Physics 310
Lecture 7a – Waveform Generators & Timers

Wed. 3/7 Thurs. 3/8 Fri. 3/9	**Ch 10:** Oscillators & Timers - Beginning **Ch 10:** Oscillators & Timers – Rest of; **Lab 7:** Oscillators & Timers **Lab 7:** Oscillators & Timers**; Quiz** Ch 10	 **HW 7:** Ch10 Pr. 2*, 6,7,8**; Lab 7 Notebook**

Equipment

Lect. 7a(short) ppt
Spring-ball solid

Handout:

- Lab #7

Waveform Shaping

- Comparator (Schmitt Trigger is better version) – sine or triangle to square wave
- Integrator – square to triangle wave
- Differentiator – square wave to "pulses"

Oscillators (just a few examples)

- Crystal oscillators – LC equivalent
- Relaxation oscillators – neon lamp, op-amp square-wave oscillator, Wein bridge

555 Timer

- Basic operation
- Circuit examples – One-shot (monostable operation), astable operation

Waveform Generators – ICL8038 is an example, external components determine frequency

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-Need a better explanation on the 555 timer if thats important for us to know..
It was only a small section in the chapter.
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You bet; however, I'm going to defer that until tomorrow; I'll deal with circuits built of more familiar components today.

10-1 Intro.

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-What application does an oscillator have in a circuit.  Like what does it do..?
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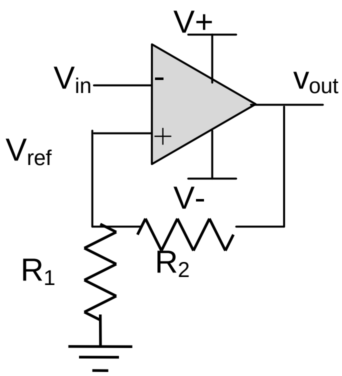
- Often you want a circuit to generate a specific waveform – maybe a sine wave to help smoothly drive a motor, maybe a square wave to blink a light or count off seconds, or maybe a sawtooth wave to drive stick-slip motion of a piezo. Here are different circuits that *generate* different outputs from scratch or *shape* given inputs into different outputs.

10-2 Waveform *Shapers*

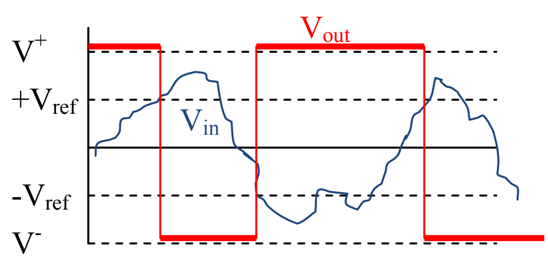
We've actually already met some of these 'shapers', though we weren't necessarily thinking them in these terms. Given one kind of periodic input they generate a different kind of periodic output.

10-2.1 Comparator (with Schmidt trigger)

- We already met the Schmitt Triggered comparator, but since that was before break (and an exam over different material), maybe this is a nice place to ease us back in.



- **Analysis**
    - Now, this is *not* a negative-feedback configuration (the negative input is not linked to the output) so we don't get to invoke the Golden Rule to analyze it, instead, we need to go back to basics:
        - $V_{out} = A(V_{in+} - V_{in-})$ unless it hits the + or – "rail" trying, i.e., V^+, V^-.
    - In this circuit, V_{in} is wired to V_{in-} and we think of the voltage on V_{in+} as a reference, sometimes referred to as V_{ref}.
    - So, if $V_{in} \begin{Bmatrix} < \\ > \end{Bmatrix} V_{ref}$ then $V_{out} = \begin{cases} V^+ \\ V^- \end{cases}$
    - But that's not all; the two resistors can be seen as making a voltage divider so that $V_{ref} = V_{out}\left(\frac{R_1}{R_1 + R_2}\right)$ so V_{ref} flips between $V_{ref+} \equiv V^+\left(\frac{R_1}{R_1 + R_2}\right)$ and $V_{ref-} \equiv V^-\left(\frac{R_1}{R_1 + R_2}\right)$ as V_{out} flips. The effect is to make Hi/Lo output that's impervious to (small) noise.



So, when the input signal's upward bound, it has to cross $+V_{ref}$, to flip the output up, and then when it's downward bound it has to cross $-V_{ref}$ to flip the output down. Note that I've illustrated a situation in which the rails are symmetric around 0, like +/-15V; that would make the reference thresholds symmetric and the square wave's high & lo values equal and opposite. One needn't have that symmetry; for example, sometimes a comparator will be wired up with the negative rail at 0, which would obviously shift up the negative reference and the 'low' value of the output would only be 0.

