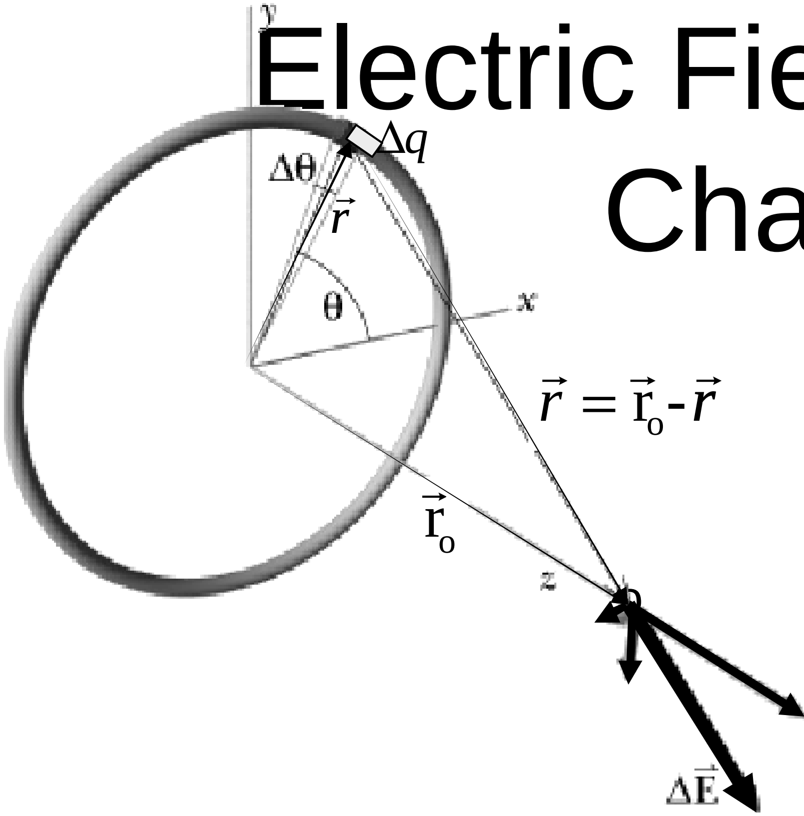
Electric Field of a Uniformly Charged Ring



note: Cylindrical Symmetry suggests Cylindrical Coordinates

Step 1: cut up charge distribution and *draw* it's contribution to the field: ΔE

Step 2: write an expression for ΔE

Step 3: Add up all ΔE's to get the total E

Step 4: Check results

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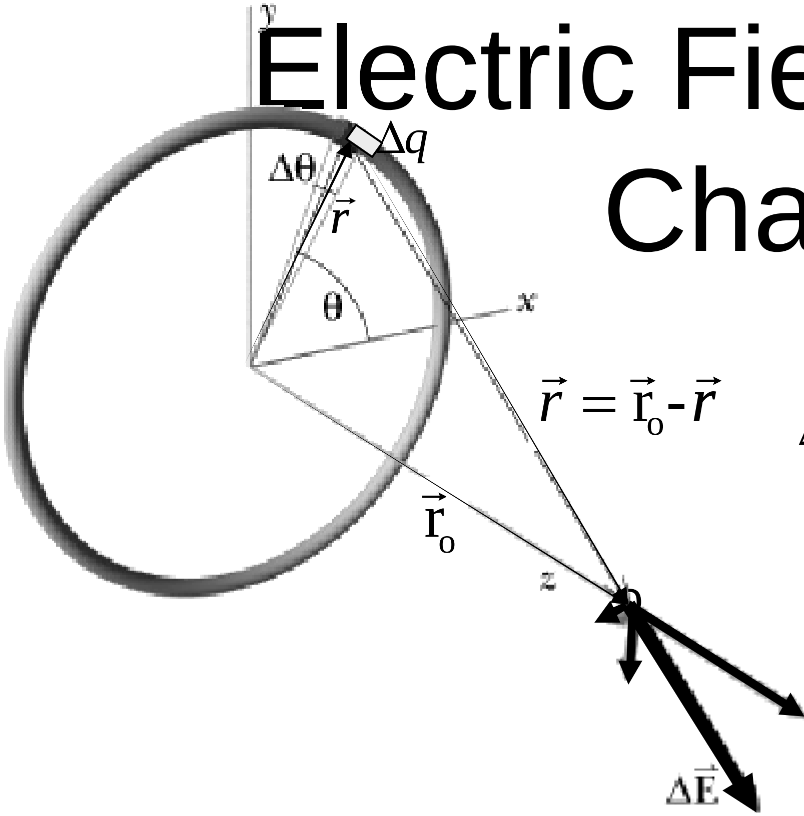

