Wed., 3/25 Thurs., 3/26 Fri., 3/27	**21.7-9** Maxwell's, Gauss's, and Ampere's **Quiz** Ch 21, **Lab 9** Ampere's Law **22.1-2,10** Intro to Faraday's & Lenz's	**RE25** **RE26**
Mon., 3/30 Tues., 3/31	**22.3-4,.7** Faraday & Emf & Inductance	**RE27** **Lab 9 Write-up HW21:**RQ.12, 15,17; P.20, 23, 26

Handouts:

- Equation Sheet
- Lab

Practice with Gauss and Ampere.

See ppt.

Integral vs. Differential formulation

Whether I say "my brother's *older* than I am" or "I'm *younger* than my brother", I'm communicating the exact same relationship. Similarly

$$\int f(x)dx = F(x) \text{ and } \frac{d}{dx}F(x) = f(x)$$

Communicate the exact same relationship. If one is true, the other's true.

In this chapter, we've developed the following four *integral* relations:

Maxwell's Equations (so far):

$$\oint \vec{E}\cdot\hat{n}\,dA = \frac{\sum q_{\text{inside}}}{e_0} \qquad \text{Gauss's law for electricity}$$

$$\oint \vec{B}\cdot\hat{n}\,dA = 0 \qquad \text{Gauss's law for magnetism}$$

$$\oint \vec{E}\cdot d\vec{\ell} = 0 \qquad \text{incomplete (will be Faraday's law)}$$

$$\oint \vec{B}\cdot d\vec{\ell} = m_0 \sum I_{\text{inside path}} \qquad \text{Ampere's law (incomplete)}$$

The first two are surface integrals and the last two are path (line) integrals. The last two equations are incomplete and we will modify them in Ch. 22.

Now, we're going to invert them and see what the *derivative* forms look like. There's *no* new information in the new forms, so why bother?

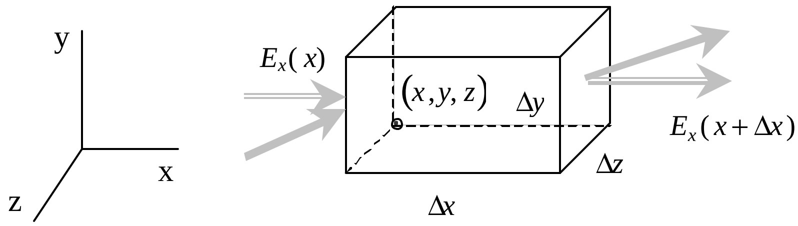
The same reason we bothered to rephrase the information in the momentum relation in the form of the energy relation: we'll get a tool that's easier to use, both mathematically and conceptually, in specific situations.

In point of fact, we *won't* be using these new tools much in *this* course, but they are part of the physics cannon (right up there with the momentum and energy principles) and you will get more familiar with their use in our advanced courses.

See PowerPoint

(1) Gauss's law

Consider a small box with edges along the coordinate axes.



Calculate the electric flux per volume in the limit that the volume goes to zero, which is the divergence of $\vec{E}$:

Divergence

- **Motivation. Return to Rain**
    - Recall that the general idea of a "flux" is a flow rate: the charge flux down a wire, dq/dt, is the current. Similarly, in the example of rain that we used to motivate the definition of flux, the rate at which water enters a room through some open windows would be a flux. $\Phi_w = \frac{dm_w}{dt} = \oint \rho_w \vec{v}_w \cdot d\vec{A}$
    - **Normalizing per Volume.** Now, if I told you that 1 kg of water rained in per minute, you'd be pretty worried – until I told you that the room was the Superdome- that volume's *huge.* 1 kg / minute leak isn't so bad as if we were talking about, say , *this* room. This example illustrates that *flux* alone doesn't tell the whole story. Sometimes you're more interested in flux *per volume.* On a per volume basis, the same flux into the Superdome is *nothing* compared to that into this room. Flux out per volume is "Divergence." (Conversely, I suppose we'd call Flux *in* per volume "Convergence"=–Divergence)

$$div \equiv \frac{\Phi_w}{Vol} = \frac{dm_w}{dt}\Big/ Vol = \frac{\oint \rho_w \vec{v}_w \cdot d\vec{A}}{Vol}$$

Math.

