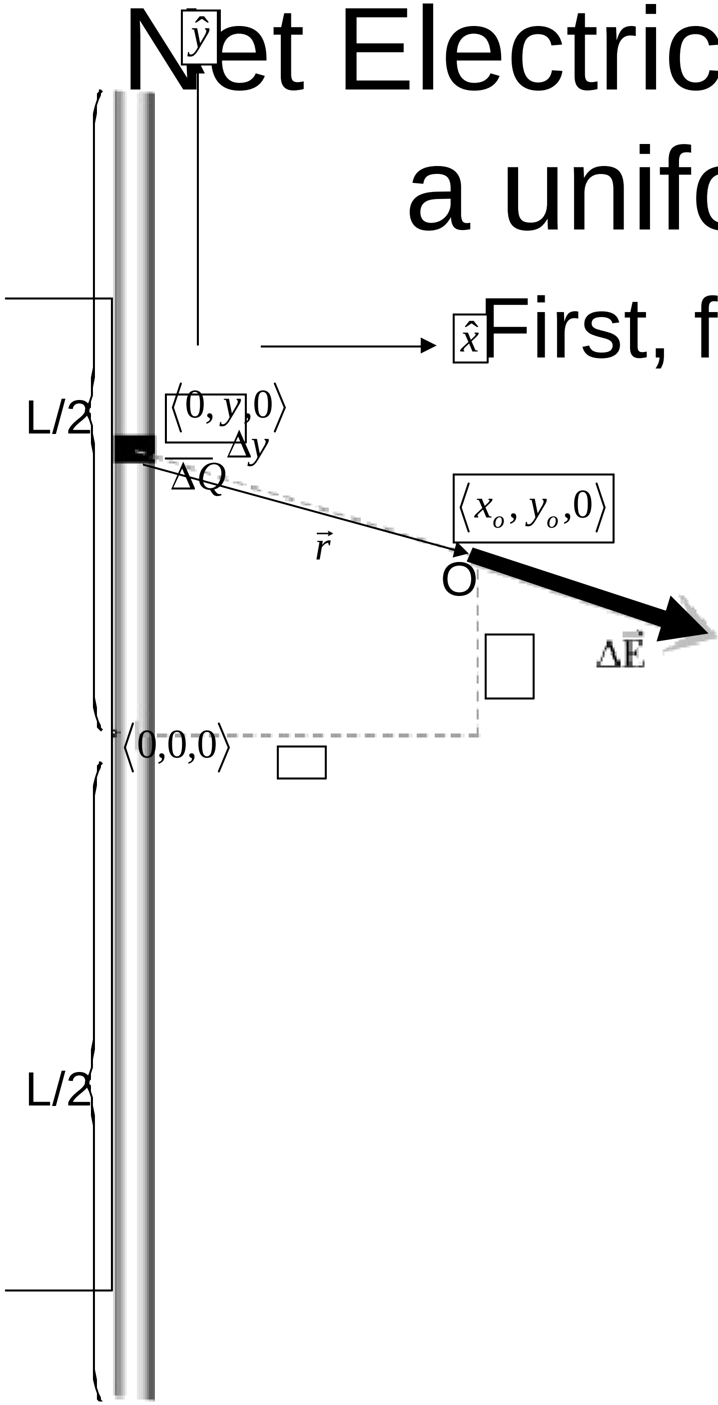
Net Electric Field at point O due to a uniformly charged thin Rod



First, find field at O due to a morsel of the rod, ΔE.

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\Delta Q}{r^2}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{\Delta Q}{r^2}\frac{\vec{r}}{|r|} = \frac{1}{4\pi\varepsilon_0}\frac{\Delta Q}{r^3}\vec{r}$$

where

$$\vec{r} = \langle \text{observation location}\rangle - \langle \text{charge location}\rangle$$

$$= \langle x_o, y_0, 0\rangle - \langle 0, y, 0\rangle = \langle x_o, (y_0 - y), 0\rangle$$