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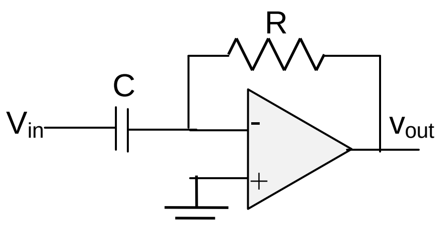
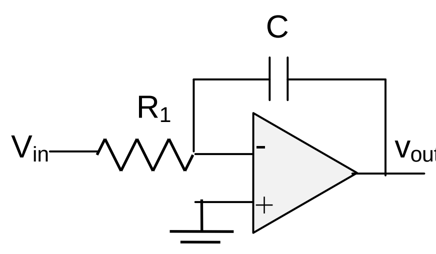
10-2.1.1 Sine-to Square Wave

- If the input is anything with a regular period (and not *too* noisy) this circuit can generate a square wave. The book shows a special case of a) sine wave input and b) V^- = ground. Because these are asymmetric limits, the reference points aren't symmetric around 0 (they're more like $-V_{ref} = 0$, $+V_{ref} = 100$mV) whereas the input signal, a sine wave, is symmetric around zero, so the output is asymmetric – it spends longer Hi than it does LO since the input spends longer below 100mV than it does above 0 V.

10-2.2 Differentiator & Integrator

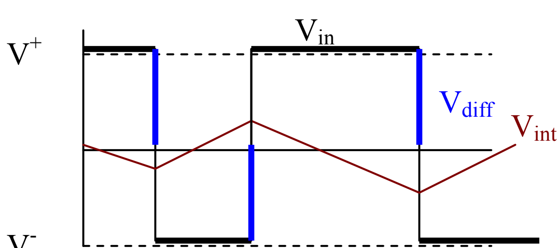


Differentiator

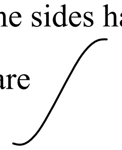
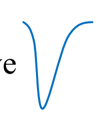


Integrator

- What if you already have a square wave, perhaps from a Schmitt triggered comparator, what other wave forms could you shape from *that*?
- **Integrator**
  - **Q**: If the input for an integrator were a +/- square wave, what would its output be?
    - **A**: A triangle wave (note: since the integrator's output is really the *negative* integral, the triangle wave would slope *down* for positive input and *up* for negative input).
  - Similarly, if the input were a +/0 square wave, then the integrator would make a jagged stair-step (decreasing during the + input and holding steady during the 0).
- **Differentiator**
  - **Q**: How about for a differentiator, what would be the output when fed a square-wave input?
  - **A** differentiator would output spikes whenever the square wave input transitioned. (like the integrator, the differentiator outputs *negative* derivative, so spike *up* for a down transition and spike *down* for up transition.)



This "spike" or "blip" is what the text refers to as a "pulse." In the ideal, a perfect square wave would have perfectly vertical sides and so the derivative would be infinite. Of course, that's impractical; usually the sides have a bit of slope and curve at the ends:

Side of square  - Derivative 

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3.  To my understanding a pulse wave is really an asymmetric square wave.  On pg
217 Diefenderfer says that a differentiator circuit is used to convert a  square
wave to a pulse wave.  Wouldn't the differentiating a square wave result in blips
rather than sustained asymmetrical voltages?
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Yes. By "pulse" he means what you mean by "blip."

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He goes on to say that if the input is a single pulse, then the output pulse
width can be modified.  How does this work?
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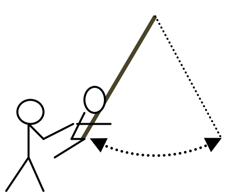
He just means that, while the input might be a square-ish pulse that lasts for, say 10 ms, with a side taking 2μs to transition from 'lo' to 'hi', then the output is a pair of + and - (curvy) pulses 10ms apart and with times of only 2μs. If you have circuitry down the line that responds to only, say positive pulses, then as far as it's concerned, this comparator swapped a 10ms pulse for a 2μs pulse.

Wave-Form *Generators*

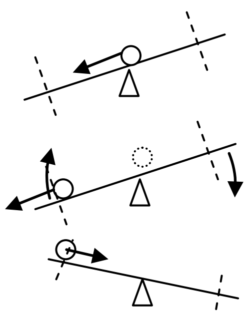
The rest of the chapter is about circuits that *don't* require external, periodic signals to drive them; these circuits *generate* their oscillating outputs from scratch. I'm doing things a little out of the book's order so I can bring more familiar types of circuits forward and postpone less familiar ones. I should note that almost all of these generators are a little peculiar; we'll be able to get the general gist of them and completely understand most of the logical steps, but they all involve one or another 'cute' step that's a little harder to reason out. So, don't feel like you have to master these, just *mostly* understand them.

Two types. In broad strokes, there are two types of oscillators – Relaxation and Resonance. To get a qualitative feel for how each type works, let's think about their mechanical analogs.