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_o}\frac{\Delta q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_o}\frac{\Delta q}{r^3}\vec{r}$$

where

$$\vec{r} = \vec{r}_o - \vec{r} = \langle 0,0,z\rangle - \langle r\cos\theta, r\sin\theta, 0\rangle$$

$$\vec{r} = \langle -r\cos\theta, -r\sin\theta, z\rangle$$

so

$$|\vec{r}| = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2 + z^2}$$

$$|\vec{r}| = \sqrt{r^2(\cos^2\theta + \sin^2\theta) + z^2}$$

$$|\vec{r}| = \sqrt{r^2 + z^2}$$

Side Note: Translate to VPython code

thus

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_o}\frac{\Delta q}{\left(r^2 + z^2\right)^{3/2}}\langle -r\cos\theta, -r\sin\theta, z\rangle$$

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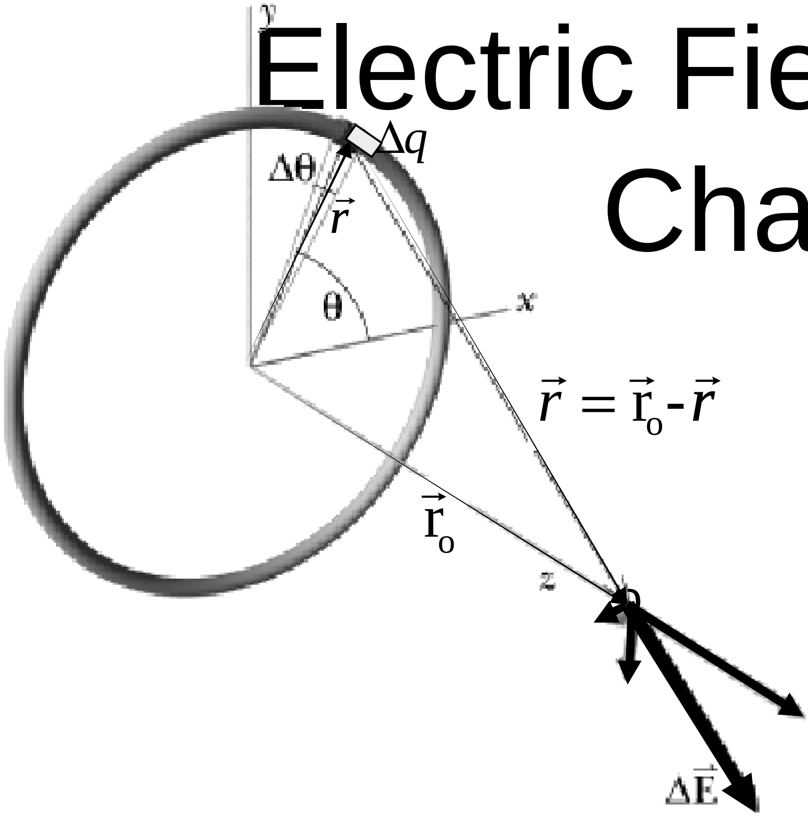

$$\vec{E} = \sum_{ring} \Delta\vec{E}$$

where

$$\Delta\vec{E} = \frac{1}{4pe_o} \frac{\Delta q}{\left(r^2 + z_o^2\right)^{3/2}} \left\langle -r\cos q, -r\sin q, z_o \right\rangle$$

so

$$\vec{E} = \sum_{q=0}^{q=2p} \frac{1}{4pe_o} \frac{\Delta q}{\left(r^2 + z_o^2\right)^{3/2}} \left\langle -r\cos q, -r\sin q, z_o \right\rangle$$

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To make an integral, need a Δθ.

$$\frac{\Delta q}{\Delta q r} = \frac{q}{2pr} \Rightarrow \Delta q = \frac{q}{2p} \Delta q$$

thus

$$\vec{E} = \sum_{q=0}^{q=2p} \frac{1}{4pe_o} \frac{\frac{q}{2p}\Delta q}{\left(r^2 + z_o^2\right)^{3/2}} \left\langle -r\cos q, -r\sin q, z_o \right\rangle$$