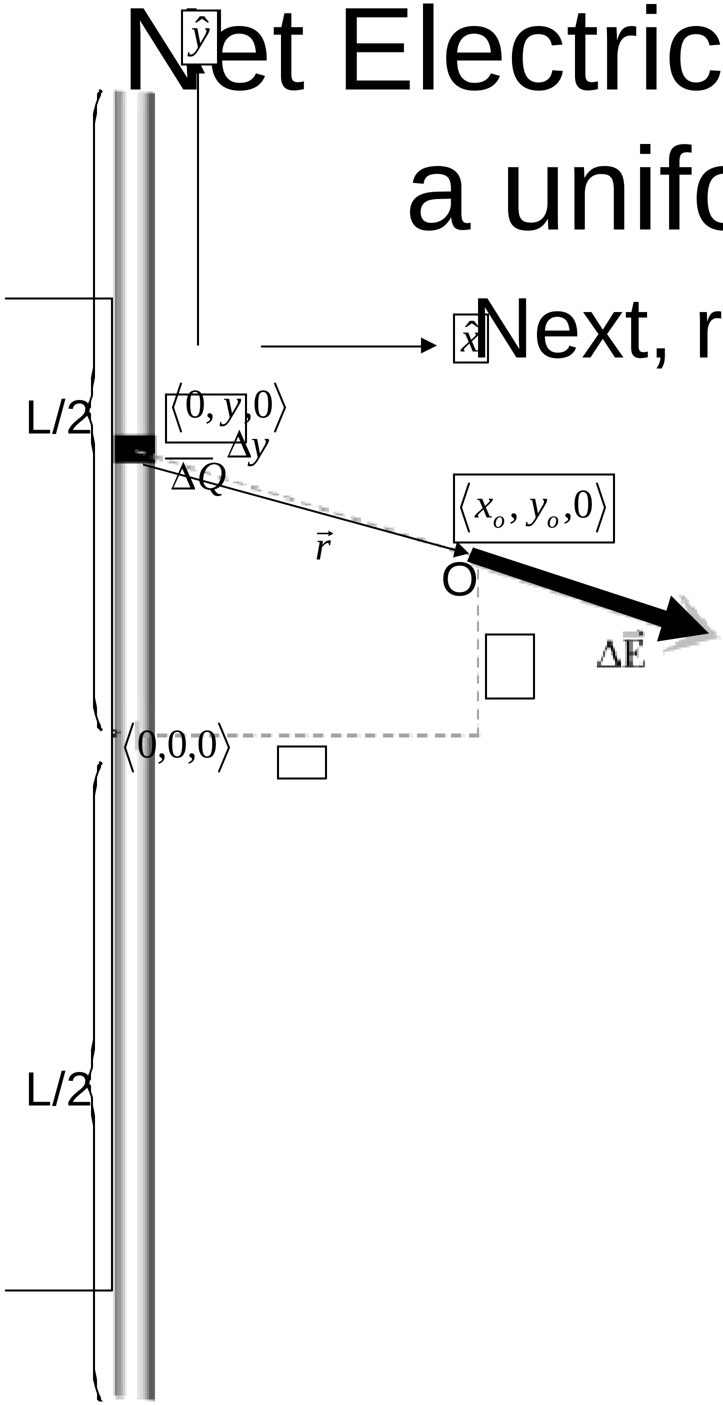
so,

$$|r| = \left[x_o^{\,2} + (y_0 - y)^2\right]^{1/2}$$

Thus

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{\Delta Q}{\left[x_o^{\,2} + (y_0 - y)^2\right]^{3/2}}\langle x_o, (y_0 - y), 0\rangle$$

Net Electric Field at point O due to a uniformly charged thin Rod



- Next, rephrase in terms of the breadth of the morsel, y.

$$\Delta\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{\Delta Q}{\left[x_o^{\ 2} + (y_0 - y)^2\right]^{3/2}}\langle x_o, (y_0 - y), 0\rangle$$

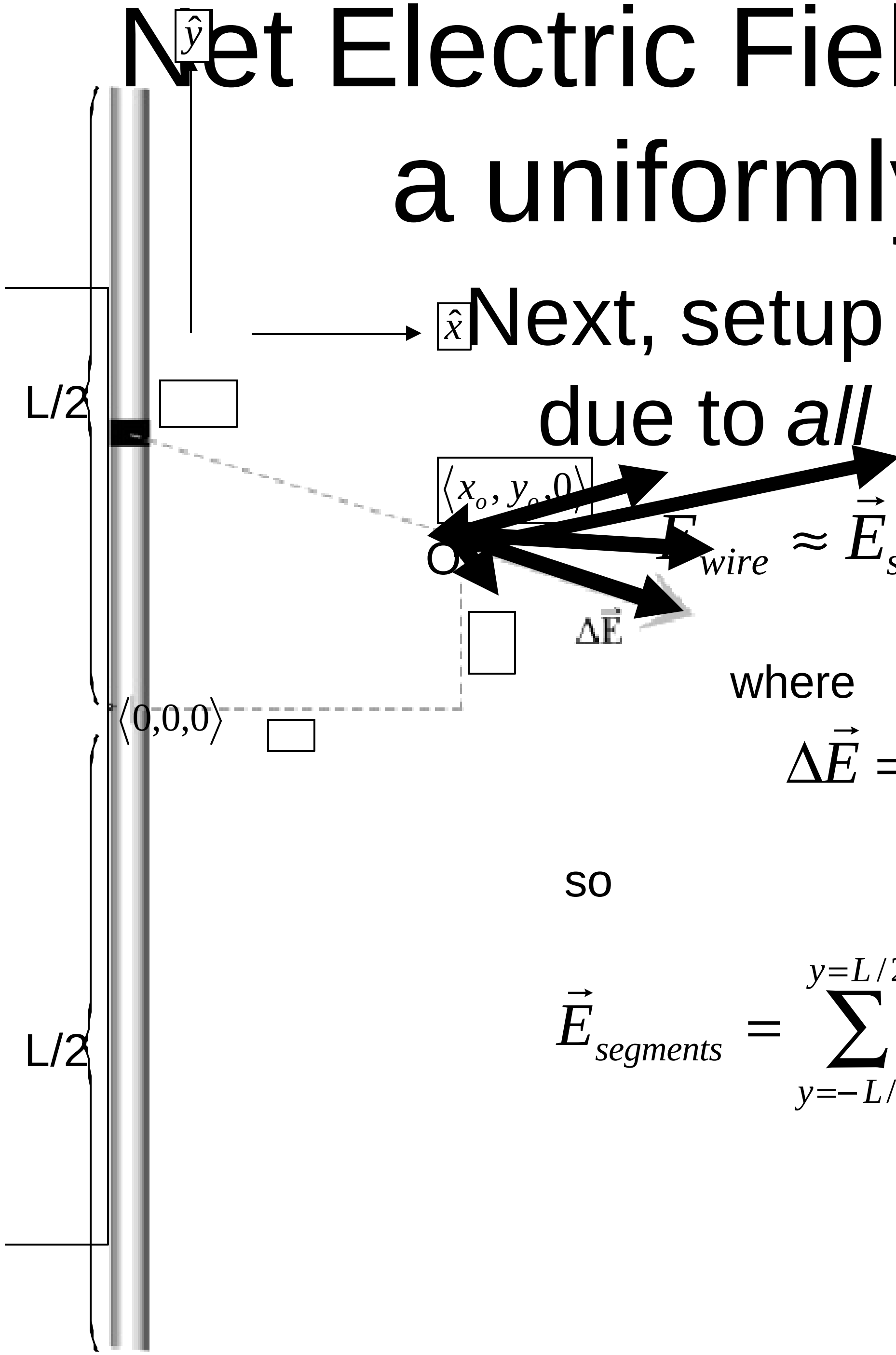
where

$$\frac{\Delta Q}{\Delta y} = \frac{Q}{L} \Rightarrow \Delta Q = \frac{Q}{L}\Delta y$$

so,

$$\Delta\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{(Q/L)\Delta y}{\left[x_o^{\ 2} + (y_0 - y)^2\right]^{3/2}}\langle x_o, (y_0 - y), 0\rangle$$

