Mon. 9/24 Tues. 9/25 Wed. 9/26 Thurs. 9/27 Fri., 9/28	**3.5** Angular Momentum for multiple particles **4.1-.3, 4.9** Work & Energy, Force as a Gradient, 2 Particle Interaction ***Science Poster Session***: *Hedco7~9pm* **4.4-.6** Curl of Conservative Force, Varying Potential, 1-D systems	 **HW3b, Project Topic** **HW4a**

Energy Principle, a.k.a. Work Energy Relation

- Mathematically, here's how the new tools are related to the old ones, using a trick that you've enjoyed a few times over the last few chapters.

$$\vec{F}_{net\to obj} = m\frac{d\vec{v}}{dt}$$

Looking at Cartesian components of this equation

$$F_{net.x} = m\frac{dv_x}{dt} = mv_x\frac{dv_x}{dx}$$

$$F_{net.x}dx = mv_x dv_x \Rightarrow \int_{x_i}^{x_f} F_{net.x}dx = m\int_{v_{x.f}}^{v_{x.i}} v_x dv_x = \tfrac{1}{2}mv_{x.f}^2 - \tfrac{1}{2}mv_{x.i}^2$$

Similarly,

$$F_{net.y}dy = mv_y dv_y \Rightarrow \int_{y_i}^{y_f} F_{net.y}dy = m\int_{v_{y.f}}^{v_{y.i}} v_y dv_y = \tfrac{1}{2}mv_{y.f}^2 - \tfrac{1}{2}mv_{y.i}^2$$

$$F_{net.z}dz = mv_z dv_z \Rightarrow \int_{z_i}^{z_f} F_{net.z}dz = m\int_{v_{z.f}}^{v_{z.i}} v_z dv_z = \tfrac{1}{2}mv_{z.f}^2 - \tfrac{1}{2}mv_{z.i}^2$$

Or adding all three equation and rewriting the right hand side

$$\int_{x_i}^{x_f} F_{net.x}dx + \int_{y_i}^{y_f} F_{net.y}dy + \int_{z_i}^{z_f} F_{net.z}dz = \tfrac{1}{2}mv_f^2 - \tfrac{1}{2}mv_i^2$$

Notice that the right hand side no longer reflects our intermediate step of having broken things into Cartesian components. We can do something similar with the left-hand side, though it may help to back up a step and think of the equation just before integrating

$$F_{net.x}dx + F_{net.y}dy + F_{net.z}dz = mv_x dv_x + mv_y dv_y + mv_z dv_z$$

Now, looking at those exotic *dx, dy, dz* and remembering they're just components of a tiny displacement vector.

Then, we could rewrite the sum of the product of like components as simply the dot product

$$\vec{F}_{net}\cdot d\vec{r}$$

Formally then we can write our equation as

$$\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{net} \cdot d\vec{r} = \tfrac{1}{2} m v_f^2 - \tfrac{1}{2} m v_i^2$$

$$W_{i \to f} = \Delta T$$

Where what we really mean by the single integral is the integral over all three components of the object's path, be they expressed in Cartesian coordinates, polar coordinates, or some other system.

Of course, we call the left-hand side the Work that the net force does on the object while pushing it along the path from initial to final position. We call the right hand side the change in kinetic energy that results. (for reasons of its own, the book uses T for kinetic energy)

(Interestingly, if you use the relativistic expression for momentum here, then you get the relativistic expression of energy (rest energy + relativistic kinetic). If you use the classical expression for momentum, then you get the classical expression for kinetic energy.)

- As you well know, this new representation of the interplay between Interaction (work) and motion (kinetic energy) will be useful. Before we really put it to use we'll get familiar with the pieces. Continuing with a common theme in the book thus far – we'll pay extra attention to actually doing the integral (which back in Phys 231 you didn't have to worry too much about.)

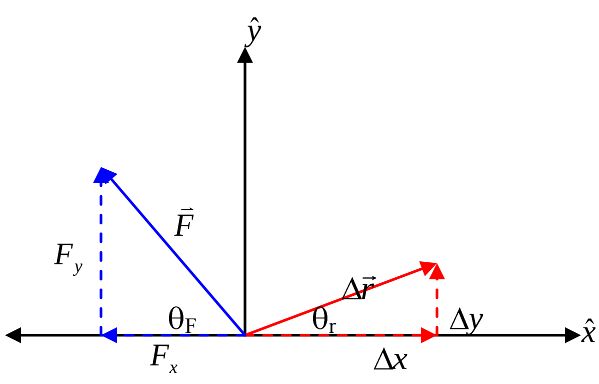
- **Interaction: (Mechanical) Work**
  - $\boxed{W_{b \to a} \equiv \int_i^f \vec{F}_{b \to a} \cdot d\vec{r}_a}$