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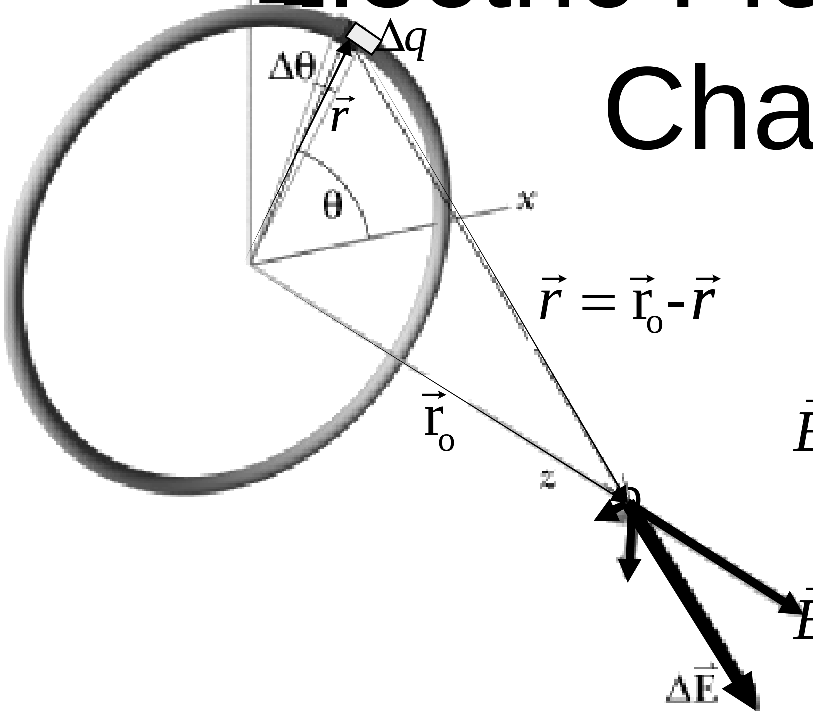

$$\vec{E} = \sum_{q=0}^{q=2p} \frac{1}{4pe_o} \frac{\frac{q}{2p}\Delta q}{\left(r^2+z^2\right)^{3/2}} \left\langle -r\cos q, -r\sin q, z \right\rangle$$

$$\vec{E} = \lim_{\Delta q \to 0} \sum_{q=0}^{q=2p} stuff\,\Delta q = \int_{q=0}^{q=2p} stuff\,dq$$

$$\vec{E} = \int_{q=0}^{q=2p} \frac{1}{4pe_o} \frac{\frac{q}{2p}}{\left(r^2+z_o^2\right)^{3/2}} \left\langle -r\cos q, -r\sin q, z_o \right\rangle dq$$

Step 1: cut up charge distribution and *draw* it's contribution to the field: ΔE

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$$\vec{E}_x = \frac{1}{4pe_o} \frac{\frac{q}{2p}(-r)}{\left(r^2+z_o^2\right)^{3/2}} \int_{q=0}^{q=2p} \cos q\,dq = 0$$

$$\vec{E}_y = \frac{1}{4pe_o} \frac{\frac{q}{2p}(-r)}{\left(r^2+z_o^2\right)^{3/2}} \int_{q=0}^{q=2p} \sin q\,dq = 0$$

$$\vec{E}_z = \frac{1}{4pe_o} \frac{\frac{q}{2p}z_o}{\left(r^2+z_o^2\right)^{3/2}} \int_{q=0}^{q=2p} dq = \frac{1}{4pe_o} \frac{\frac{q}{2p}z}{\left(r^2+z_o^2\right)^{3/2}} 2p$$

