Wed. 9/19 Thurs. 9/20 Fri. 9/21	**3.1-.2** Momentum & Rockets **3.3-.4** Center of Mass and Angular Momentum for 1 particle	HW3a (3.A-.C)
Mon. 9/24 Tues. 9/25 Wed. 9/26	**3.5** Angular Momentum for multiple particles **4.1-.3, 4.9** Work & Energy, Force as a Gradient, 2 Particle Interaction ***Science Poster Session****: Hedco7~9pm*	**HW3b, Project Topic**

Equipment: string, straw, tape, balloon, clip

Carts and track

Hoover discs (charged up)

Conservation of Momentum: (again)

The total momentum of a system is:

$$\vec{P} = \sum_{\alpha} \vec{p}_{\alpha},$$

and the total external force on the system is:

$$\vec{F}^{ext} = \sum_{\alpha} \vec{F}^{ext}_{\alpha}.$$

The rate of change of the system's total momentum is determined by the total *external* force on the system:

$$\dot{\vec{P}}_{system} = \vec{F}^{ext}.$$

This leads to the *Principle of Conservation of Momentum*:

$$\boxed{\text{If } \vec{F}^{ext} = 0, \text{ then } \vec{P} = \text{constant.}}$$

Keep in mind that momentum is a <u>vector</u>!

A quick refresher on collisions.

Example 1: An *elastic collision* is one in which the kinetic energy is conserved. Not all collisions are elastic. Pool ball collisions are almost perfectly elastic. So, what is the angle between the paths of the balls immediately afterwards (friction with the table and spinning combine to subsequently curve the path)?
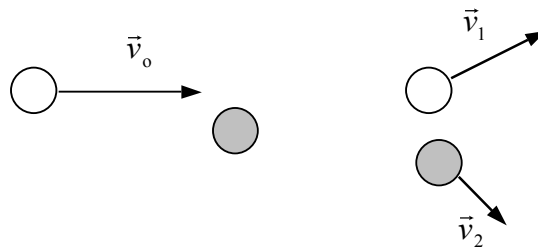
By conservation of momentum, $m\vec{v}_o = m\vec{v}_1 + m\vec{v}_2$ or, assuming they both have the same mass (as pool balls would), $\vec{v}_o = \vec{v}_1 + \vec{v}_2$, which means the velocity vectors form a triangle (see the diagram below).
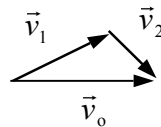
Since the collision is elastic, $\frac{1}{2}mv_o^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}mv_2^2$ or $v_o^2 = v_1^2 + v_2^2$.

So, we have both

$$v_o^2 = v_1^2 + v_2^2$$

And

$$\vec{v}_o = \vec{v}_1 + \vec{v}_2$$

But if we simply treated this as a vector math problem, and we squared the second equation, we'd have

$$v_o^2 = v_1^2 + 2\vec{v}_1 \cdot \vec{v}_2 + v_2^2$$
$$v_o^2 = v_1^2 + 2|v_1||v_2|\cos\theta_{1\rightarrow2} + v_2^2$$

Comparing this and what we got from the constancy of kinetic energy (i.e. elasticity of collision)

$$v_o^2 = v_1^2 + v_2^2$$

They can be reconciled only if the cross term is zero. What are the three ways that can be true?

One or the other simply isn't moving, or if the angle between the two out-going velocities, $\vec{v}_1$ and $\vec{v}_2$, is 90°.

Note: while I *can* say what the angle is between these two vectors, I'd need more information to say what the angle is between one of them and the initial velocity. If I knew the where the point of collision was on the surfaces of the two balls, then I could say what the direction of the collision force was relative to each ball's center of mass, and that would help me figure it out.