Now for a little math.

$$\operatorname{div}\left(\vec{E}\right) = \lim_{\Delta V \to 0} \frac{\oint \vec{E}\cdot\hat{n}\,dA}{\Delta V}$$

$$= \lim_{\Delta V \to 0} \left\{ \frac{\left[E_x(x+\Delta x) - E_x(x)\right]\Delta y \Delta z}{\Delta x \Delta y \Delta z} + \frac{\left[E_y(y+\Delta y) - E_y(y)\right]\Delta x \Delta z}{\Delta x \Delta y \Delta z} + \frac{\left[E_z(z+\Delta z) - E_z(z)\right]\Delta x \Delta y}{\Delta x \Delta y \Delta z} \right\}$$

$$= \lim_{\Delta x \to 0} \frac{\left[E_x(x+\Delta x) - E_x(x)\right]}{\Delta x} + \lim_{\Delta y \to 0} \frac{\left[E_y(y+\Delta y) - E_y(y)\right]}{\Delta y} + \lim_{\Delta z \to 0} \frac{\left[E_z(z+\Delta z) - E_z(z)\right]}{\Delta z}$$

$$= \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}$$

The other side of Gauss's law over the volume in the limit that the volume goes to zero is:

$$\lim_{\Delta V \to 0} \left(\frac{1}{\varepsilon_0} \frac{\sum q_{\text{inside}}}{\Delta V} \right) = \frac{\rho}{\varepsilon_0},$$

where ρ is the charge density. The differential form of Gauss's law is:

$$\operatorname{div}\left(\vec{E}\right) = \vec{\nabla}\cdot\vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\varepsilon_0}$$

Note that this is a scalar equation. In the second form, the "del" operator is $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.

Relativistic

- The book argues that divergence is inherently relativistically correct. The argument goes something like this.
  - **Must handle moving sources.**
    - To be relativistically correct means to be able to correctly handle *moving* charges. So, we must argue that it handles them correctly. The difficulty with moving charges is that it takes time t = R/c for field to emanate from a charge to a point a distance R away. In the mean time, the charge has moved a distance r = vt = Rv/c where v < c.
    - **Coulomb's Law's failure (no retardation time).** Coulomb's Law's problem is that it tries to say what the field is a distance R away is at a given instant based on where the charge is at that same instant (*without* allowing for the fact that that isn't where the charge was when the field was emanated).
      - **Can *generalize* Coulomb's Law:** applying Coulomb's law in the frame in which the source is stationary, and transforming the expression to the frame in which the source is moving, you can have a relativistcally correct version.
    - You might imagine that the integral form of Gauss's Law has the same problem – we'll return to that objection.

- **Divergance's work-around (no need for retardation time).** To find the divergence, we imagine shrinking the volume / area / R down to zero. In the process, we shrink t = R/c to zero and r = vt to zero. So, the divergence talks about the field *at* the source and how it depends on the source at that same time. Since it takes no time for the field to emanate, the source hasn't moved, so the divergence relation holds regardless of whether the charge is moving. Voila – the divergence relation is inherently insensitive to charge motion – relativistically correct, though perhaps it would be more appropriate to call it "relativistically agnostic."
- **Integral Form.** Now for the integral form. Since the integral form is simply the mathematical inverse of the differential form, the two forms express the exact same relationship between source and field (just like "I'm younger than my brother" and "my brother's older than I am.") So if one way of phrasing the relationship is relativistically correct, then so must be the other.