Review of Work

- **Quantitatively:**

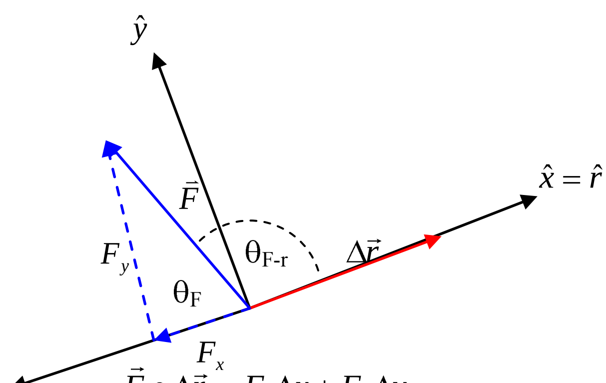
The scalar "dot" product

Representation 1:



$$\vec{F} \bullet \Delta\vec{r} = F_x \Delta x + F_y \Delta y + F_z \Delta z$$

Representation 2:



$$\vec{F} \bullet \Delta\vec{r} = F_x \Delta x + F_y \Delta y$$

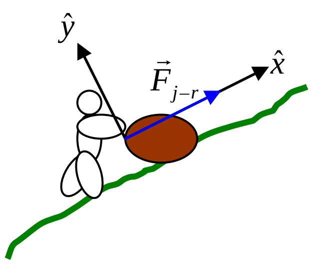
$$\vec{F} \bullet \Delta\vec{r} = F_x |\Delta\vec{r}| + 0$$

$$\vec{F} \bullet \Delta\vec{r} = |\vec{F}|\, Cos(\theta_{F-r})\, |\Delta\vec{r}|$$

$$\vec{F} \bullet \Delta\vec{r} = |\vec{F}||\Delta\vec{r}|\, Cos(\theta_{F-r})$$

Note that in the case illustrated above, F_x is negative. Looking at the 2nd representation, the one in terms of Cos(), you can see that, if the two vectors are parallel to each other, you get the full product of their magnitudes; if they're fully *anti*-parallel, you get the negative of their magnitudes – this is like what you'd get if you multiplied two scalars (5*3 = 15, -5*3 = -15). Now, if they're completely *perpendicular* to each other you get 0.

- **Why call this mathematical tool "work", how does it compare with our everyday definition? Work – Conceptual**
    - **Everyday meaning: Work is a combination of *effort* and *achievement*.**
    - **Ex.**
        - Prof Teaches Physics, Students Learn Physics
            - Prof does + work on students
        - Prof Teaches Physics, Students Don't Learn Anything
            - Prof does 0 work on students
        - Prof Doesn't Teach Physics, Students Learn Physics anyway
            - Prof does 0 work on students
        - Prof Teaches Physics, Students Get more Confused about Physics
            - Prof does – work on students
        - Prof Teaches Physics, Students Learn French
            - Prof does 0 work on students
- **Mathematical Physics Definition agrees with Everyday Conceptual**
    - **Everyday meaning agrees with mathematical, physics meaning**
    - **Ex.**



Joe Pushes rock up-hill

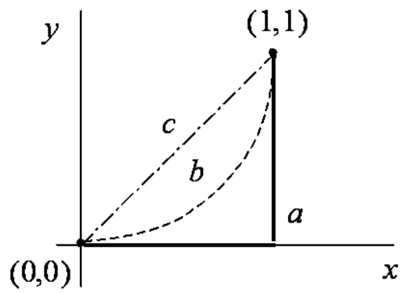
$$W_{j-r} = F_{x.j-r}\Delta x + F_{y.j-r}\Delta y + F_{z.j-r}\Delta z$$

- Rock goes up hill
    - Joe does + work on rock
    - $W_{j-r} = F_{x.j-r}\Delta x + 0\cdot 0 + 0\cdot 0 > 0$
- Rock stays put
    - Joe does 0 work on rock
    - $W_{j-r} = 0\cdot \Delta x + 0\cdot 0 + 0\cdot 0 = 0$
- Rock rolls down hill
    - Joe does – work on rock
    - $W_{j-r} = F_{x.j-r}\cdot(-|\Delta x|) + 0\cdot 0 + 0\cdot 0 < 0$
- Rock slides sideways (Mary is pushing that way)
    - Joe does 0 work on rock
    - $W_{j-r} = F_{x.j-r}\cdot 0 + 0\cdot 0 + 0\cdot \Delta z = 0$

Path Independence.