Example 2: *completely inelastic collision* is at the opposite extreme; the objects stick together afterwards. Suppose two asteroids collide and stick together. Before the collision, one has a mass of 1.25 kg and a speed of 4 m/s and the other has a mass of 4 kg and a speed of 3 m/s in a direction perpendicular to the first. What is the final velocity of the combined asteroid?
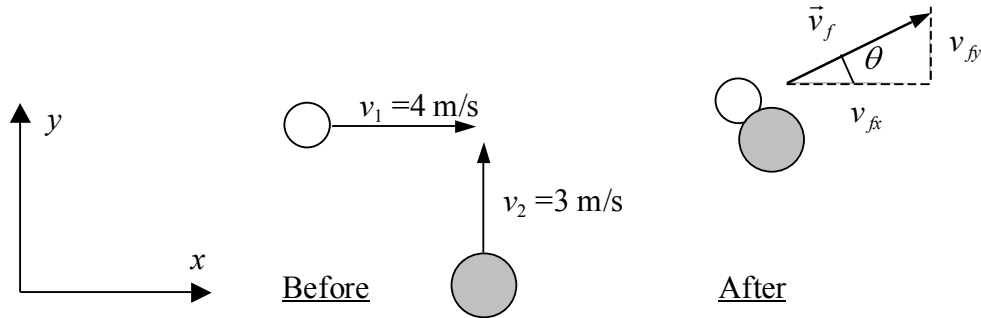
Choose the coordinate system shown in the diagram above. Conservation of momentum gives:

$$m_1 v_1 = m_t v_{fx} \qquad \text{and} \qquad m_2 v_2 = m_t v_{fy}$$

$$v_{fx} = \frac{m_1 v_1}{m_t} = \frac{(1.25 \text{ kg})(4 \text{ m/s})}{5.25 \text{ kg}} = 0.95 \text{ m/s} \qquad v_{fy} = \frac{m_2 v_2}{m_t} = \frac{(4 \text{ kg})(3 \text{ m/s})}{5.25 \text{ kg}} = 2.3 \text{ m/s}$$

The final velocity is $\vec{v}_f = (0.95 \text{ m/s}, 2.3 \text{ m/s})$ or the final speed is:

$$v_f = \sqrt{(0.95 \text{ m/s})^2 + (2.3 \text{ m/s})^2} = 2.5 \text{ m/s},$$

at an angle from the initial velocity of the 1.25-kg asteroid:

$$\theta = \tan^{-1}(2.3 \text{ m/s}/0.95 \text{ m/s}) = 67.6°.$$
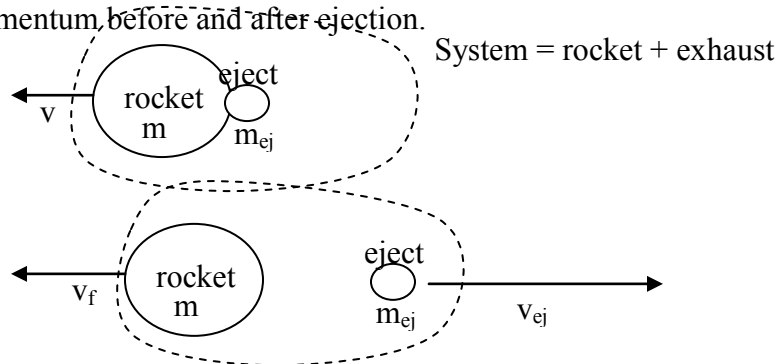
Linear perfectly inelastic separation:

A rocket accelerates by essentially forcefully ejecting mass, kind of a reverse collision. Let's define our system as the rocket plus the mass that's being ejected from it and think about how we'd divvy up the momentum before and after ejection.