- **Resonance.**
  - **Mechanical Analog.** For the Resonance oscillator, think of a mass bobbing on a spring or a kid on a swing swinging back and forth. In either case, there's a frequency at which the system naturally oscillates back and forth without any external help from you or me, that's the 'resonance frequency.' Of course, as you're quite familiar, once you set your mass bobbing or kid swinging, it's oscillations will begin decaying – while the period of oscillation remains constant, thanks to friction, drag, etc., the amplitude of oscillation gets smaller and smaller until the thing's essentially stopped moving. So, if you want it to *keep* oscillating, you need to give it well timed nudges. In terms of energy, while friction is sucking energy out of the system, your nudges are counter acting that by putting energy back into the system. For the kid on the swing, you give a push once each swing; of course, you make a point of pushing *forward* when the kid's moving *forward* (not when the kid's moving *backward* – that would be counterproductive.)



  - **Basic Ingredients.** So, there's a system that naturally oscillates at its resonance frequency, and there's someone giving nudges at the right time to keep the system going in spite of energy loss due to friction. Those are the two pieces you'll see in all the circuits that use resonance to generate an oscillating current & voltage.
- **Relaxation.**

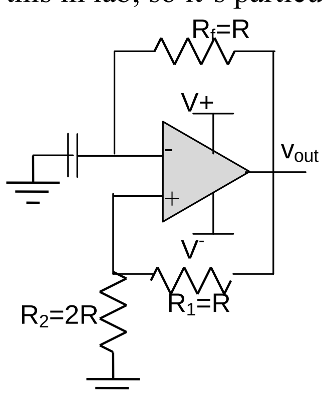


- **Mechanical Analog.** This example is a little contrived, but say you have plank set on a pivot, teter-totter-like, with a little trough cut along its length and a marble sitting in the trough. Near each end of the plank you’ve drawn a ‘finish line.’ You tip one end of the plank up and the marble runs to toward the other, but when it hits the finish line there, you tip the plank back the other way, so the marble now runs toward the other end; of course, when it hits the finish line there, you tip the plank back again.
- **Basic ingredients.** So, there’s a system that naturally moves at some rate from high (gravitational) potential toward low potential. Then someone switches the potentials (tips the plank the other way) whenever the system crosses a threshold. Between the natural rate at which the system ‘relaxes’ toward lower potential and the choice of threshold levels, a period of oscillation is determined.

Other Relaxation Oscillators (electronic ping-pong)
While the book covers these last in section 10-4, I think you’re probably best prepared to understand them up front.
`-What does it mean when an oscillator is relaxed?`
The defining characteristic of “relaxation” oscillators is that they all have some way in which charge / voltage builds up, hits some threshold, and then discharges, i.e. “relaxes.”

Differentiator-Comparator

Essentially merge the Differentiator and the Schmidt Triggered Comparator. You’ll build this in lab, so it’s particularly good getting a feel for how it works.



First let’s consider the *negative* feedback loop

$$\Delta V_f = -IR$$

$$\overbrace{V_{out} - V_-}^{\Delta V_f} = -IR$$

$$V_- = V_{out} + IR$$

$$I = -\frac{dq_{cap}}{dt} = -C\frac{dV_-}{dt}$$

$$V_- = V_{out} - RC\frac{dV_-}{dt}$$

For the moment, let’s imagine V_{out} is a constant (we’ll slightly justify that in a moment), then this is solved by

$$V_- = V_{out}(1 - e^{-t/(RC)})$$

Now for the *positive* feedback loop

$$v_{out} = A(V_+ - V_-)$$

$$V_+ = v_{out}\frac{2R}{3R} = v_{out}\frac{2}{3}$$

So,

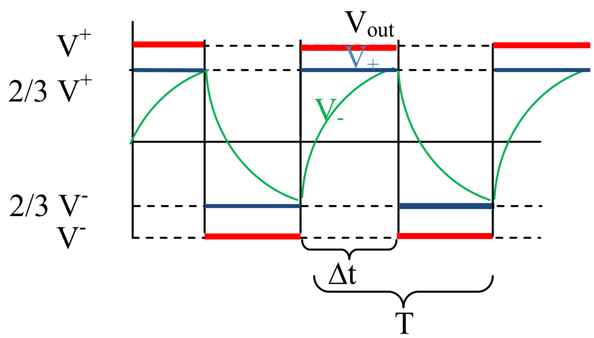
$$v_{out} = A\left(v_{out}\frac{2}{3} - V_-\right) \Rightarrow \begin{cases} V+ & if \quad V_- < \tfrac{2}{3}v_{out} \\ A\left(v_{out}\frac{2}{3} - V_-\right) & if \quad V_- \approx \tfrac{2}{3}v_{out} \\ V- & if \quad V_- < \tfrac{2}{3}v_{out} \end{cases}$$

Now, it may be tempting then to flip this around and say that $v_{out} = \frac{V_-}{2/3 - 1/A} \approx \tfrac{3}{2}V_-$ *however,* that's only true when v_{out} is close enough to V_- for the huge gain factor, A, to not send v_{out} off to one of the rails. This is something of an unstable equilibrium situation and, for practical purposes, happens a negligible fraction of the time.