The *line integral* of the force over a path is defined as the *work* done by the force moving between points 1 and 2. This result is called the *Work-KE Theorem*. The work may depend on the path taken between the endpoints!

Example 1: Find the work (line integrals) for $\vec{F} = y\hat{x} - x\hat{y}$ moving from (0,0) to (1,1) along the three different paths shown below. For path *b*, $y = x^2$.



(a) Path *a* is made of two parts. **Piece-wise example:** So it's easy to see how to generalize, I'm going to be very systematic / slow and plodding through this fairly simple example. If you have taken or are going to take our upper-level E&M course, you'll encounter similar path integrals for which similar care is necessary.

$$W_a = \int_i^f \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$W_{a.1} + W_{a.2} = \int_i^1 \vec{F}(\vec{r}) \cdot d\vec{r} + \int_1^f \vec{F}(\vec{r}) \cdot d\vec{r}$$

$$W_{a.1} + W_{a.2} = \int_i^1 \vec{F}(\vec{r}) \cdot d\vec{x} + \int_1^f \vec{F}(\vec{r}) \cdot d\vec{y}$$

$$W_{a.1} + W_{a.2} = \int_i^1 F_x(y) dx + \int_1^f F_y(x) dy$$

$$W_{a.1} + W_{a.2} = \int_i^1 y dx + \int_1^f (-x) dy$$

$$W_{a.1} + W_{a.2} = \int_{x,y=0,0}^{x,y=1,0} y dx + \int_{x,y=1,0}^{x,y=1,1} (-x) dy$$

$$W_{a.1} + W_{a.2} = \int_{x=0}^{x=1} 0 dx + \int_{y=0}^{y=1} (-1) dy = 0 - \int_{y=0}^{y=1} dy = -1$$

(b) Path *b* now has a non-trivial relationship between its two components, $y=x^2$.

$$W_b = \int_i^f \vec{F}(x,y) \cdot d\vec{r} = \int_i^f F_x(y) dx + \int_i^f F_y(x) dy = \int_i^f y dx + \int_i^f (-x) dy$$

so far, not so very different from the previous problem, but now let's use the way that the x and y components of the object's position are related: $y=x^2$, so we could mathematically rewrite

$$W_b = \int_i^f x^2 dx + \int_i^f \left(-\sqrt{y}\right) dy = \tfrac{1}{3}x^3\Big|_i^f - \tfrac{2}{3}y^{3/2}\Big|_i^f = \tfrac{1}{3} - \tfrac{2}{3} = -\tfrac{1}{3}$$

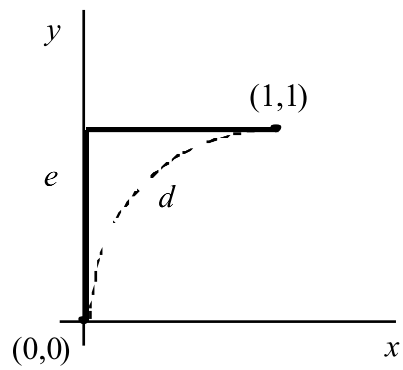
(c) **Exercise:** For path c, $x = y$ so:

$$W_c = \int_i^f \vec{F}(x,y)\cdot d\vec{r} = \int_i^f F_x(y)dx + \int_i^f F_y(x)dy = \int_i^f y dx + \int_i^f \left(-x\right) dy$$

Make the substitution $x = y$

$$W_c = \int_0^1 x\, dx + \int_0^1 \left(-y\right) dy = 0$$

Exercises: For the same force, find the line integrals for paths d and e shown below. On path d, $x = y^2$.



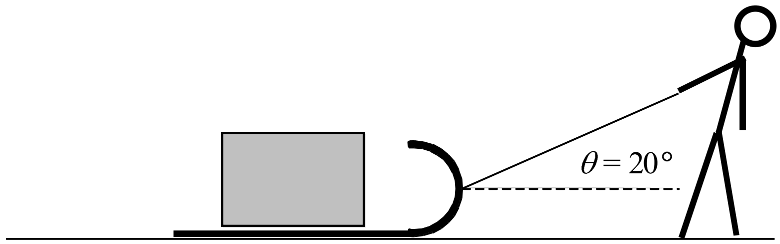
d) You can do this just like we did path b, or we can take a very small step of abstraction and parameterize both x and y in terms of a new variable, u. Path b can be described parametrically as $x = u^2$ and $y = u$ for u from 0 to 1, so $du = 2u\, du$ and $dy = du$.