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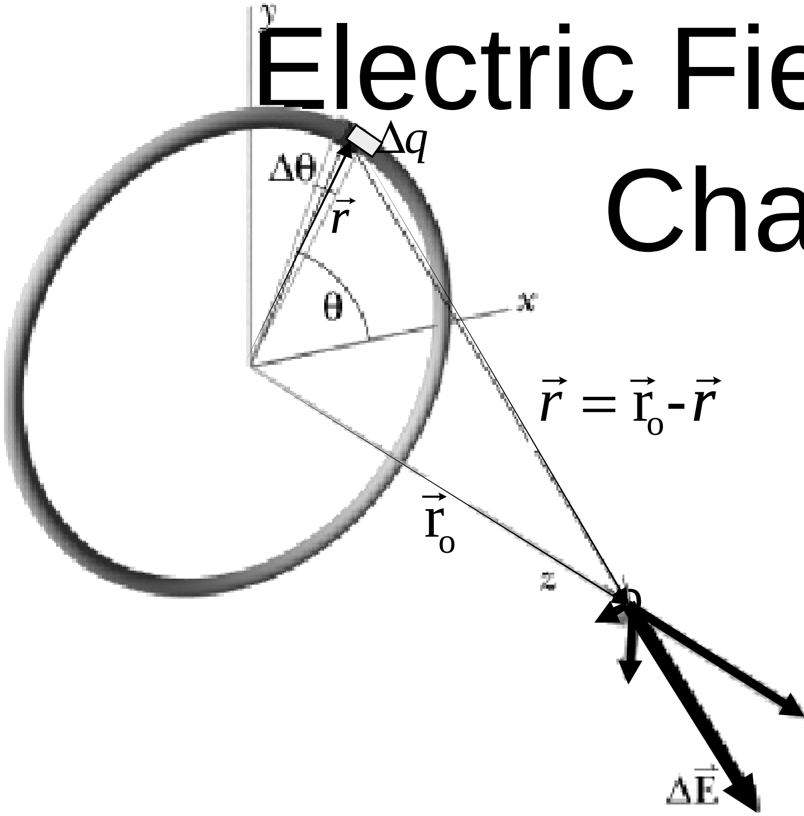

$$\vec{E}_x = 0 \qquad \vec{E}_y = 0 \qquad \vec{E}_z = \frac{1}{4pe_o}\frac{qz_o}{\left(r^2 + z_o^{\,2}\right)^{3/2}}$$

Why?

Units?

Logic?

Limits?

Step 1: cut up charge distribution and *draw* it's contribution to the field: ΔE

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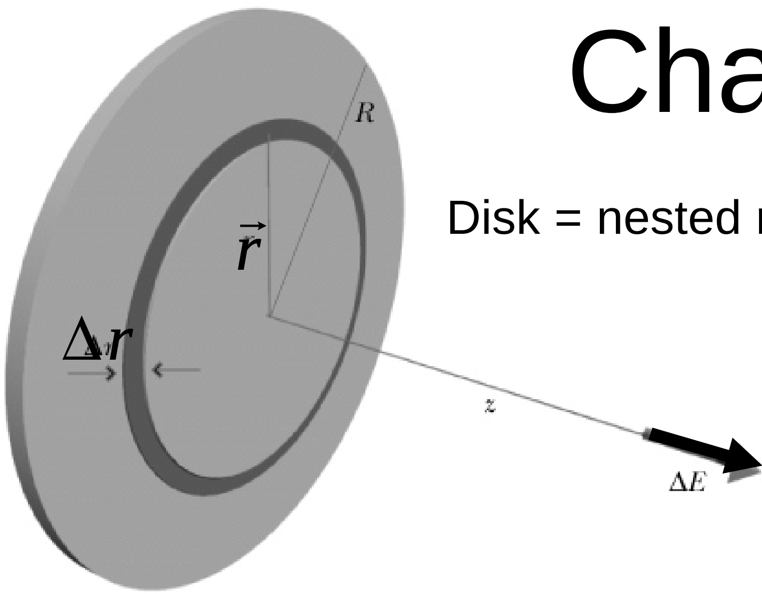
Electric Field of a Uniformly Charged Disk



Disk = nested rings

$$\Delta E_z = \frac{1}{4pe_0}\frac{q_{ring} z_o}{\left(r^2 + z_o^{\ 2}\right)^{3/2}}$$

where

$$q_{ring} = Q\frac{(\text{area of ring})}{(\text{area of disk})} = Q\frac{2pr\Delta r}{pR^2}$$

