left(r_{i-cm}\dot{\phi}\right)^2 = \tfrac{1}{2}\sum m_i \left(r_{i-cm}\right)^2 \dot{\phi}^2 = \tfrac{1}{2} I\dot{\phi}^2$$

For a rigid body rotating about an axis in a fixed direction (we generalize this in Ch. 10):

$$T = T_{CM} + T_{rot} = \tfrac{1}{2} MV^2 + \tfrac{1}{2} I\omega^2,$$

and $U = U^{ext}$ because the internal energy U^{int} does not change when the relative positions of the particles does not change.

Example 1: (related to Ex. 4.9) If they start from rest, which will make it to the end of a ramp faster, a cylinder (disk) or a thin ring? Do the masses or radii matter?



Define the PE to be zero at the bottom of the ramp, so initially $U_{\text{o}} = Mgh$ and finally $U_f = 0$. The initial KE is $T_{\text{o}} = 0$ and the final KE is $T_f = T_{CM} + T_{rot} = \tfrac{1}{2} Mv^2 + \tfrac{1}{2} I\omega^2$. For an object that is rolling without slipping, $\omega = v/R$, where R is the radius. The moment of inertia for a cylinder is $I_{cyl} = \tfrac{1}{2} MR^2$ and for a thin ring it is $I_{ring} = MR^2$. The final KE in the two cases are $T_{f,cyl} = \tfrac{3}{4} Mv^2$ and $T_{f,ring} = Mv^2$. Conservation of energy, $T_{\text{o}} + U_{\text{o}} = T_f + U_f$, gives:

$$\text{cylinder:} \quad Mgh = \tfrac{3}{4} Mv^2 \quad v_c = \sqrt{4gh/3}$$

$$\text{ring:} \quad Mgh = Mv^2 \quad v_r = \sqrt{gh}$$

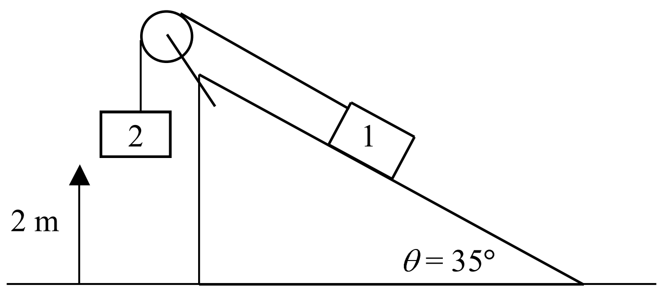
The cylinder will be going faster at <u>any</u> point along the ramp, so it will reach the bottom first. This result does not depend on the mass or radius, just how the moment of inertia depends on the shape. For any round object, it will be $I = (\text{shape factor}) MR^2$.

Question: How would a solid sphere compare to the other two shapes?

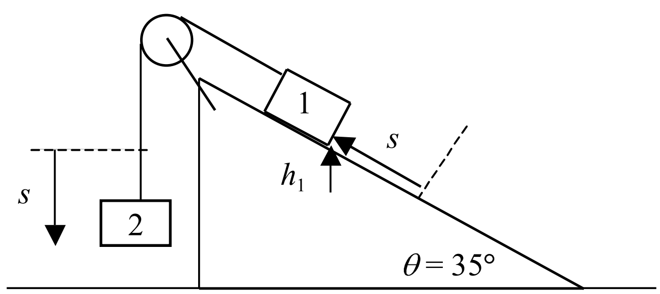
Answer: It would beat both because a larger fraction of its mass is near the axis of rotation. (Its moment of inertia is $I = \frac{2}{5} MR^2$, but you don't need to know that.)

Last time, we'd looked at this problem:

<u>Example 1</u>: Suppose the masses $m_1 = 4$ kg and $m_2 = 6$ kg are initially at rest. Ignore friction and assume that the mass of the pulley is small. What will the speed of m_2 be when it hits the ground?



This system is described by one parameter, *s*, the distance that the masses have moved. They are tied together, so they move the same amount (until 2 hits the floor).