Then, we'd say that v_{out} flips sign, from one rail to the other, when V_- crosses 2/3^rds^ of its value.

- Now, just to get a foot hold, let's say that V_{out} is at the positive rail, V+.
- Then V_+=2/3 V+ and V_- grows exponentially toward V+.
- But once it grows enough that $V_->V_+$=2/3 V+ then v_{out} drops to the negative rail, V-, and so V_+=2/3 V- and V- starts exponentially decaying to V-…until $V_-<V_+$=2/3 V-
- at which point things flip again…

The behavior looks something like this



What's the period or frequency of this oscillation? Let's find the time to charge from the bottom to the top, Δt, and double that.

Over this time interval,

$v_- = -\tfrac{2}{3}V^+ + \tfrac{5}{3}V^+(1 - e^{-t/(RC)})$ has the right exponential growth and the right limits at t=0 and infinity

so it's as charged as it's going to get at

$$v_{-} = \tfrac{2}{3}V^{+} = -\tfrac{2}{3}V^{+} + \tfrac{5}{3}V^{+}\left(-e^{-\Delta t/(RC)}\right)$$
$$\tfrac{4}{3}V^{+} = \tfrac{5}{3}V^{+}\left(-e^{-\Delta t/(RC)}\right)$$
$$\tfrac{4}{5} = \left(-e^{-\Delta t/(RC)}\right)$$
$$e^{-\Delta t/(RC)} = \tfrac{1}{5}$$
$$\Delta t = RC\ln 5$$

Thus the

Full period is twice this

$T = 2\Delta t = 2RC\ln 5$

And the frequency is

$$f = 1/T = \frac{1}{2RC\ln 5}$$

(note: the text is missing this factor of ln(5).)

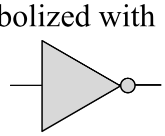
Going back through this argument in slightly more general terms (where the resistance values aren't necessarily as shown) $f = \dfrac{1}{2R_f C\ln\left(2\dfrac{R_2}{R_1}+1\right)}$

Double Inverter Oscillator

- The next oscillator circuit uses a component you haven't met yet; fortunately its function is pretty easy to understand.
- **Crash Course in Inverters.** The "Inverter" is a logic chip that's essentially a Comparator with the *input* and *reference* flipped.
  - When the *input* signal crosses *below* a lower threshold, the inverter's output goes to the *positive* rail;
  - when the *input* crosses *above* an upper threshold, the output goes to a *negative* rail (which is often simply ground.)

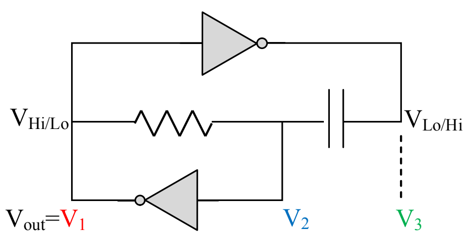
  While a Comparator can be used to achieve this function, an even simpler bit of circuitry works: something akin to the Transistor Switch. Just like the op-amp, under the hood, the inverter is built of transistors. Like the op-amp, it *draws* negligible current in it input (very large input impedance) and can source as much current as you'd reasonable want (very small output impedance).

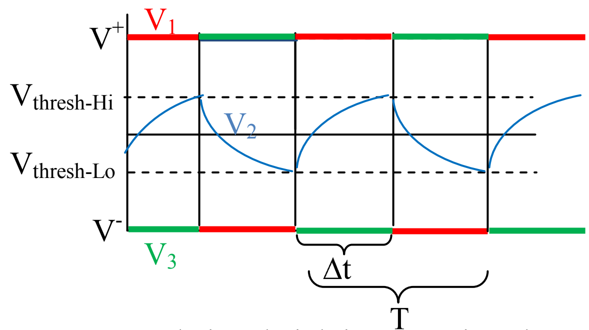
  - **Symbol.** As you're slowly beginning to see, circuit diagrams involving op-amps and the such often bother showing only the details that are necessary for conveying the *logic* of the circuit (what it does to the *signal*), not the other connections that 'power' the chip. In that spirit, an Inverter is symbolized with

  - This is pretty abstract, I grant. If it helps you to remember it, the lineage is that the Follower is represented by the same symbol but without the dot on the end. That makes some sense because a follower typically is an Op-Amp (triangle) that just passes the signal through (line in, line out). An inverter does the same thing, but gives you the logical opposite. The dot/circle/knot represents the logical NOT. It's kind of juvenile, but together the triangle and circle say "here take my input…*NOT.*"

Okay, so here's an oscillator that uses two inverters.