Net Electric Field at point O due to a uniformly charged thin Rod



- Next, setup sum over bits of field at O due to *all* morsels (all along wire).

$$\vec{E}_{wire} \approx \vec{E}_{segments} = \sum_{y=-L/2}^{y=L/2} \Delta\vec{E}$$

where

$$\Delta\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{(Q/L)\Delta y}{\left[x_o^{\,2} + (y_0 - y)^2\right]^{3/2}}\langle x_o, (y_0 - y), 0\rangle$$

so

$$\vec{E}_{segments} = \sum_{y=-L/2}^{y=L/2} \frac{1}{4\pi\varepsilon_0}\frac{(Q/L)\Delta y}{\left[x_o^{\,2} + (y_0 - y)^2\right]^{3/2}}\langle x_o, (y_0 - y), 0\rangle$$

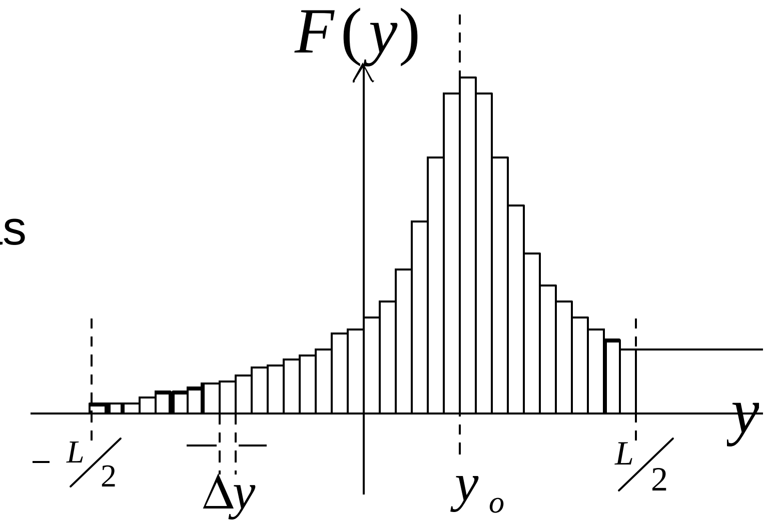
Math-land digression: What the Sum means

$$E_{segments.x} = \sum_{y=-L/2}^{y=L/2} \frac{1}{4pe_0} \frac{(Q/L)\Delta y}{\left[x_o^{\ 2} + (y_0 - y)^2\right]^{3/2}} x_o$$

$$E_{segments.x} = \sum_{y=-L/2}^{y=L/2} F(y)\Delta y$$ = sum of rectangles' areas

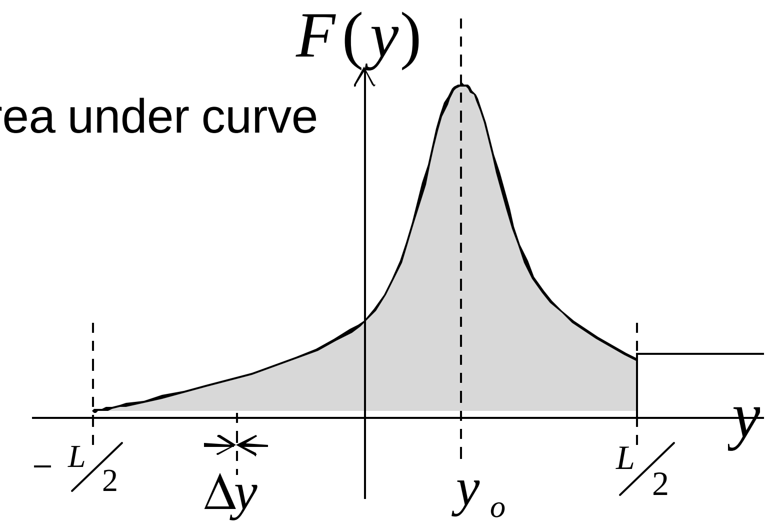
where

$$F(y) = \frac{1}{4pe_0} \frac{(Q/L)x_o}{\left[x_o^{\ 2} + (y_0 - y)^2\right]^{3/2}}$$