The total mechanical energy of the system is:

$$E = T_1 + U_1 + T_2 + U_2 = \tfrac{1}{2}(m_1 + m_2)\dot{s}^2 - m_1 g s \sin\theta + m_2 g s$$

The initial condition of the system is $s(0) = 0$ and $\dot{s}(0) = 0$, so $E_o = 0$. Conservation of mechanical energy gives:

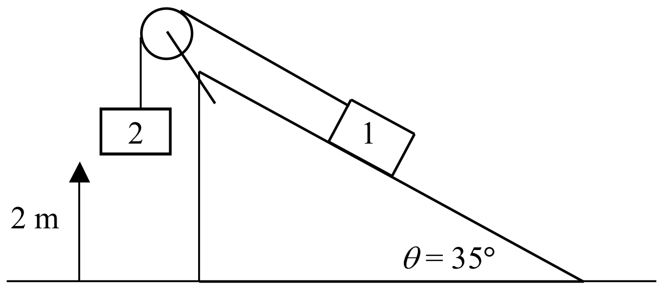
$$0 = \tfrac{1}{2}(m_1 + m_2)\dot{s}^2 - m_1 g s \sin\theta + m_2 g s .$$

Solving for the speed and putting in the final condition $s = 2$ m gives:

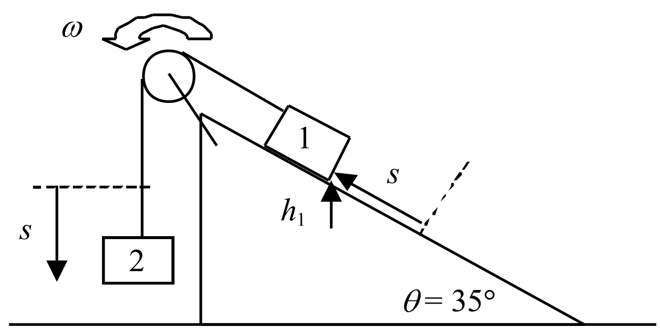
$$\dot{s}=\sqrt{\frac{2gs(m_1\sin\theta-m_2)}{m_1+m_2}}=\sqrt{\frac{2(9.8\,\mathrm{m/s^2})(2\,\mathrm{m})[(6\,\mathrm{kg})\sin 35^\circ-(4\,\mathrm{kg})]}{4\,\mathrm{kg}+6\,\mathrm{kg}}}=$$

Now, what if we say the pulley *has* mass?

Example 2: Suppose the masses $m_1 = 4$ kg and $m_2 = 6$ kg are initially at rest. Ignore friction, but assume the pulley is a cylinder of mass $m_p = 1$ kg and radius $R = 0.2$ m. Also, the rope does not slip over the pulley. What will the speed of m_2 be when it hits the ground?



This system is described by one parameter, *s*, the distance that the masses have moved. They are tied together, so they move the same amount (until 2 hits the floor).



The angular speed of the pulley is related to the speed of the rope ($\dot{s}$) by $\omega = \dot{s}/R$. The moment of inertia of the cylinder is $I = \frac{1}{2}m_pR^2$. The total mechanical energy of the system is:

$$E = T_1 + T_2 + U_1 + U_2 + T_p = \tfrac{1}{2}(m_1+m_2)\dot{s}^2 + \tfrac{1}{2}I\omega^2 + -m_1gs\sin\theta + m_2gs$$

$$E = \tfrac{1}{2}(m_1+m_2)\dot{s}^2 + \tfrac{1}{2}\left(\tfrac{1}{2}m_pR^2\right)(\dot{s}/R)^2 + -m_1gs\sin\theta + m_2gs$$

$$E = \tfrac{1}{2}\left(m_1+m_2+\tfrac{1}{2}m_p\right)\dot{s}^2 - m_1gs\sin\theta + m_2gs$$

The initial condition of the system is $s(0)=0$ and $\dot{s}(0)=0$, so $E_o = 0$. Conservation of mechanical energy gives:

$$0 = \tfrac{1}{2}\left(m_1+m_2+\tfrac{1}{2}m_p\right)\dot{s}^2 - m_1gs\sin\theta + m_2gs.$$