Similarly, we can get a differential version of Gauss's law for magnetism:

$$\mathrm{div}\left(\vec{B}\right)=\vec{\nabla}\cdot\vec{B}=\frac{\partial B_x}{\partial x}+\frac{\partial B_y}{\partial y}+\frac{\partial B_z}{\partial z}=0$$

Recall that the electric field is minus the gradient of the electric potential:

$$\vec{E}=-\vec{\nabla}V=-\left(\frac{\partial V}{\partial x},\frac{\partial V}{\partial y},\frac{\partial V}{\partial z}\right).$$

Gauss's law can also be written in terms of the electric potential as:

$$\vec{\nabla}\cdot\left(-\vec{\nabla}V\right)=\frac{\partial}{\partial x}\left(-\frac{\partial V}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial V}{\partial y}\right)+\frac{\partial}{\partial z}\left(-\frac{\partial V}{\partial z}\right)=\frac{\rho}{\varepsilon_0}$$

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=-\frac{\rho}{\varepsilon_0},$$

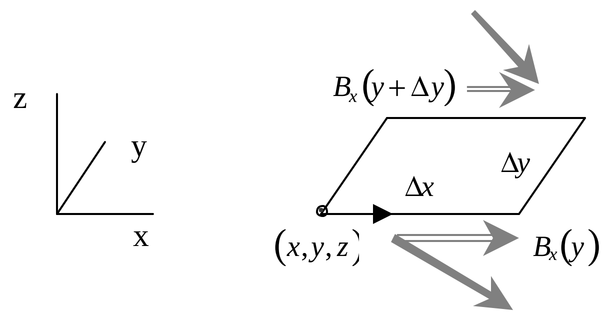
which is known as Poisson's equation and $\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}$ is the Laplacian operator. In empty space ($\rho=0$):

$$\nabla^2 V=\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}+\frac{\partial^2 V}{\partial z^2}=0,$$

which is called Laplace's equation.

(2) Ampere's law

Consider a small rectangular loop in the *xy* plane.



Calculate the path integral of the magnetic field as the area of the loop goes to zero, which is a component of the curl of $\vec{B}$:

$$\begin{aligned}\left[\mathrm{curl}\left(\vec{B}\right)\right]_z &= \lim_{\Delta A\to 0}\frac{\oint \vec{B}\cdot d\vec{\ell}}{\Delta A}\\ &= \lim_{\Delta A\to 0}\left[\frac{B_x(y)\Delta x+B_y(x+\Delta x)\Delta y-B_x(y+\Delta y)\Delta x-B_y(x)\Delta y}{\Delta x\Delta y}\right]\\ &= \lim_{\Delta y\to 0}\left[\frac{B_x(y)-B_x(y+\Delta y)}{\Delta y}\right]+\lim_{\Delta x\to 0}\left[\frac{B_y(x+\Delta x)-B_y(x)}{\Delta x}\right]\\ &= \frac{\partial B_y}{\partial x}-\frac{\partial B_x}{\partial y}\end{aligned}$$

The other side of Ampere's law in the limit that the area goes to zero is:

$$\lim_{\Delta A \to 0}\left(\mu_0 \frac{\sum I_{\text{inside,in } z \text{ direction}}}{\Delta A}\right) = \mu_0 J_z,$$

where J_z is the current density (current per area) in the z direction. If we looked at loops in the xz and yz planes, we would find that:

$$\text{curl}\left(\vec{B}\right) = \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}.$$

This is still incomplete like the integral version. Similarly, we have an incomplete expression:

$$\text{curl}\left(\vec{E}\right) = \vec{\nabla} \times \vec{E} = 0.$$

This will be non-zero if the magnetic field is changing, which is the topic of Ch. 22…

Differential forms of the equations: this is a look ahead to PHYS 332

$\text{div}\left(\vec{E}\right) = \vec{\nabla} \cdot \vec{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \frac{\rho}{\epsilon_0}$	Gauss's law for electricity
$\text{div}\left(\vec{B}\right) = \vec{\nabla} \cdot \vec{B} = \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z} = 0$	Gauss's law for magnetism
$\text{curl}\left(\vec{E}\right) = \vec{\nabla} \times \vec{E} = 0$	incomplete (will be Faraday's law)
$\text{curl}\left(\vec{B}\right) = \vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$	Ampere's law (incomplete)

In all of the above, the "del" operator (must be applied to something) is $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.