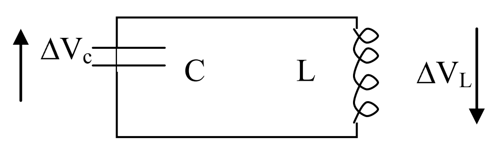


Remember, the inverters draw negligible current, so their sole job is to monitor the voltages at their inputs, and apply the appropriate voltage at the output.

10-3 Resonant Oscillators

10-3.1 Amplified LC

Recall this simple circuit from our distant past (for the moment, ignoring the fact that there's *always* resistance).



$$V_c = -V_L$$

$$Q/C = -L\frac{di}{dt}$$

$$i = -LC\frac{d^2 i}{dt^2}$$

Hm… what function of time would be equal to its negative second derivative? Sine or Cosine!

$i = i_o \sin(\omega t)$ where $\omega = \dfrac{1}{\sqrt{LC}}$.

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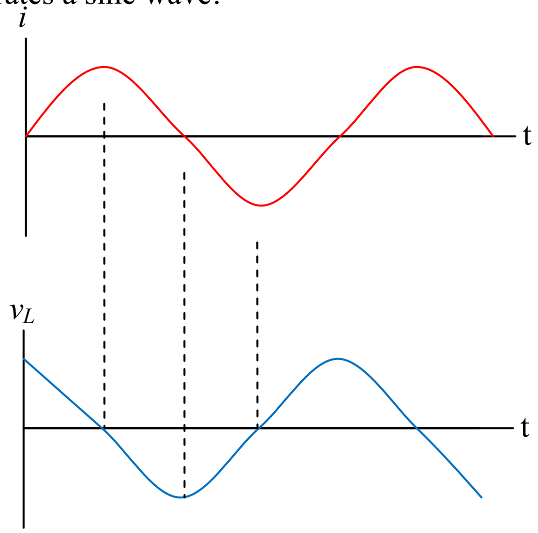
So, if you started off with a charged capacitor and then connected it to an inductor, the current (and thus the voltage too) would oscillate back and forth *sinusoidally* at this frequency – it generates a sine wave!



The mechanical analog is a mass on a spring – rather than tracking the motion of the charges, in this circuit, track the motion of that mass – the spring/capacitor may be initially compressed/+charged, so it starts pushing on the mass/charges through the inductor which has some inertia so it slows this change in motion. But by the time the spring/capacitor becomes unstretched/uncharged, the momentum of the mass / inductor, keeps things moving so that the spring/capacitor gets even more stretched/-charged. Eventually the mass/charge stops moving and starts going back the other way. The system oscillates back and forth.

Okay, now for the unfortunate part: there is *always* resistance, and that serves to drain energy out of the system / decay the sine wave. So, if we want to use this as a sine wave generator, then we need to put energy *back into* the system, boost up the amplitude. That's where an amplifier comes in handy – amplify the voltage and send it back to drive the voltage. Some of the circuits that follow are implementations of this kind of thinking – take something that naturally resonates at a single frequency, and then amp it up to offset any losses.

For the mass on a spring, this would be like you're giving it a little nudge down/up each time it's moving down/up. The exact timing of the nudge isn't too important, as long as it's down when headed down, up when headed up.

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1.  I may be looking to far into this but I want to make sure I understand.  On
Pg. 218 is the diagram of the LC circuit.  In the section where he says that one
could overcome power loss by resupplying energy,  he states that the energy
should be resupplied at T1, T4, and T8.  Would supplying the energy at T1 be
incorrect?  T1 does not appear to be at the same state as T4 and T8.
Quantitatively, would we want to resupply the energy at the point in time when
current is 0?
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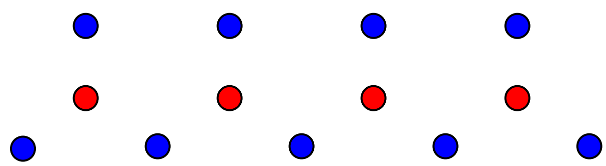
You are right that time 1 doesn't look quite like times 4 & 8; what matters is that the extra charge gets driven onto the capacitor while it's charging up and/or driven off while it's discharging.

10-3.2 Piezo Crystal Oscillator

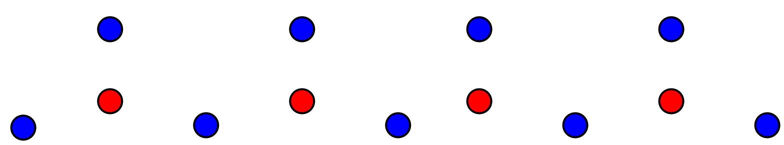
- Nature has made a handy LC circuit: a piezo-electric crystal. These are used all over the place in clocks and watches. A Piezo-electric is a material that, when compressed or expanded generates a voltage difference, proportional to the expansion/contraction, across its sides. Now, since every chunk of material has a resonance frequency at which it would easily vibrate when struck - tuning fork is an extreme example, but everything will do it, then the idea is to “strike” the piezo with a voltage, then it will oscillate sinusoidally and, meanwhile, generate a sinusoidally oscillating voltage.