$$W_d = \int_0^1 F_x\left(u^2,u\right) 2u\, du + \int_0^1 F_y\left(u^2,u\right) du = \int_0^1 \left(u\right) 2u\, du + \int_0^1 \left(-u^2\right) du = \left[\frac{u^3}{3}\right]_0^1 = +\frac{1}{3}$$

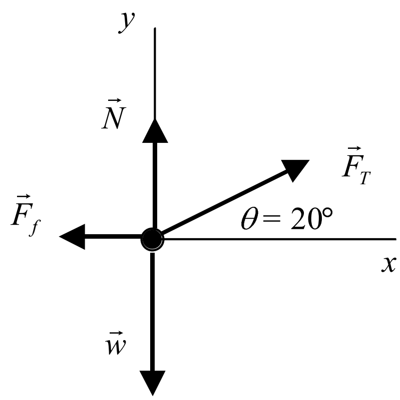
(e) Path e is made of two parts. In the first, $x = 0$ as y goes from 0 to 1 so $d\vec{r} = dy\,\hat{y}$. In the second, $y = 1$ as x goes from 0 to 1 so $d\vec{r} = dx\,\hat{x}$. So here's the quick solution.

$$W_e = \int_0^1 F_y(0,y)\, dy + \int_0^1 F_x(x,1)\, dx = \int_0^1 0\, dy + \int_0^1 (1)\, dx = +1.$$

Example 2: A sled and load with a total mass of 80 kg is pulled with a force of 180 N at an angle of 20° above horizontal. The sled is initially at rest. If the coefficient of kinetic friction is $\mu = 0.2$, what will the speed be after the sled has moved forward 5 m? (In practice, it is very difficult to apply a constant force!)



The force diagram for the sled (treating it as a particle) is below.



The sled does <u>not</u> accelerate in the *y* direction, so the size of the normal force is found from:

$$N + F_T \sin\theta - mg = 0,$$

$$N = mg - F_T \sin\theta = (80\text{ kg})(9.8\text{ m/s}^2) - (180\text{ N})\sin 20^\circ = 722\text{ N}.$$

The size of the frictional force is $F_f = \mu N = 0.2(722\,\text{N}) = 144\ \text{N}$. The Work-KE Theorem gives:

$$\Delta T = \tfrac{1}{2} m v_f^2 - 0 = W_{net} = -F_f \Delta x + F_T \cos 20^\circ \Delta x.$$

The final speed is:

$$v_f = \sqrt{(F_T \cos 20^\circ - F_f) 2\Delta x / m} = \sqrt{[(180\text{ N})\cos 20^\circ - 144\text{ N}]2(5\text{ m})/(80\text{ kg})} = 1.8\text{ m/s}.$$

4.4 Particle Interactions: Potential Energy

- We introduced the idea of energy thinking that it would be just another way to quantify motion. Now we're going to stretch our definition, to quantify the *potential* to move or change identity – potential energy.
- I asked you to read 4.9 because that supports a slightly different approach to potential energy – looking at it as a *shared* property for an interaction.
- **Potential Energy.** While mathematically very similar to work, conceptually it's significantly different, and it's the way physicists tend to conceptualize the more

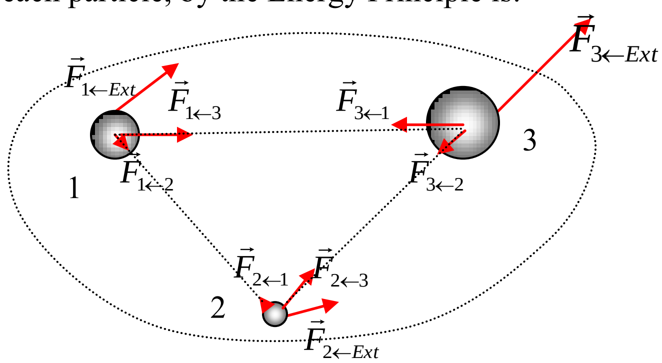
fundamental interactions. Therefore, we'll spend quite some time exploring the nuances of Potential Energy.

4.4.1 Choosing a system

- At this point in our development of Newton's 2nd Law, we considered a system of three interacting particles, and used that to generalize to the Newton's 2nd Law for Systems of Particles. We'll follow a parallel path now.
- For concreteness, say we are interested in how a gas cloud orbiting a star behaves. We'll call the gas cloud our "system" and the star (and everything else in the universe for that matter) "external" to our system.