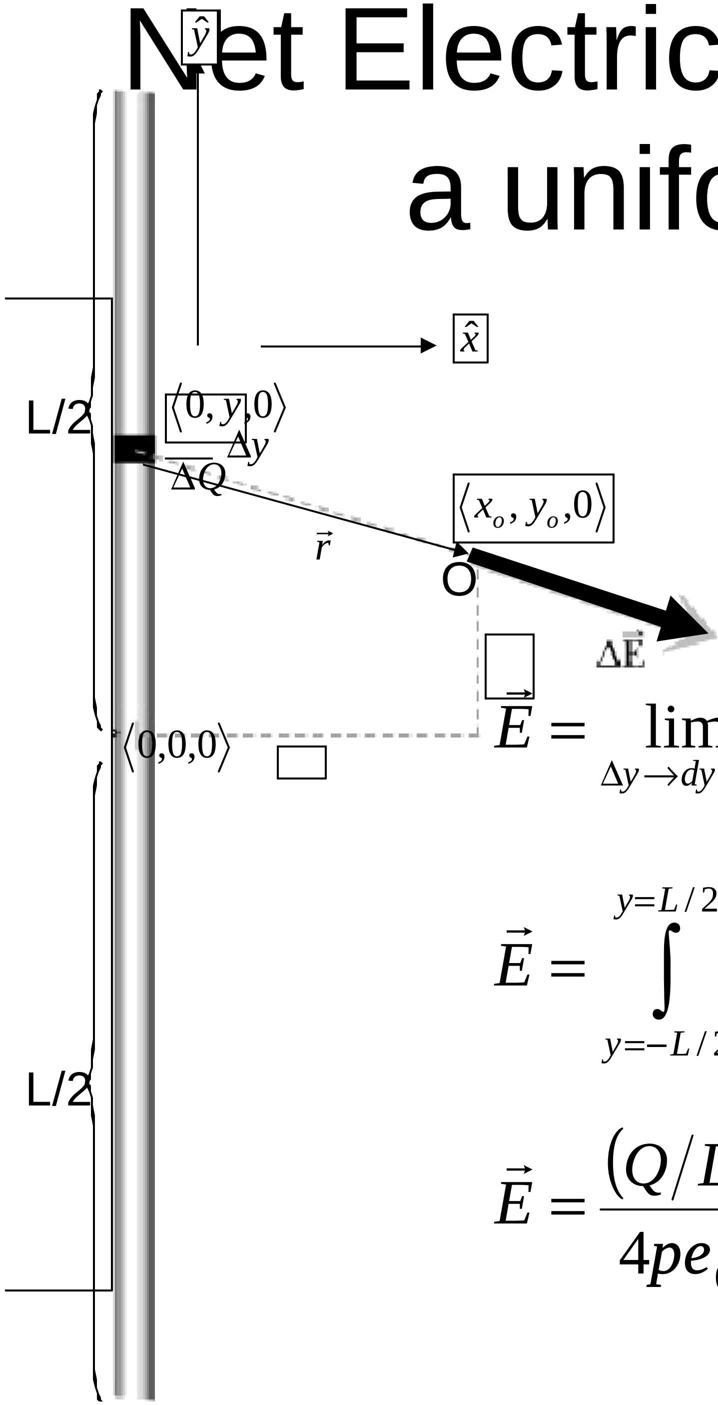


$$E_{wire.x} = \lim_{\Delta y \to 0} E_{segmetns.x} = \lim_{\Delta y \to 0} \left(\sum_{y=-L/2}^{y=L/2} F(y)\Delta y \right)$$ = area under curve

$$\boxed{E_{wire.x} = \int_{y=-L/2}^{y=L/2} F(y)\,dy}$$



Net Electric Field at point O due to a uniformly charged thin Rod



Take Differential Limit:

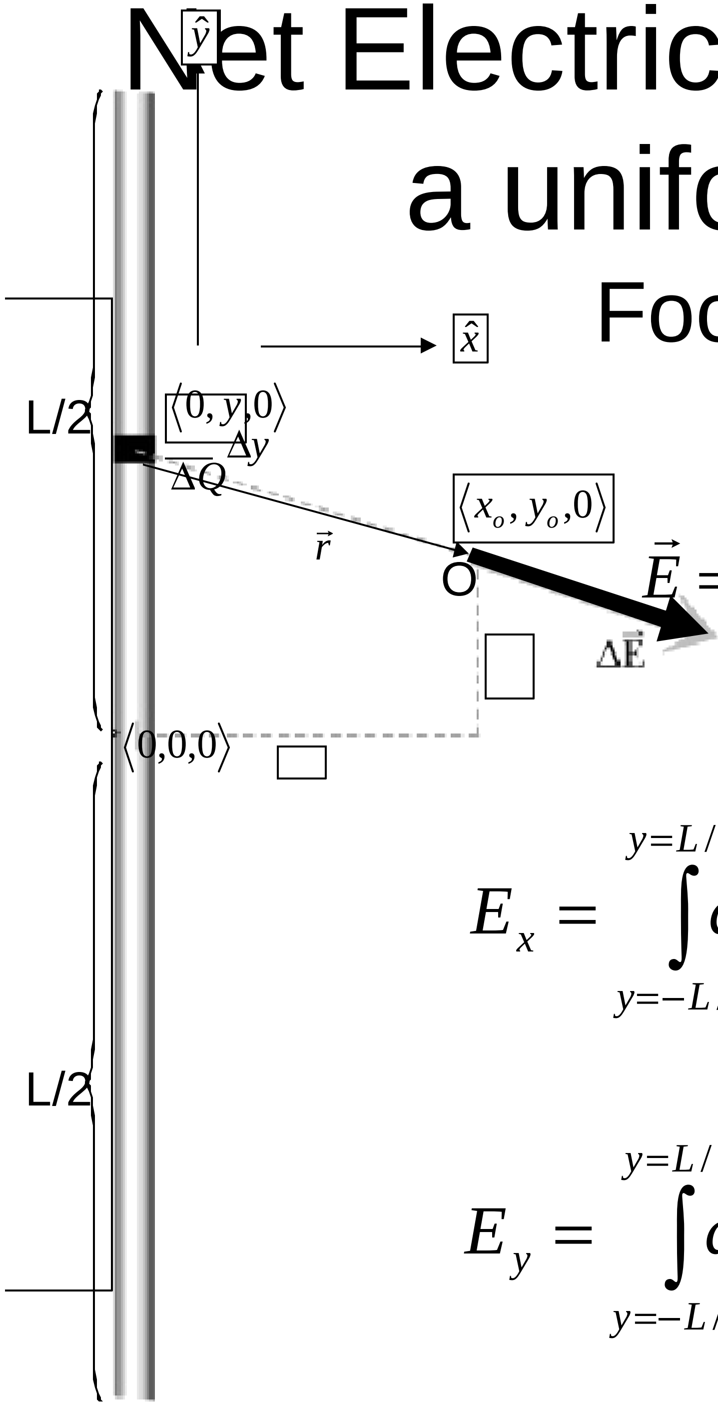
sum becomes integral

$$\vec{E} = \lim_{\Delta y \to dy \to 0} \left(\sum_{y=-L/2}^{y=L/2} \frac{1}{4pe_0} \frac{(Q/L)\Delta y}{\left[x_o^{\ 2} + (y_0 - y)^2\right]^{3/2}} \langle x_o, (y_0 - y), 0\rangle \right)$$

$$\vec{E} = \int_{y=-L/2}^{y=L/2} \left(\frac{1}{4pe_0} \frac{(Q/L)dy}{\left[x_o^{\ 2} + (y_0 - y)^2\right]^{3/2}} \langle x_o, (y_0 - y), 0\rangle \right)$$

$$\vec{E} = \frac{(Q/L)}{4pe_0} \int_{y=-L/2}^{y=L/2} \left(\frac{dy}{\left[x_o^{\ 2} + (y_0 - y)^2\right]^{3/2}} \langle x_o, (y_0 - y), 0\rangle \right)$$

Net Electric Field at point O due to a uniformly charged thin Rod



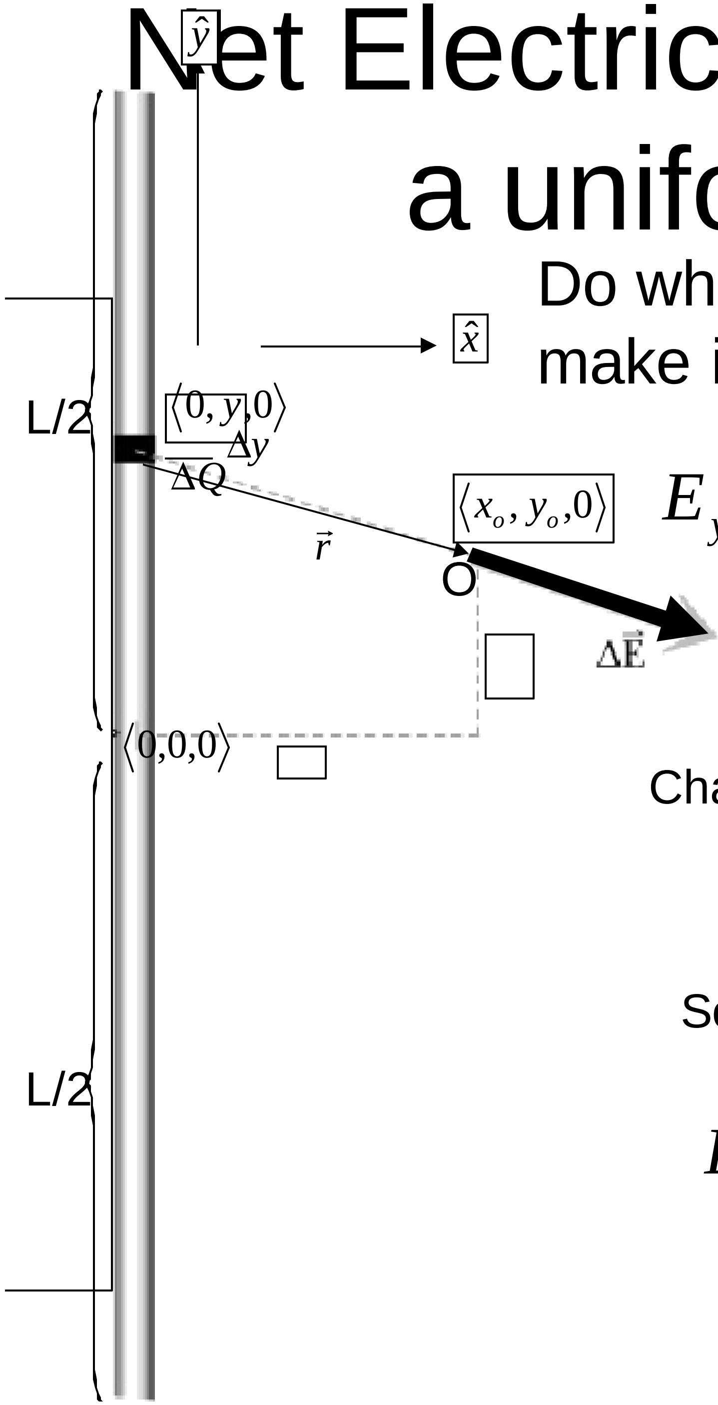
Focus on x and y components individually

$$\vec{E} = \frac{(Q/L)}{4pe_0} \int_{y=-L/2}^{y=L/2} \left(\frac{dy}{\left[x_o^{\,2} + (y_0 - y)^2\right]^{3/2}} \langle x_o, (y_0 - y), 0 \rangle \right)$$

$$E_x = \int_{y=-L/2}^{y=L/2} dE_x = \frac{(Q/L)x_o}{4pe_0} \int_{y=-L/2}^{y=L/2} \frac{dy}{\left[x_o^{\,2} + (y_0 - y)^2\right]^{3/2}}$$

$$E_y = \int_{y=-L/2}^{y=L/2} dE_y = \frac{(Q/L)}{4pe_0} \int_{y=-L/2}^{y=L/2} \frac{(y_0 - y)dy}{\left[x_o^{\,2} + (y_0 - y)^2\right]^{3/2}}$$