How Piezos work:

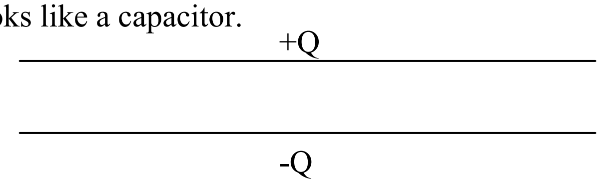
- We think of crystals as being built of atomic scale bricks, call the “unit cell”. Just to picture one, one common “unit cell” is body-centered-cubic: imagine a cube with an atom on each corner and one in the middle. Then imagine building a big, thick slab of such cubes stacked on/under/by/infront/behind each other. That’s a chunk of crystal. To a large extent, what properties the “unit cell” has, simply scale up to the whole chunk.
- Now a Piezo is special in two ways. First, there’s a charge distribution within the unit cell, and second the atomic bonds that hold it together are stiffer one way than another.
- Imagine this 2-D crystal (it doesn’t *really* work in 2-D, but you’ll get the basic idea), where, say, blue is positive and red is negative. If the charges are distributed just right, so that the positives have, say about half as much charge as do the negatives, then the crystal is net neutral and over large chunks the even and odd distributions average out.



- Now say you compress the crystal; the splayed bonds will bend more easily than the aligned bonds will compress, so it will look like this. Aside from getting wider and shorter, notice that the red negatives are closer to the bottom than they used to be. If, in the unsquashed version, the + and – charge distributions averaged out, in this squashed version, they cannot, and there will be a net negative charge on the bottom and a net positive charge on the top.

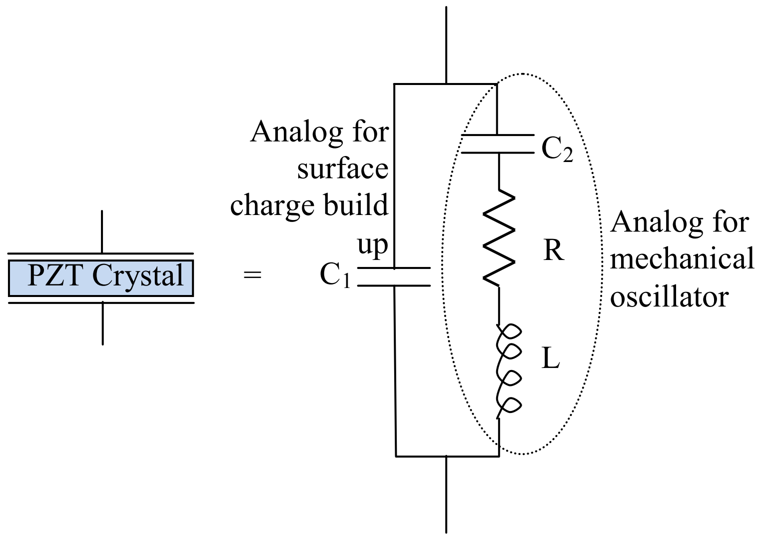


Electrically, this looks like a capacitor.



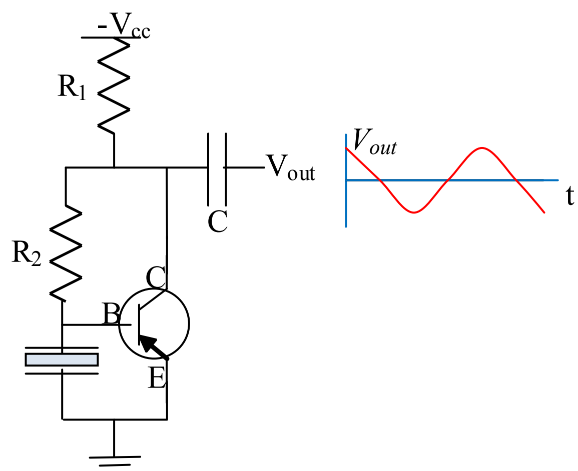
When you compress and decompress the Piezo, the charge on the “capacitor plates” and thus voltage developed increases and decreases. We can model the interaction between this mechanical & voltage oscillation with the “plates’” charging and discharging. Conveniently, a mechanical dampened harmonic oscillator has an electronic analog in an LCR circuit (the resistor sucks energy out of the system, like drag, the capacitor can store up and spend down energy, like a spring, the inductor resists change like the inertia of a mass.) So, while the Piezo's voltage *really* may be oscillating in response to mechanical oscillations, that behavior can be modeled in terms of the equivalent LCR's voltage!