Before:

$$P_i = v(m + m_{ej})$$

After

$$P_f = v_f m - m_{ej} v_{ej}$$

So the change in momentum for the whole system is

$$\Delta P_{system} = P_f - P_i = vm - m_{ej}v_{ej} - v(m + m_{ej}) = (v_f - v)m - m_{ej}(v_{ej} + v) = (v)m - m_{ej}(v_{ej} + v)$$

The sum of the two speeds in the brackets is essentially how quickly the rocket and ejected mass are moving away from each other, that's what's known as the "exhaust speed", v_{ex}, how quickly the exhaust moves away from the rocket.

$$P_f - P_i = \left(\Delta v \, m - m_{ej} \, v_{ex} \right)$$

Now, what we're really interested in is what the rocket is up to, so we'll rephrase a few things to be a little more rocket-centric. The ejected mass is how much mass the rocket is *loosing* :

$$m_{ej} = -\Delta m$$

$$\Delta P_{system} = \left(\Delta v \, m - \left(\Delta m \, v_{ex} \right) \right.$$

Alright, we've rephrased the change in the *system's* total momentum in terms of the change in the rocket's speed and the rocket's mass. Of course, according to Newton's 2nd Law, this change in momentum is related to the net force acting on the system via

$$F_{net.ext \to system} = \frac{\Delta P_{system}}{\Delta t} = \left(\frac{\Delta v}{\Delta t} \right) m + \left(\frac{\Delta m}{\Delta t} \right) v_{ex}$$

$$F_{net.ext \to system} = \left(\frac{dv}{dt} \right) m + \left(\frac{dm}{dt} \right) v_{ex}$$

CASE 1: If there is **no external force**, then:

$$\left(\frac{dv}{dt} \right) m + \left(\frac{dm}{dt} \right) v_{ex} = F_{net.ext \to system} = 0$$

$$\left(\frac{dv}{dt} \right) m = -\left(\frac{dm}{dt} \right) v_{ex}$$

or

$$m\dot{v} = -\dot{m}v_{ex}.$$

The expression on the right hand side is referred to as the rocket's "*thrust*".

The equation above looks like Newton's second law with the "thrust" as the force, but the mass *m* is changing

This can be integrated (remember that *m* is not constant):

$$dv = -v_{ex} \frac{dm}{m}$$

$$\int_{v_o}^{v} dv' = v - v_o = -v_{ex} \int_{m_o}^{m} \frac{dm'}{m'} = -v_{ex} \ln(m')\big|_{m_o}^{m}$$

$$v - v_o = v_{ex} \ln(m/m_o).$$

CASE 2: If a rocket is moving upward **near the earth** (farther away the weight varies) under the influence of gravity, then calling upward the positive direction:

$$\frac{dP_{system}}{dt} = F_{net.ext \to system}$$

$$\left(\frac{dv}{dt}\right)m + \left(\frac{dm}{dt}\right)v_{ex} = -\left(m + m_{ej}\right)g = -\left(m - dm\right)g$$

Making the substitution that the ejected mass is the differentially small change in mass of the rocket, and knowing we're going to be looking in the differential limit, that term vanishes leaving just

$$\left(\frac{dv}{dt}\right)m + \left(\frac{dm}{dt}\right)v_{ex} = -mg$$

$$dv + \left(\frac{dm}{m}\right)v_{ex} = -g\,dt$$

Integrating, we have

$$\left(-v_o\right) + v_{ex}\ln\left(\frac{m}{m_o}\right) = -g\left(-t_o\right)$$

$$v = v_o - g\left(-t_o\right) + v_{ex}\ln\left(\frac{m_o}{m}\right)$$

If we're starting from launch ($v_o=0$)

$$v = -gt - v_{ex}\ln\left(\frac{m}{m_o}\right)$$

Constant rate of mass exhaust: Suppose the mass of the rocket decreases at a constant rate $\dot{m} = -k$ (exhaust is ejected at a constant rate k). The mass of the rocket is

$$m(t) = m_o - kt,$$

so the speed of the rocket in that case is:

$$v(t) = v_o - gt + v_{ex}\ln\left(\frac{m_o}{m(t)}\right) = v_o - gt + v_{ex}\ln\left(\frac{m_o}{m_o - kt}\right).$$

If we knew the function $m(t)$ ahead of time, we could've put that in and integrated over v and t.