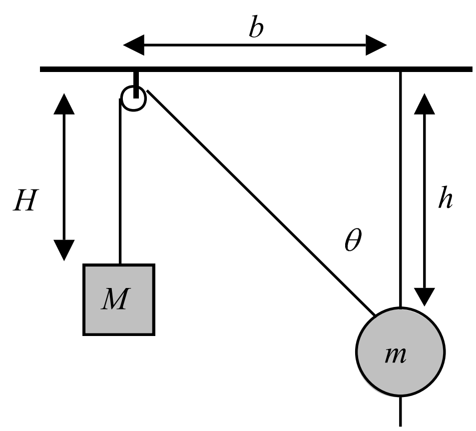
Solving for the speed and putting in the final condition $s = 2$ m gives:

$$\dot{s} = \sqrt{\frac{2gs(m_1 \sin\theta - m_2)}{m_1 + m_2 + \frac{1}{2}m_p}} = \sqrt{\frac{2(9.8\,\text{m/s}^2)(2\,\text{m})[(6\,\text{kg})\sin 35^\circ - (4\,\text{kg})]}{4\,\text{kg} + 6\,\text{kg} + \frac{1}{2}(\ \text{kg})}} = ,$$

which is slight smaller than what we found yesterday for a massless pulley. The pulley's PE does not change, so it does not have a large effect.

More Problems

Example 2: (Prob. 4.36) The ball (mass *m*) has a hole through it and slides on a frictionless vertical rod. A light string of length *l* passes over a small frictionless pulley and attaches to another mass *M*. The positions of the objects can be specified by the angle *θ*. (a) Write an expression for the potential energy $U(\theta)$. (b) Find whether or not the system has an equilibrium position and for what values of *m* and *M*. Are any equilibrium positions stable?



(a) The length of the string from the pulley to the ball is $b/\sin\theta$. The heights are $h = b/\tan\theta$ and $H = l - b/\sin\theta$. Since these distance are measured below a reference point, the PE is:

$$U = -mgh - MgH = -mg(b/\tan\theta) - Mg(l - b/\sin\theta)$$

$$\boxed{U(\theta) = gb\left(\frac{M}{\sin\theta} - \frac{m}{\tan\theta}\right) - Mgl = \frac{gb}{\sin\theta}(M - m\cos\theta) - Mgl}.$$

The last term is a constant that could be "defined away."

(b) The derivative of *U* is:

$$\frac{dU}{d\theta} = \frac{gb}{\sin^2\theta}\left[\sin\theta(m\sin\theta) - (M - m\cos\theta)(\cos\theta)\right].$$

Use the relation $\sin^2\theta + \cos^2\theta = 1$ to get:

$$\frac{dU}{d\theta} = \frac{gb}{\sin^2\theta}(m - M\cos\theta).$$

The condition for equilibrium, $dU/d\theta = 0$, yields the condition

$$m - M\cos\theta = 0$$

$$\cos\theta = \frac{m}{M}.$$

This only has solutions if $m \le M$. When $m = M$, the answer is $\theta = 0$, which is not possible for a finite length of string. Therefore, if $m < M$ there is an equilibrium point at:

$$\theta_o = \cos^{-1}\left(\frac{m}{M}\right).$$

The requirement that $m < M$ makes sense because in equilibrium the tension of the string must be equal to Mg, but upward force on *m* is only a fraction of the tension. Therefore, *m* must be smaller if they are to both be in equilibrium.