So,

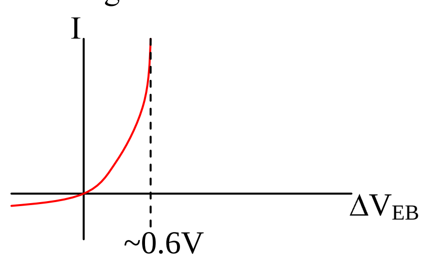


The LCR branch has a natural frequency of $\omega_o = \dfrac{1}{\sqrt{LC}}$ or $f_o = \dfrac{1}{2\pi\sqrt{LC}}$. Though obvious, it's worth pointing out that this equivalent circuit (and so this crystal) oscillates *sinusoidally.*

- **Sample Piezo-Crystal Circuits**
  - Now we'll look at two circuits that use the periodicity of Piezo crystals to generate periodic waveforms.
  - **Pierce Crystal Oscillator**
    - This uses a piezo to subtly oscillate the voltage between a bi-polar transistor's Base and Emitter. As the piezo naturally oscillates sinusoidally, you might not be surprised that the circuit's output is a nice sine wave. Here's what the circuit looks like, but we probably need to refresh our memory on Transistors before we get to seriously figuring this out.



- **Transistor Background.**
  - **PNP not NPN.** First, a few things might look backwards to you. The V_{cc} voltage supply line is explicitly marked as *negative* (rather than positive) and the arrow in the transistor (indicating the direction of charge-carrier motion if V_{cc} were positive) points from the Emitter to the Base (rather than the other way around). The reason things look backwards is that we'd mostly looked at circuits with NPN structures, but this circuit uses a PNP transistor (Bass is p-doped to have acceptors and the Emitter and Collector are n-doped to have donors). So, the voltages are flipped, and current flows in the opposite direction; those are the only practical differences.
  - **ΔV_{EB} Not Constant.** Since this relies on a property of transistor's that we've approximated away every single time we've analyzed transistor circuits, it's worth reviewing a tad. Up to this point, it's been appropriate to approximate the voltage difference between a bi-polar transistor's Base and Emitter as a constant ~ 0.6V. A plot of that would be a boring flat line – regardless of I, V was 0.6Volts. Of course, that's a simplification. Actually, the current-voltage relation's more of an exponential.



  - Except for when passing miniscule currents, the Emitter and Bass have a voltage difference very close to 0.6V. Thus we've typically approximated it as such. However, there is a low-current / low-voltage range for which voltage changes significantly with current, and vice-versa.

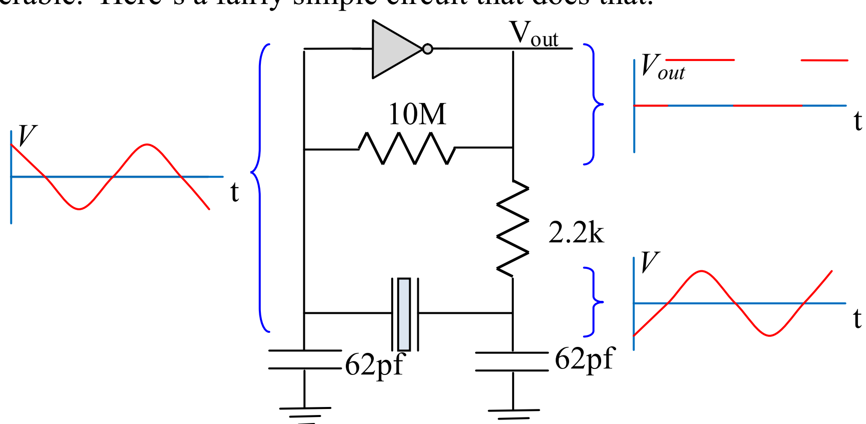
▪ **Circuit Analysis.** With this behavior in mind, let's return to the Pierce Crystal Oscillator circuit and figure out how it works.

- **Superficial.** Without being too rigorous, since the Piezo naturally oscillates, ΔV_{EB} oscillates, which means that I_B oscillates; of course, $I_C = \beta I_B$, so that means that I_C oscillates (and it does so with much greater amplitude than I_B since β>>1), which means that V_C oscillates, and thus both V_{out} and V_{EB} oscillates, which returns us to the beginning of this chain of reasoning. The resulting sine-wave signal's frequency can be controlled to very high precision, say10^{-5} to 10^{-7} fractional error. That means a watch timed by a quartz crystal might get off by 1 second every day or every month.

- **Rigorous**. A more rigorous analysis would use phasors and the piezo's RCL equivalent (on page 5 of these notes) to model the piezo's role. It would also have to model the emitter current's exponential dependence on the voltage between it and the base. That doesn't sound like much fun, but the basic pieces are then:
  - $v_c - V_{cc} = i_1 R_1$
  - $v_B - v_c = i_2 R_2$
  - $i_1 - i_2 = i_c$
  - $i_2 = i_B + i_p$
  - $i_E = i_o \left(e^{eV_{E-B}/kT} - 1\right)$
  - $i_c = i_E - i_B = \beta i_B$
  - $v_B = i_p Z_p$
  - $Z_p = \dfrac{1}{\frac{1}{Z_{C_1}} + \frac{1}{Z_{C2}+Z_R+Z_L}}$

Crystal Oscillator Square Wave Circuit

- The Pierce Crystal Oscillator makes nice sine-waves, but often a square wave is preferable. Here's a fairly simple circuit that does that.