4.4.2 Interacting Energy in a Multiplarticle System

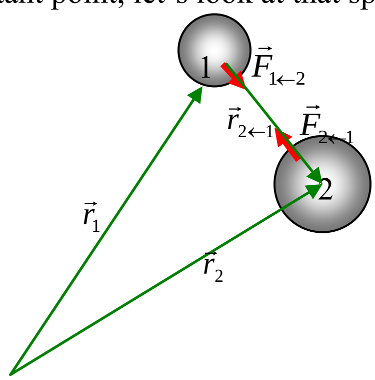
- **Maintaining a system: A word to the wise**. Like the choice of coordinate axes, choosing the boarder between "internal" and "external" to a system is at your convenience and somewhat arbitrary. But, once you've made your choice in a problem, it is important to remain consistent; else sign errors will be introduced. Take for example a ball falling to the Earth. You could define the Ball as your system or you could define the Ball + Earth as your system, but, changing midstream will mess up your math.
- Okay, imagine our "dust cloud" of three particles interacting with each other and some external objects (perhaps a star), and drifting through space. Then the change in energy of each particle, by the Energy Principle is:



$$\Delta E_1 = \vec{F}_{1\leftarrow net}\cdot\Delta\vec{r}_1 = \left(\vec{F}_{1\leftarrow 2}+\vec{F}_{1\leftarrow 3}+\vec{F}_{1\leftarrow ext}\right)\Delta\vec{r}_1 = W_{1\leftarrow 2}+W_{1\leftarrow 3}+W_{1\leftarrow ext}$$
$$\Delta E_2 = \vec{F}_{2\leftarrow net}\cdot\Delta\vec{r}_2 = \left(\vec{F}_{2\leftarrow 1}+\vec{F}_{2\leftarrow 3}+\vec{F}_{2\leftarrow ext}\right)\Delta\vec{r}_2 = W_{2\leftarrow 1}+W_{2\leftarrow 3}+W_{2\leftarrow ext}$$
$$\Delta E_2 = \vec{F}_{3\leftarrow net}\cdot\Delta\vec{r}_3 = \left(\vec{F}_{3\leftarrow 1}+\vec{F}_{3\leftarrow 2}+\vec{F}_{3\leftarrow ext}\right)\Delta\vec{r}_3 = W_{3\leftarrow 1}+W_{3\leftarrow 2}+W_{3\leftarrow ext}$$
$$\Delta\left(E_1+E_2+E_3\right) = W_{internal}+W_{ext\rightarrow system}$$

- **Contrast with Multiparticle Momentum Principle: internal interactions matter.** It's important to note that, while the *forces* between two masses are equal and opposite, thus *they* cancel if we simply sum up all internal forces, the *work* done by these forces is *not* equal and opposite, so they *do not* cancel. Consider the simple case of two equal masses, initially at rest, gravitationally attracting each other. They exert equal and opposite forces on each other and undergo equal and opposite displacements. That means that they do *equal work* on each other. ($\vec{F}\cdot d\vec{r}$) Far from canceling, this means together, double the work is done!

- So, unlike internal forces, internal work does not cancel out. This gives rise to a qualitative difference between the change in energy of a system and the change in its momentum. If you recall, the momentum principle for a system of particles is $\frac{d\vec{p}_{total}}{dt} = \vec{F}_{ext.net}$. This means that you need only worry about external forces when you predict how the system's total momentum will change. However, the energy of a system can be changed due to external or internal interactions. Practically speaking, this makes energy much harder to keep track of. For example, imagine two identical cars crashing head-on. They come to a dead halt. Before, there was no net momentum and there was plenty of obvious energy, after, there was no net momentum, and no *obvious* energy – where did it go – internal work (crumpling fenders, etc.)
- We used a system of 3 particles in the argument above because something rather significant and distracting would happen for a system of just 2 particles. Now that we've made the important point, let's look at that special case.



- $W_{1,2} = W_{1\leftarrow 2} + W_{2\leftarrow 1}$
  $W_{1,2} = \left(\vec{F}_{1\leftarrow 2}\cdot\Delta\vec{r}_1 + \vec{F}_{2\leftarrow 1}\cdot\Delta\vec{r}_2\right)$
  - By the reciprocity principle, $\vec{F}_{1\leftarrow 2} = -\vec{F}_{2\leftarrow 1}$
- $W_{1,2} = \left(\vec{F}_{1\leftarrow 2}\cdot\Delta\vec{r}_1 - \vec{F}_{1\leftarrow 2}\cdot\Delta\vec{r}_2\right) = \vec{F}_{1\leftarrow 2}\cdot\left(\Delta\vec{r}_1 - \Delta\vec{r}_2\right)$
  $W_{1,2} = \vec{F}_{1\leftarrow 2}\cdot\Delta\vec{r}_{1\leftarrow 2} = \left|\vec{F}_{1\leftarrow 2}\right|\left|\Delta\vec{r}_{1\leftarrow 2}\right|\cos\theta$