Step 1: cut up charge distribution and *draw* it's contribution to the field: ΔE

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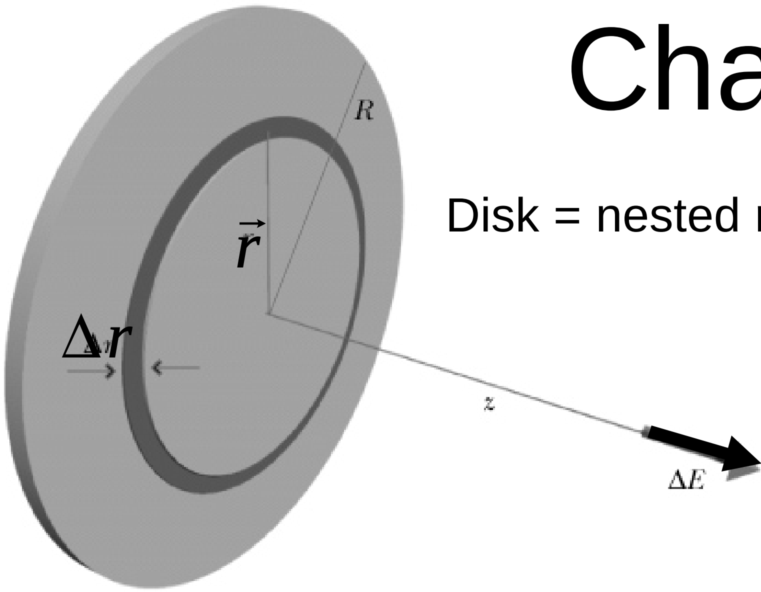
Step 4: Check results

so

$$\Delta E_z = \frac{1}{4pe_0}\frac{\left(Q\frac{2pr\Delta r}{pR^2}\right)z_o}{\left(r^2 + z_o^{\ 2}\right)^{3/2}}$$

Electric Field of a Uniformly Charged Disk



Disk = nested rings

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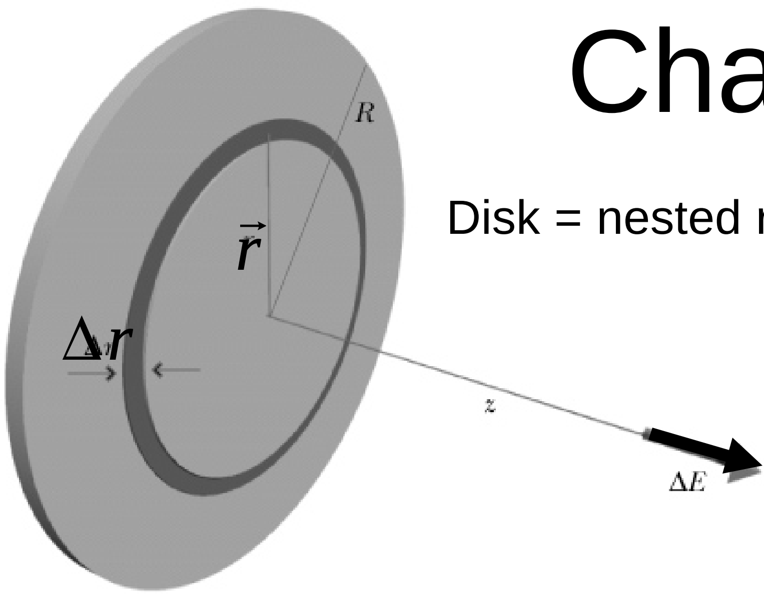
so

$$\Delta E_z = \frac{1}{4pe_0}\frac{\left(Q\frac{2pr\Delta r}{pR^2}\right)z_o}{\left(r^2 + z_o^{\ 2}\right)^{3/2}}$$

$$\Delta E_z = \frac{1}{2e_0}\frac{Q}{pR^2}\frac{z_o r\Delta r}{\left(r^2 + z_o^{\ 2}\right)^{3/2}}$$

Electric Field of a Uniformly Charged Disk



Disk = nested rings $$E_z = \sum_{disk} \Delta E_z$$

where

$$\Delta E_z = \frac{1}{2e_0} \frac{Q}{pR^2} \frac{z_o r \Delta r}{\left(r^2 + z_o^2\right)^{3/2}}$$

so

$$E_z = \sum_{r=0}^{r=R} \frac{1}{2e_0} \frac{Q}{pR^2} \frac{z_o r \Delta r}{\left(r^2 + z_o^2\right)^{3/2}}$$

$$E_z = \frac{1}{2e_0} \frac{Qz_o}{pR^2} \int_{r=0}^{r=R} \frac{r\,dr}{\left(r^2 + z_o^2\right)^{3/2}}$$