Rocket Example/Exercise: (Ex. 9.12 of Thornton) A Saturn V rocket has an initial mass of 2.8×10^6 kg including 2.1×10^6 kg of fuel for the first stage. Fuel is burned at a nearly constant rate of 1.4×10^4 kg/s and the exhaust speed is 2600 m/s relative to the rocket. How high is the rocket when the first stage burns out?

Find the height at the burnout time:

So, we're given that
$$m_o = 2.8 \times 10^6 \, kg$$

$$m_{fuel} = 2.1 \times 10^6 \, kg$$

$$\dot{m} = -k = -1.4 \times 10^4 \, kg$$

$$-\dot{m} t_b = m_f$$

$$t_b = \frac{m_f}{|\dot{m}|} = \frac{2.1 \times 10^6 \, \text{kg}}{1.4 \times 10^4 \, \text{kg/s}} = 150 \, \text{s} \, .$$

Pick up from here and find an expression for the height as a function of time under this condition Since the rocket start at rest:

$$v(t) = -gt + v_{ex} \ln \left(\frac{m_o}{m_o - kt} \right)$$

$$v(t) = -gt - v_{ex} \ln \left(\frac{m_o - kt}{m_o} \right) = -gt - v_{ex} \ln \left(1 - \left(\frac{k}{m_o} \right) t \right) \, .$$

Integrate this again to get the height:

$$\int_0^y dy' = -\int_0^t \left[gt' + v_{ex} \ln \left(1 - \frac{k}{m_o} t' \right) \right] dt' \, .$$

Two of the integrals are easy. The third one benefits from a change of variables.

$$u \equiv 1 - \frac{k}{m_o} t'$$

So the limits of integration become $u_{\min} = 1$ and $u_{\max} = 1 - \frac{k}{m_o} t$

And $\dfrac{du}{dt'} = -\dfrac{k}{m_o} \Rightarrow dt' = -\dfrac{m_o}{\dot{m}} du$

Use $\dot{m} = -k$ or $dm = -k \, dt$ to change variables for the second integral on the right:

$$y = -\tfrac{1}{2} gt^2 - v_{ex} \int_0^t \ln \left(1 - \frac{k}{m_o} t \right) dt'$$

$$y = -\tfrac{1}{2} gt^2 + v_{ex} \frac{m_o}{k} \int_{u_{\min}}^{u_{\max}} \ln(u) \, du$$

$$y = -\tfrac{1}{2}gt^2 + v_{ex}\frac{m_o}{k}\left[\ln(u) - u\right]_1^{1-\frac{k}{m_o}t=\frac{m}{m_o}}$$

$$y = -\tfrac{1}{2}gt^2 + v_{ex}\frac{m_o}{k}\left[\left(\frac{m}{m_o}\right)\left(\ln\left(\frac{m}{m_o}\right) - 1\right) - \left(\ln\left(1\right) - 1\right)\right]$$

$$y = -\tfrac{1}{2}gt^2 + v_{ex}\frac{m_o}{k}\left[\left(\frac{m}{m_o}\right)\left(\ln\left(\frac{m}{m_o}\right) - 1\right) + 1\right]$$

$$y = -\tfrac{1}{2}gt^2 + v_{ex}\left(\frac{m_o - m}{k}\right) + v_{ex}\frac{m}{k}\ln\left(\frac{m}{m_o}\right)$$

$$y = -\tfrac{1}{2}gt^2 + v_{ex}\left(\frac{m_{\text{fuel}}}{k}\right) + v_{ex}\frac{m}{k}\ln\left(\frac{m}{m_o}\right)$$

$$y = -\tfrac{1}{2}g\left(\frac{m_{\text{fuel}}}{k}\right)^2 + v_{ex}\left(\frac{m_{\text{fuel}}}{k}\right) + v_{ex}\frac{m}{k}\ln\left(\frac{m}{m_o}\right)$$