4.4.2.1 Potential Energy and The Energy Principle for a Multiparticle System

- One way of looking at the internal interactions of a system such as our three dust particles is as the self-contained *potential* to change energy. For example, two magnets held at some distance have the *potential* to fly toward each other. Or a boulder perched precipitously high above a valley floor has an almost tangible potential to come crashing down.
- By a little mathematical manipulation, we can shift to the perspective in which internal work is reconsidered as change in *potential* energy, ΔU.
- $\Delta\left(E_1 + E_2 + E_3\right) = W_{internal} + W_{ext\rightarrow system}$
  $\Delta\left(E_1 + E_2 + E_3\right) + \left(-W_{internal}\right) = W_{ext\rightarrow system}$
- $\boxed{\Delta U \equiv -W_{internal}}$ *

- * Provided the work satisfies a couple of conditions, it makes sense to think in terms of potential.
    - **Redeemable Energy.** This 'energy of configuration' is redeemable for kinetic energy upon rearrangement of the configuration.
        - ***Conservative Force.** There are some criteria for the energy being redeemable, we often say that the force must be "conservative":
            1. it only depends on the particle's position $\vec{r}$ (<u>not</u> on velocity, time, etc.),
            2. the work done by the force moving between any two points does <u>not</u> depend on the path.

            If these criteria aren't met, then the concept of 'potential' energy isn't very applicable. An obvious place you encounter non-conservative forces is friction – not that energy gets destroyed or created, but simply rubbing something backwards isn't going to put the energy back where it came from.

-
    - **Conceptual difference between work and potential.**
        - This may seem like a silly mathematical game to play, defining change in potential energy this way. While it's mathematically trivial, it's conceptually significant; it encourages us to look at interactions and systems in a different way.
        - Work is something you *do.* Potential is something you *have.* If you have a work-study arrangement, this may hit home: at the beginning of the semester, you *have* the *Potential* to earn, say $1,500. Mid-way through the semester you have *done Work,* earned $1,000, and now you have the potential for only $500 more. Doing work *reduces* potential, thus the negative sign.

    - **Two Particle Potential**
        - Returning to our two dust particles,
        - $\Delta U_{1,2} = -W_{1,2} = -\vec{F}_{1\leftarrow 2} \cdot \Delta\vec{r}_{1\leftarrow 2} = -\left|\vec{F}_{1\leftarrow 2}\right|\left|\Delta\vec{r}_{1\leftarrow 2}\right|\cos\theta$
    - A significant result is that the change in potential energy is negative the force dotted into the change in separation of the two particles. This representation gives us a convenient conceptual foothold and a jumping off point for further mathematical developments.
    - **Potential Energy is Shared.** This representation stresses that potential energy is a *shared* quantity. We most often speak of the potential energy shared by a system, not of that of each individual member. Thus, for each interaction, there is one shared potential term.
- **Total (system) Energy.**
    - Defining the *total* energy of the *system* as the energies of the individual particles plus this newly christened potential energy: $\boxed{E_{system} = E_1 + E_2 + E_3 + U}$

- $\Delta E_{system} = W_{system \leftarrow ext}$
- This reads "the energy of the system changes when work is done upon it."

4.4.3 Conservation of Energy

- Consider an isolated system, i.e., one on which no external work is being done, then $\Delta E_{system} = 0$. We say that in this case the system's energy is *conserved.* Energy may be transferred between parts of the system, and it may change forms (potential and kinetic) via internal interactions, but it is neither lost nor gained, created nor destroyed.
- In general, if you take into consideration all objects interacting with each other, include them all within your system, then the energy of that group can't change.
- For that matter, the universe as a whole is, presumably, an isolated system (there's nothing else to interact with), so the total energy in the universe is conserved.
- Physicists like conservation laws (conservation of energy, conservation of momentum, conservation of charge…) for the simple reason 0's are easy to work with.