Net Electric Field at point O due to a uniformly charged thin Rod



Do what you must to either *do* the integral, or make it recognizable as one that's done for you.

$$E_y = \int_{y=-L/2}^{y=L/2} dE_y = \frac{(Q/L)}{4pe_0} \int_{y=-L/2}^{y=L/2} \frac{(y_0 - y)dy}{\left[x_o^{\ 2} + (y_0 - y)^2\right]^{3/2}}$$

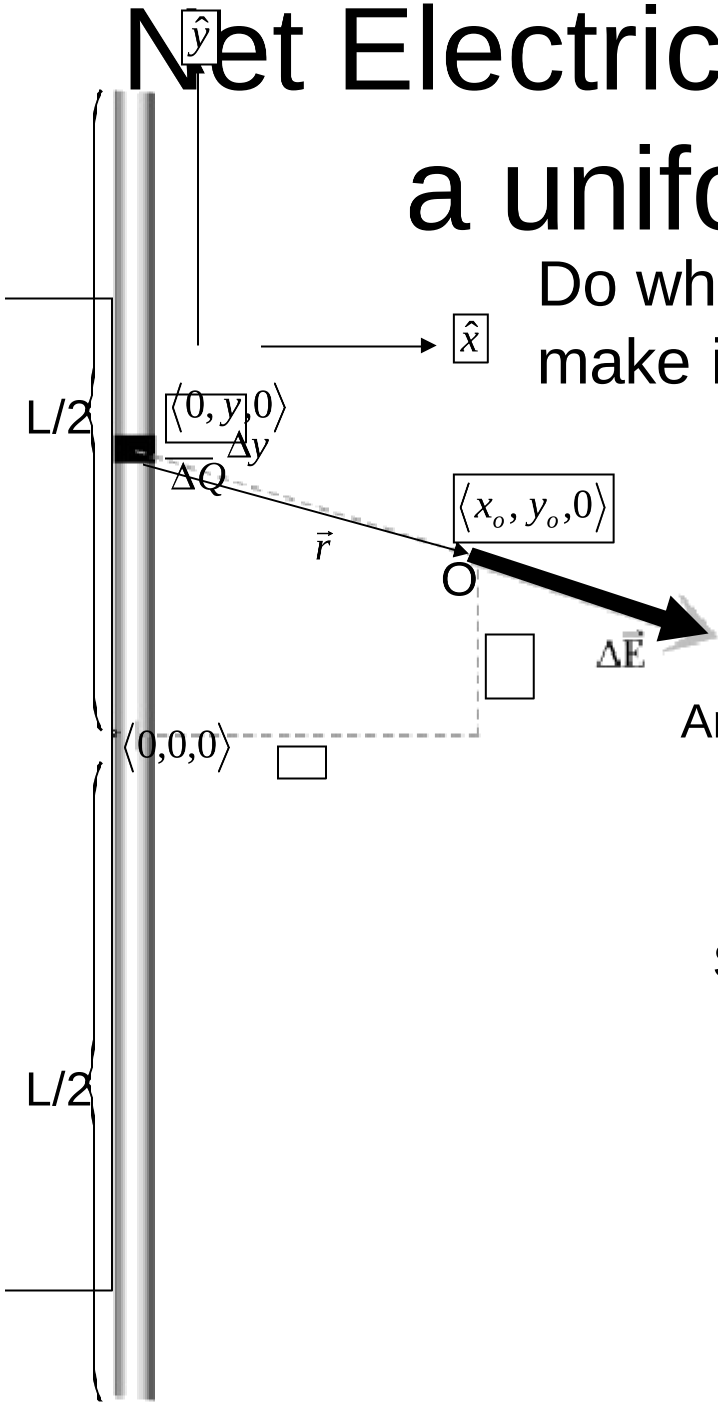
Change of variables:

$$\tilde{y} \equiv (y_o - y) \Rightarrow y = y_o - \tilde{y} \quad \text{and} \quad dy = -d\tilde{y}$$

So,

$$E_y = \frac{(Q/L)}{4pe_0} \int_{\tilde{y}=y_o+L/2}^{\tilde{y}=y_o-L/2} \frac{-\tilde{y}d\tilde{y}}{\left[x_o^{\ 2} + \tilde{y}^2\right]^{3/2}}$$

Net Electric Field at point O due to a uniformly charged thin Rod



Do what you must to either *do* the integral, or make it recognizable as one that's done for you.

$$E_y = \frac{(Q/L)}{4pe_0}\int_{\tilde{y}=y_o-L/2}^{\tilde{y}=y_o+L/2}\frac{\tilde{y}\,d\tilde{y}}{\left[x_o{}^2+\tilde{y}^2\right]^{3/2}}$$