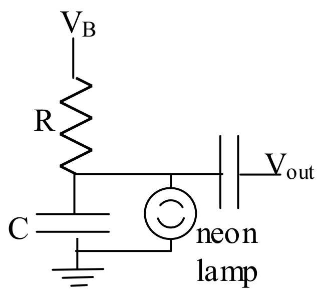
- This is similar to a Sine-Wave / Square-Wave converter with the Sine-Wave generator built in. While the crystal oscillates sinusoidally, the inverter puts out a square wave, and (getting oomf from the power line) drives the crystal kind of like if you had a steel mass on a spring and each time it crossed the center you flipped a magnet below it – driving it up & down & up…

10-4 Relaxation Oscillators

Many oscillators have some aspect of charging to a threshold and then 'relaxing' or discharging. Backing out from the realm of electronics, you have lots of experience with this kind of oscillator. Think of a dripping sink – water slowly builds up at the end of the sink until there's enough that the weigh outweighs the attraction to the spout (the threshold) – then water drops (the relaxation) and a new droplet begins to form (there's actually a chaotic element to the dripping sink so you can't perfectly predict when the next drop will fall; we won't be wanting that in our electronics). We'll now consider a few electronic oscillators like this.

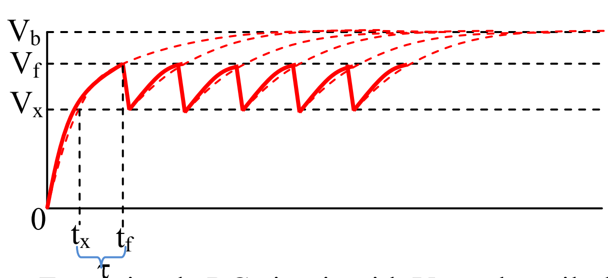
10-4.1 Neon Lamp Oscillator

- A simple strobe lamp exhibits this kind of behavior.



Sparks Background. Central to this circuit's operation is a fact that you learned back in Phys 232 – it takes a certain electric field strength to *start* a spark, but once it's initiated, the spark can be self-sustaining even at a slightly reduced field strength. Of course, the central phenomenon of a spark is the flow of ions through the air; it takes a high local field to actually ionize an atom, but once an ion is generated, it can, with a lower field, be accelerated enough so that it ionizes another atom upon collision.

Turning on and off the lamp. Now, a Neon lamp works by something like controlled sparking. As the capacitor on the left of the circuit charges up, so do the electrodes in the neon lamp, and thus the voltage difference between the lamp's two electrodes builds up. When it gets high enough, around 60 V for most neon lamps, there's a big enough electric field that the neon can be ionized, and so it can conduct an electrical current. That current *dis*-charges the electrodes, and so the field weakens. But that's okay; as long as there are some ions in the tube, they can be accelerated enough that they ionize the neutral atoms when they collide, so the current continues even as the plates discharge. Eventually, the field gets weak enough that the ions aren't sufficiently accelerated to smash apart more ions and so there aren't any more ions and the lamp stops passing current. So, the capacitor and the two electrodes start charging up again. The process repeats itself, over and over.



For a simple RC circuit with V_b as the rail, the voltage across the capacitor is a function of time,

$$V(t) = V_b \left(1 - e^{-t/(RC)}\right)$$

So, it would charge from 0 to V_x in time t_x,

$$t_x = -RC\ln\left(1-\frac{V_x}{V_b}\right) = -RC\ln\left(\frac{V_b - V_x}{V_b}\right) = RC\ln\left(\frac{V_b}{V_b - V_x}\right)$$

Similarly, it would take time t_f to charge all the way to V_f from 0 Volts.

$$t_f = RC\ln\left(\frac{V_b}{V_b - V_f}\right)$$

Then the time it takes to charge from V_x to V_f is just

$$\tau = t_f - t_x = RC\ln\left(\frac{V_b}{V_b - V_f}\right) - RC\ln\left(\frac{V_b}{V_b - V_x}\right) = RC\ln\left(\left(\frac{V_b - V_x}{V_b - V_f}\right)\right)$$

Now, to the extent that this charge-up time is *much* longer than the charge-down time, this is roughly the total time for one cycle. Then the *frequency* is roughly one over this.

$$f \approx \frac{1}{\tau} = \frac{1}{RC\ln\left(\frac{V_b - V_x}{V_b - V_f}\right)}$$

Obviously, this approximation breaks down when the RC time constant is quite small, so the charge time is about as short as the discharge time. Then we'd need to think of a way to quantify that and add it into our expressions for the period and the frequency.