4.4.4 Changing Potential energy involves a change of configuration: shape or size

- We'll make use of this relation in a few ways. The first is getting a qualitative feel for the implications.
    - Consider two H atoms bound together in a H_2 molecule, if they translate together, keeping the same separation, the same force, then the potential energy doesn't change.
    - If the two particles orbit each other, like a planet around the sun in a perfectly circular orbit, maintaining the same separation, same magnitude of force, and same relative angle between the two, then no change in potential energy.
    - The particles must move relative to each other – the dimensions of the system must change for there to be a change in potential energy.
    - **Ex.** When a bumper crumples in a collision, its internal potential energy changes.
- Though easy to see for just two particles, the same basic conclusion holds for a system of several particles. The potential energy of a perfectly rigid object is constant, it changes if the object flexes – grows, shrinks, bends.
- **Conceptualizing Potential Energy.**
    - Here are two ways of thinking about potential energy.
        - **Energies of Motion, Configuration, and Existence.** While Kinetic Energy is the Energy of Motion, Rest Energy is the energy associated with simply Existing, Potential Energy is the Energy of Configuration.
        - **Redeemable Energy.** This 'energy of configuration' is redeemable for kinetic energy upon rearrangement of the configuration.
            - ***Conservative Force.** There are some criteria for the energy being redeemable, we often say that the force must be "conservative":
                3. it only depends on the particle's position $\vec{r}$ (<u>not</u> on velocity, time, etc.),
                4. the work done by the force moving between any two points does <u>not</u> depend on the path.

If these criteria aren't met, then the concept of 'potential' energy isn't very applicable. An obvious place you encounter non-conservative forces is friction – not that energy gets destroyed or created, but simply rubbing something backwards isn't going to put the energy back where it came from.

Conservative Forces and Potential Energy:

A force $\vec{F}$ acting on a particle is *conservative* if:

5. it only depends on the particle's position $\vec{r}$ (not on velocity, time, etc.),
6. the work done by the force moving between any two points does not depend on the path.

If a force is conservative, we can define the change in *potential energy* associated with exerting a force over a distance as:

$$\Delta U(\vec{r}) = -W(\vec{r}_1 \rightarrow \vec{r}_2) = -\int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r}') \cdot d\vec{r}'$$

Oftne there's a convenient, universal refernc epoint for an interaction and we conventionally say the poentinal *at* a point is

$$U(\vec{r}) \equiv -W(\vec{r}_o \rightarrow \vec{r}) = -\int_{\vec{r}_o}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}' .$$

This would not be a good definition if the force was not conservative.

Suppose there is just one conservative force. Substitute the definition of PE into the relationship between kinetic energy and work gives:

$$\Delta T = W(\vec{r}_1 \rightarrow \vec{r}_2) = -\Delta U$$

$$\Delta(T + U) = 0.$$

That means that the *total energy* defined as:

$$E = T + U$$

is constant. If there is more than one conservative force, there is more than one form of potential energy:

$$E \equiv T + U \equiv T + U_1(\vec{r}) + \cdots + U_n(\vec{r}).$$

If there are some conservative forces and some nonconservative forces, the work can be split into two parts:

$$\Delta T = W = W_{cons} + W_{nc},$$

and there is a change in potential energy $-\Delta U$ associated with the first type of work. We can find the change in mechanical energy:

$$\Delta T = -\Delta U + W_{nc},$$
$$\Delta E = \Delta T + \Delta U = W_{nc},$$

but it is no longer conserved! If there is any work done by a nonconservative force, it changes the mechanical energy of a system.

> <u>Example 3</u>: What is the change in potential associated with the gravitational force (weight) of a particle of mass *m* moving from (0,0,0) to (x,y,z)? Choose the *y* axis to point vertically <u>upward</u>. Assume the force is conservative (i.e. that work by it doesn't depend on the path).
>
> The force is $\vec{F}_{grav} = \vec{w} = m\vec{g}$, where $\vec{g} = (0,-g,0)$ points downward. We must find the line integral from the reference point (0,0,0) to an arbitrary point (*x*,*y*,*z*). Use the path:
>
> $$(0,0,0) \xrightarrow{d\vec{r}=dx\,\hat{x}} (x,0,0) \xrightarrow{d\vec{r}=dz\,\hat{z}} (x,0,z) \xrightarrow{d\vec{r}=dy\,\hat{y}} (x,y,z),$$
>
> so the line integral (in 3 segments) is:
>
> $$\Delta U_{grav}(\vec{r}) = -\int_{\vec{r}_0}^{\vec{r}} \vec{F}(\vec{r}') \cdot d\vec{r}' = -\int_0^x 0\,dx' - \int_0^z 0\,dz' - \int_0^y (-mg)\,dy' = mgy\,.$$
>
> The potential energy only depends on the height relative to the reference point. If the particle is above (below) the reference point, the KE is positive (negative).