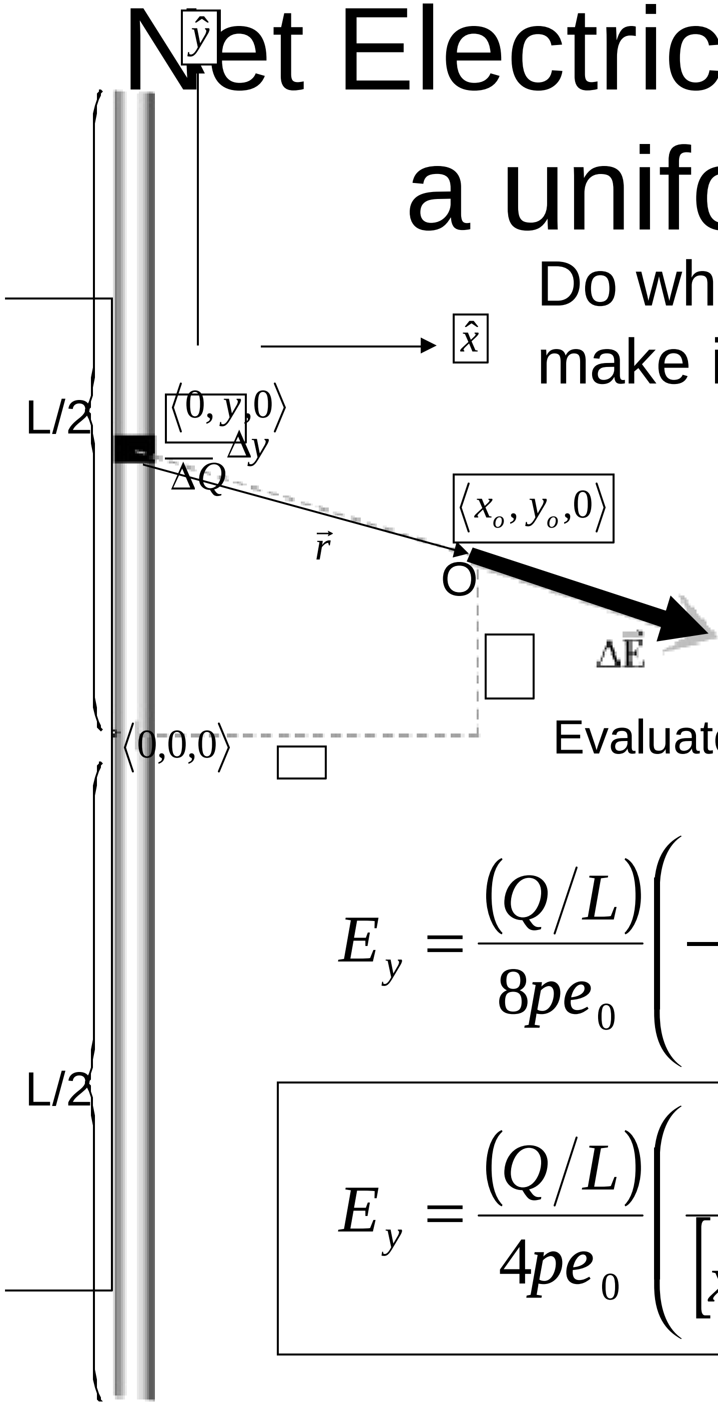
Another change of variables:

$$\breve{y} \equiv \tilde{y}^2 \Rightarrow d\breve{y} = 2\tilde{y}\,d\tilde{y} \Rightarrow \tilde{y}\,d\tilde{y} = \tfrac{1}{2}\,d\breve{y}$$

So,

$$E_y = \frac{(Q/L)}{4pe_0}\int_{\breve{y}=(y_o-L/2)^2}^{\breve{y}=(y_o+L/2)^2}\frac{\frac{1}{2}\,d\breve{y}}{\left[x_o{}^2+\breve{y}\right]^{3/2}}$$

Net Electric Field at point O due to a uniformly charged thin Rod



Do what you must to either *do* the integral, or make it recognizable as one that's done for you.

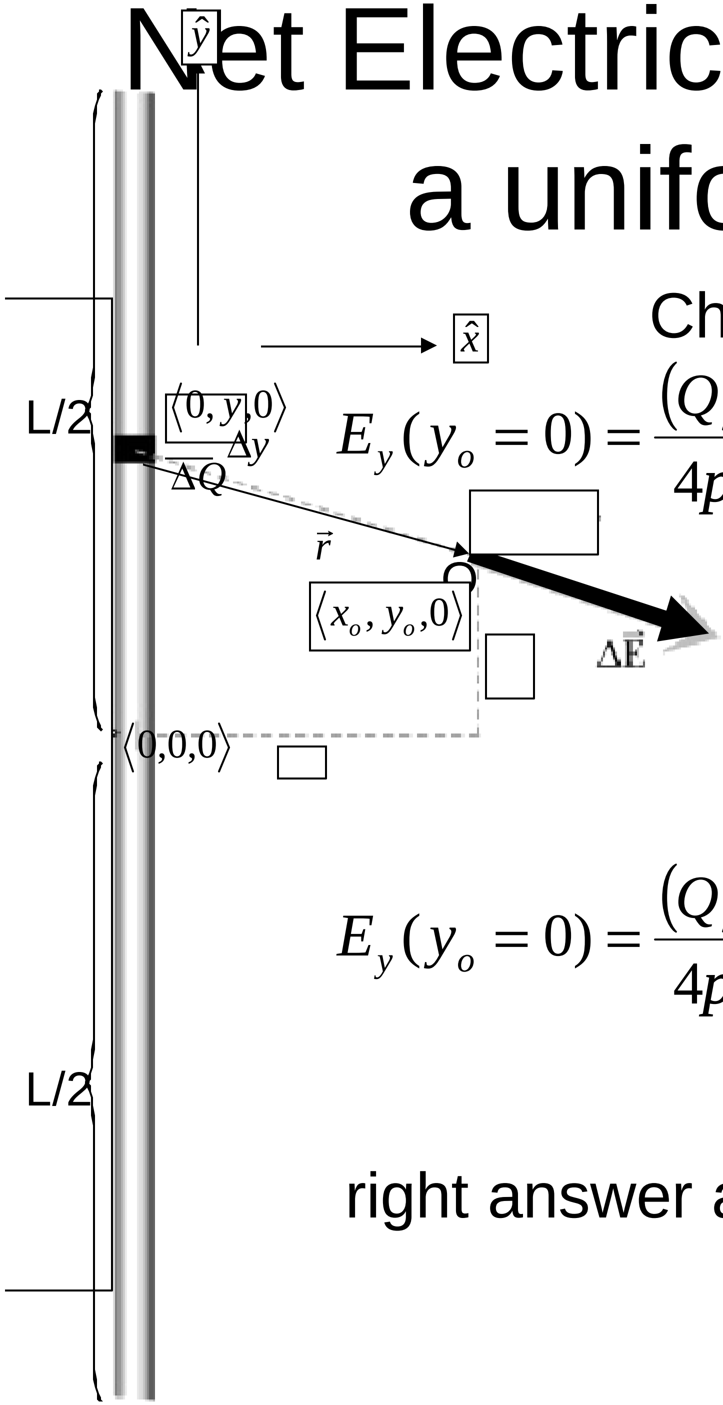
$$E_y = \frac{(Q/L)}{8\pi\varepsilon_0}\int_{\breve{y}=(y_o-L/2)^2}^{\breve{y}=(y_o+L/2)^2}\frac{d\breve{y}}{\left[x_o^{\;2}+\breve{y}\right]^{3/2}}$$