Take the second derivative of *U* to check the stability:

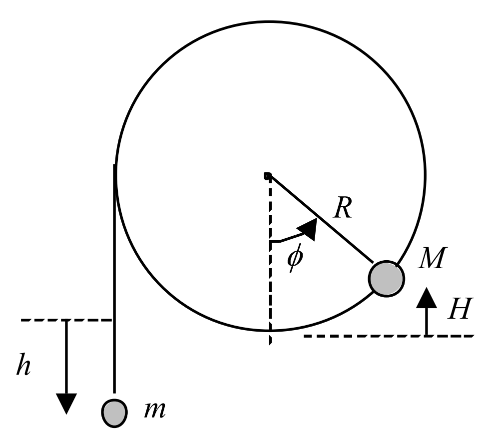
$$\frac{d^2U}{d\theta^2} = \frac{gb}{\sin^4\theta}\left\{\sin^2\theta(M\sin\theta) - [m - M\cos\theta](2\sin\theta\cos\theta)\right\}$$

At the equilibrium point, the term in square brackets is zero, so:

$$\left.\frac{d^2U}{d\theta^2}\right|_{\theta_o} = \frac{gbM}{\sin\theta_o}$$

Since we know $0 \le \theta < 90^\circ$, $\sin\theta_o$ is positive. Therefore, $\left(d^2U/d\theta^2\right)_{\theta_o} > 0$ and the equilibrium is stable.

Example 3: (Prob. 4.37) A massless (or very light) wheel of radius *R* is mounts on a horizontal axis. A mass *M* is attached to the rim of the wheel and a mass *m* is hung by a string wrapped around the rim. (a) Write an expression for the total PE as a function of the angle ϕ. Choose $U = 0$ when $\phi = 0$. (b) Find any positions of equilibrium and discuss their stability. (c) Suppose the system start at rest at $\phi = 0$. For what values of the ratio m/M will the system oscillate?



(a) As the wheel turns through an angle ϕ, mass M rises by $H = R(1-\cos\phi)$ (this works for any angle!) and mass m descends by $h = R\phi$ (the arclength unwound). The total PE is:

$$U(\phi) = MgH - mgh = MgR(1-\cos\phi) - mgR\phi.$$

(b) The condition for stability is:

$$0 = dU/d\phi = MgR\sin\phi - mgR$$

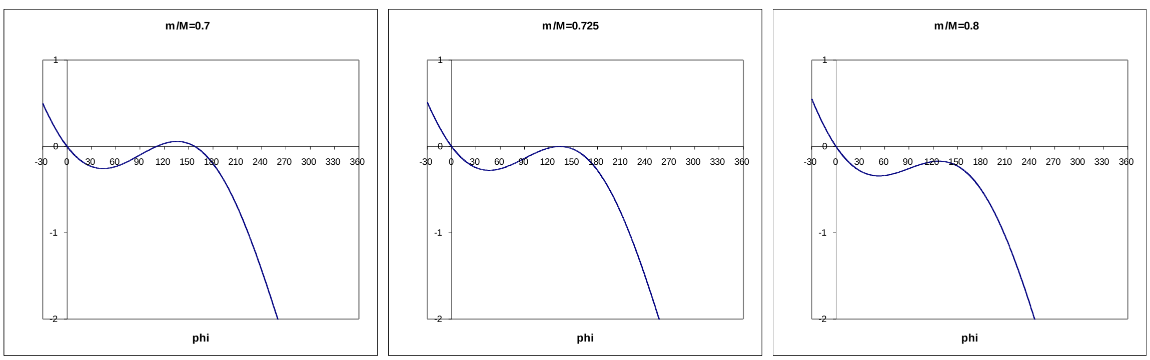
$$\sin\phi = m/M.$$

This only has solutions if $m \le M$. If $m = M$, there is one solution at $\phi = \pi/2$. If $m < M$, there are two solutions, one with $\phi < \pi/2$ (M below the axis) and one with $\phi > \pi/2$ (M above the axis). The stability is determined from the second derivative:

$$d^2U/d\phi^2 = MgR\cos\phi.$$

This is positive (negative) and the equilibrium is stable (unstable) for $\phi < \pi/2$ ($\phi > \pi/2$). For equal masses, the equilibrium at $\phi = \pi/2$ is a saddle point because $d^2U/d\phi^2 = 0$

(c) For the given initial conditions, the total energy of the system is $E = 0$. Plot the potential $U(\phi)$. The system will oscillate if there are turning points on both sides of the equilibrium. If $m/M < 0.725$, this condition is met.



Use the Excel spreadsheet "Prob4.37.xls" to show this.