4.4.5 Force as Gradient of potential energy

- So Force and Potential energy are related to each other via an integral: Potential is the Integral of Force. We really should be able to phrase that relationship the other way around: Force as a Derivative of Potential. Here's how we go about phrasing the relationship that way. Let's back up and consider an itsy-bitsy change in the potential associated with changing the separation of objects just a tad.

  $$dU_{1\&2} = -\vec{F}_{2\to1} \cdot d\vec{r}_{1\to2} = -(F_{1\leftarrow2x}dx_{1\leftarrow2} + F_{1\leftarrow2y}dy_{1\leftarrow2} + F_{1\leftarrow2z}dz_{1\leftarrow2})$$

- I'll drop these subscripts to make things a little easier on the eye

  $$dU = -(F_x dx + F_y dy + F_z dz)$$

- Now, how much would the potential energy change if we only changed the *x* separation, and held the y and z separation constant? In that case, *dy*=*dz* = 0, so we've just got

  $$dU\big|_{y,z=const} = -(F_x dx + F_y 0 + F_z 0) = -F_x dx$$

  Or

  $$\left.\frac{dU}{dx}\right|_{y,z=const} = -F_x$$

  An shorthand notation meaning 'hold other variables constant' is using curved d's:

  $$\frac{\partial U}{\partial x} = -F_x$$

  We call this the *partial* derivative of *U* with respect to *x*.

- Similarly, if we ask "how much will *U* change if we only change the *y* separation, or if we only change the *z* separation, we get

$$\left.\frac{dU}{dy}\right|_{x,z=const} = \frac{\partial U}{\partial y} = -F_y \text{ and } \left.\frac{dU}{dz}\right|_{x,y=const} = \frac{\partial U}{\partial z} = -F_z$$

So, then

$$\vec{F} = \left(F_x\hat{x} + F_y\hat{y} + F_z\hat{z}\right)$$

Can be rephrased as

$$\vec{F}_{1\leftarrow 2} = \left(\left(-\frac{\partial U}{\partial x}\right)\hat{x} + \left(-\frac{\partial U}{\partial y}\right)\hat{y} + \left(-\frac{\partial U}{\partial z}\right)\hat{z}\right) = -\left(\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}\right)$$

This (and other operations) can be written more compactly using the *del operator*:

$$\vec{\nabla} = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z},$$

which is a vector operator (vector – it has components so use "arrow" notation, operator = it is applied to a function). The force can be written as minus the *gradient* of the potential energy:

$$\vec{F} = -\left(\frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}\right)U$$

$$\vec{F} = -\vec{\nabla}U$$

Example 4: Check that you can recover the force from $U_{grav} = mgy$.

The force is:

$$\vec{F} = -\vec{\nabla}U = -\left(\frac{\partial U}{\partial x}\hat{x} + \frac{\partial U}{\partial y}\hat{y} + \frac{\partial U}{\partial z}\hat{z}\right) = -mg\,\hat{y}$$

which agrees with the way we defined the coordinates in Example 3.

Example 5: If the potential energy for a particle is $U = cr^n$ in spherical coordinates, what is the corresponding force?

The potential can be written in terms of Cartesian coordinates as:

$$U = c\left(x^2 + y^2 + z^2\right)^{n/2}.$$

The corresponding force is:

$$\vec{F} = -\vec{\nabla}U = \left(-\partial U/\partial x\right)\hat{x} + \left(-\partial U/\partial y\right)\hat{y} + \left(-\partial U/\partial z\right)\hat{z},$$

where:

$$-\partial U/\partial x = -c(n/2)(2x)\left(x^2 + y^2 + z^2\right)^{n/2-1} = -cnx\left(x^2 + y^2 + z^2\right)^{(n-2)/2} = -cnxr^{n-2}.$$

Similarly, $-\partial U/\partial y = -cnyr^{n-2}$ and $-\partial U/\partial z = -cnzr^{n-2}$, so:

$$\vec{F} = -cnr^{n-2}\left(x\hat{x} + y\hat{y} + z\hat{z}\right) = -cnr^{n-2}\vec{r} = -cnr^{n-2}\left(r\hat{r}\right) = -cnr^{n-1}\hat{r}.$$

It is easier to use the gradient in spherical polar coordinates from the back cover:

$$\vec{\nabla} f = \hat{r}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}.$$

Since U only depends on r, the force only has a radial component:

$$\vec{F} = -\vec{\nabla} U = -\hat{r}\frac{\partial U}{\partial r} = -cnr^{n-1}\hat{r}.$$

Exercise: Find the force associated with the potential $U = Ax^2 + Bxy + Cz + D$.

The corresponding force is:

$$\vec{F} = -\vec{\nabla} U = (-\partial U/\partial x)\hat{x} + (-\partial U/\partial y)\hat{y} + (-\partial U/\partial z)\hat{z},$$

$$\vec{F} = -(2Ax + By)\hat{x} - (Bx)\hat{y} - (C)\hat{z}.$$

Notice that the value of the constant D does not matter.

Next two classes:

- Friday – Conservative Forces & 1-D Systems
- Monday – Curvilinear 1-D Systems & Central Forces