Put in all of the numbers with a final mass
$$m_o - m_f = 2.8\times10^6\text{ kg} - 2.1\times10^6\text{ kg} = 0.7\times10^6\text{ kg}:$$

$$y(t_b) = -\frac{9.8\,\text{m/s}^2}{2}(50\,\text{s})^2 + (600\,\text{m/s})(50\,\text{s}) + (600\,\text{m/s})\frac{0.7\times10^6\text{ kg}}{1.4\times10^4\text{ kg/s}}\ln\left(\frac{0.7\times10^6\text{ kg}}{2.8\times10^6\text{ kg}}\right)$$

$$y(t_b) = 9.95\times10^4\text{ m} \approx 100\text{ km}.$$

The actual height is about 2/3 of that, but we didn't account for air resistance (or the variation in the gravitational force with altitude). Note that the rocket will continue to go higher.

Horizontal Rocket balloon. Say we tape a straw to the back of a balloon and slip that onto a horizontal string, then let it go. The main force opposing its motion is drag, and it's slow enough that linear drag should do. So Newton's 2nd law should look like

$$\frac{dP_{system}}{dt} = F_{net.ext\rightarrow system}$$

$$\left(\frac{dv}{dt}\right)m + \left(\frac{dm}{dt}\right)v_{ex} = -bv$$

We'll make some simiplifying approximations like the balloon is a simple sphere of un-changing radius (so b is constant) and it ejects air at a constant rate so

$$\dot{m} = -k$$

$$m = m_o - kt$$

How are the speed and mass of the balloon related?

$$\left(\frac{dv}{dt}\right)m - kv_{ex} = -bv$$

$$\left(\frac{dv}{dt}\right) = -\frac{b}{m}\left(v - \frac{k}{b}v_{ex}\right)$$

Define $u \equiv v - \frac{k}{b}v_{ex}$

In terms of that, our equation is

$$\frac{du}{dt} = -\frac{b}{m}u$$

$$\frac{du}{u} = -b\frac{dt}{m}$$

So we can set up the integrals

$$\int_{u_i}^{u}\frac{du'}{u'} = -b\int_{0}^{t}\frac{dt'}{m}$$

The left hand side is easy to integrate; the right hand side needs one more step – phrasing m in terms of t or vice versa. I'll do the latter.

$$m = m_o - kt \text{ so } t = \frac{m_o - m}{k} \Rightarrow dt = -\frac{1}{k}dm$$

$$\int_{u_i}^{u}\frac{du'}{u'} = \frac{b}{k}\int_{m_o}^{m}\frac{dm'}{m'}$$

$$\ln\left(\frac{u}{u_o}\right) = \frac{b}{k}\ln\left(\frac{m}{m_o}\right)$$

$$\ln\left(\frac{u}{u_o}\right) = \ln\left(\left(\frac{m}{m_o}\right)^{\frac{b}{k}}\right)$$

$$\frac{u}{u_o} = \left(\frac{m}{m_o}\right)^{\frac{b}{k}}$$

The time has come to translate back from u to v: $u \equiv v - \frac{k}{b}v_{ex}$ so $u_o \equiv 0 - \frac{k}{b}v_{ex}$

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$$\frac{v - \frac{k}{b}v_{ex}}{-\frac{k}{b}v_{ex}} = \left(\frac{m}{m_o}\right)^{\frac{b}{k}}$$

$$1 - v\left(\frac{b}{kv_{ex}}\right) = \left(\frac{m}{m_o}\right)^{\frac{b}{k}}$$

$$v = v_{ex}\frac{k}{b}\left(1 - \left(\frac{m}{m_o}\right)^{\frac{b}{k}}\right)$$

Next two classes:

- Friday – Center of Mass & Angular Momentum of a Particle
- Monday – Angular Momentum for Systems of Particles