Evaluate Integral:

$$E_y = \frac{(Q/L)}{8\pi\varepsilon_0}\left(-2\frac{1}{\left[x_o^{\;2}+\breve{y}\right]^{1/2}}\right)\Bigg|_{\breve{y}=(y_o-L/2)^2}^{\breve{y}=(y_o+L/2)^2}$$

$$E_y = \frac{(Q/L)}{4\pi\varepsilon_0}\left(\frac{1}{\left[x_o^{\;2}+(y_o-L/2)^2\right]^{1/2}}-\frac{1}{\left[x_o^{\;2}+(y_o+L/2)^2\right]^{1/2}}\right)$$

Net Electric Field at point O due to a uniformly charged thin Rod



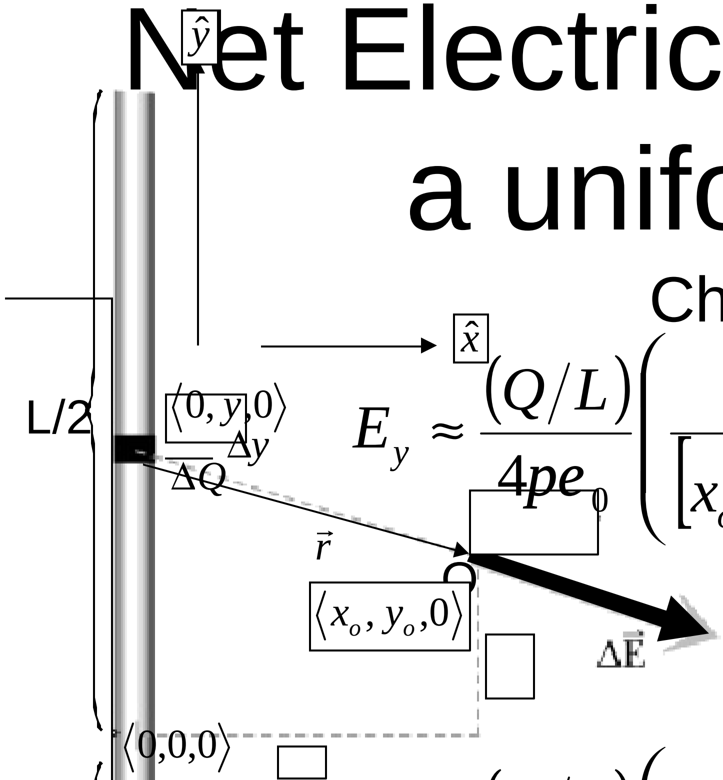
Check: right answer at $y_o = 0$?

$$E_y(y_o = 0) = \frac{(Q/L)}{4\pi\varepsilon_0}\left(\frac{1}{\left[x_o^{\;2} + (y_o - L/2)^2\right]^{1/2}} - \frac{1}{\left[x_o^{\;2} + (y_o + L/2)^2\right]^{1/2}}\right)$$

$$E_y(y_o = 0) = \frac{(Q/L)}{4\pi\varepsilon_0}\left(\frac{1}{\left[x_o^{\;2} + (L/2)^2\right]^{1/2}} - \frac{1}{\left[x_o^{\;2} + (L/2)^2\right]^{1/2}}\right) = 0$$

right answer at $y_o = 0$? Yes.

Net Electric Field at point O due to a uniformly charged thin Rod



Check: right answer at y_o>>L?

$$E_y \approx \frac{(Q/L)}{4\pi\varepsilon_0}\left(\frac{1}{\left[x_o^{\,2}+y_o^{\,2}-y_oL\right]^{1/2}}-\frac{1}{\left[x_o^{\,2}+y_o^{\,2}+y_oL\right]^{1/2}}\right)$$

Apply binomial expansion which says if ε<<1, then $(1+\varepsilon)^n \approx 1+n\varepsilon$

$$E_y \approx \frac{(Q/L)}{4\pi\varepsilon_0}\left(\frac{1}{\left[x_o^{\,2}+y_o^{\,2}\right]^{1/2}}\frac{y_oL}{2\left(x_o^{\,2}+y_o^{\,2}\right)}-\frac{1}{\left[x_o^{\,2}+y_o^{\,2}\right]^{1/2}}\frac{-y_oL}{2\left(x_o^{\,2}+y_o^{\,2}\right)}\right)$$

$$E_y \approx \frac{1}{4\pi\varepsilon_0}\left(\frac{Q}{\left(x_o^{\,2}+y_o^{\,2}\right)}\frac{y_o}{\left[x_o^{\,2}+y_o^{\,2}\right]^{1/2}}\right)$$

$$E_y \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\left(\frac{y_o}{|r|}\right)$$