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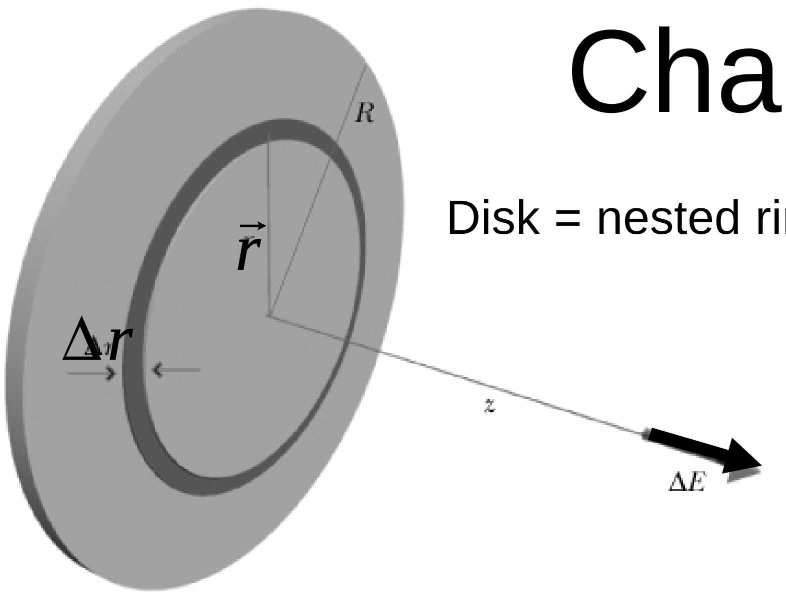

Disk = nested rings

$$E_z = \frac{1}{2\epsilon_0}\frac{Qz_o}{\pi R^2}\int_{r=0}^{r=R}\frac{r\,dr}{\left(r^2+z_o^2\right)^{3/2}}$$

Change of variables

$$u \equiv r^2 + z_o^2$$

So limits become

$$u_{\min} = z_o^2 \qquad u_{\max} = R^2 + z_o^2$$

Differential bit becomes

$$du \equiv 2r\,dr \Rightarrow r\,dr = \tfrac{1}{2}du$$

Integral becomes

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$$E_z = \frac{1}{4\epsilon_0}\frac{Qz_o}{\pi R^2}\int_{u=z_o^2}^{u=R^2+z_o^2}\frac{du}{u^{3/2}} = \frac{1}{4\epsilon_0}\frac{Qz_o}{\pi R^2}\left(\frac{-2}{u^{1/2}}\right)\Bigg|_{u=z_o^2}^{u=R^2+z_o^2}$$

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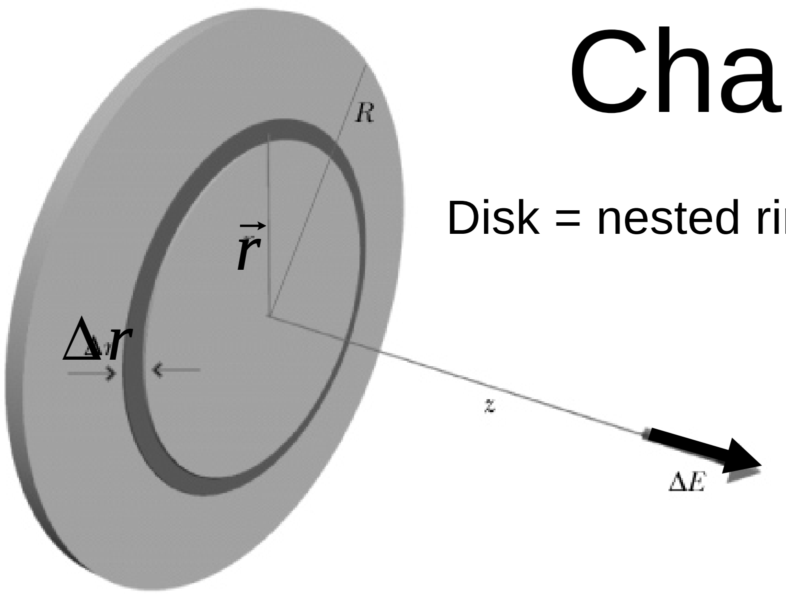

Disk = nested rings

$$E_z = \frac{1}{2\epsilon_0}\frac{Qz_o}{\pi R^2}\left(\frac{1}{z_o} - \frac{1}{\left(R^2 + z_o{}^2\right)^{1/2}}\right)$$

Units?

Logic?

Limits?

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