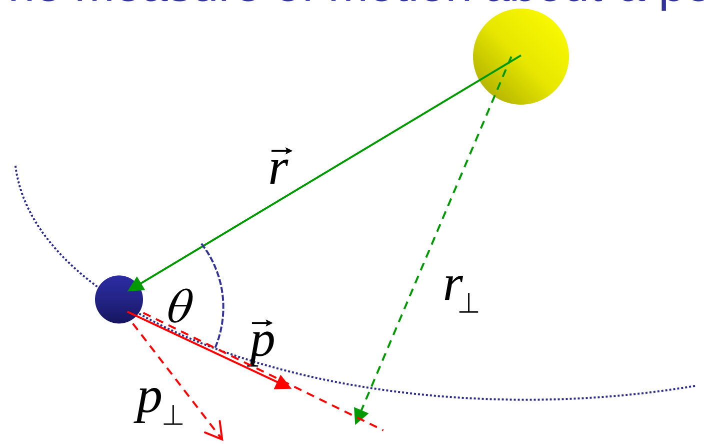
Mon. Tues. Wed. Lab Fri.	11.4-.6, (.13) Angular Momentum Principle & Torque 11.7 - .9, (.11) More Motion With & Without Torque L11 Rotation Course Evals 11.10 Quantization, Quiz 11	RE 11.c EP11 RE 11.d RE 11.e
Mon.	Review for Final (1-11)	HW11: Pr's 39, 57, 64, 74, 78
Sat. 9 a.m.	Final Exam (Ch. 1-11)	

Using Angular Momentum

The measure of motion *about* a point



Magnitude and Direction

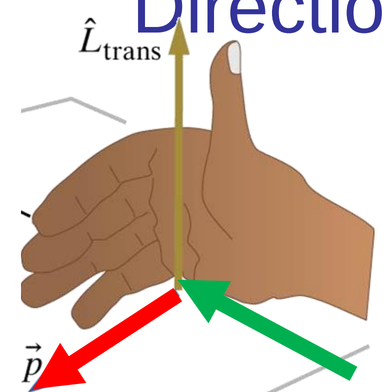
$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = \langle (p_z r_y - p_y r_z), (p_x r_z - p_z r_x), (p_y r_x - p_x r_y) \rangle$$

Magnitude

$$|L| = |p_\perp||r| = |p||r_\perp| = |p||r|\sin(\theta)$$

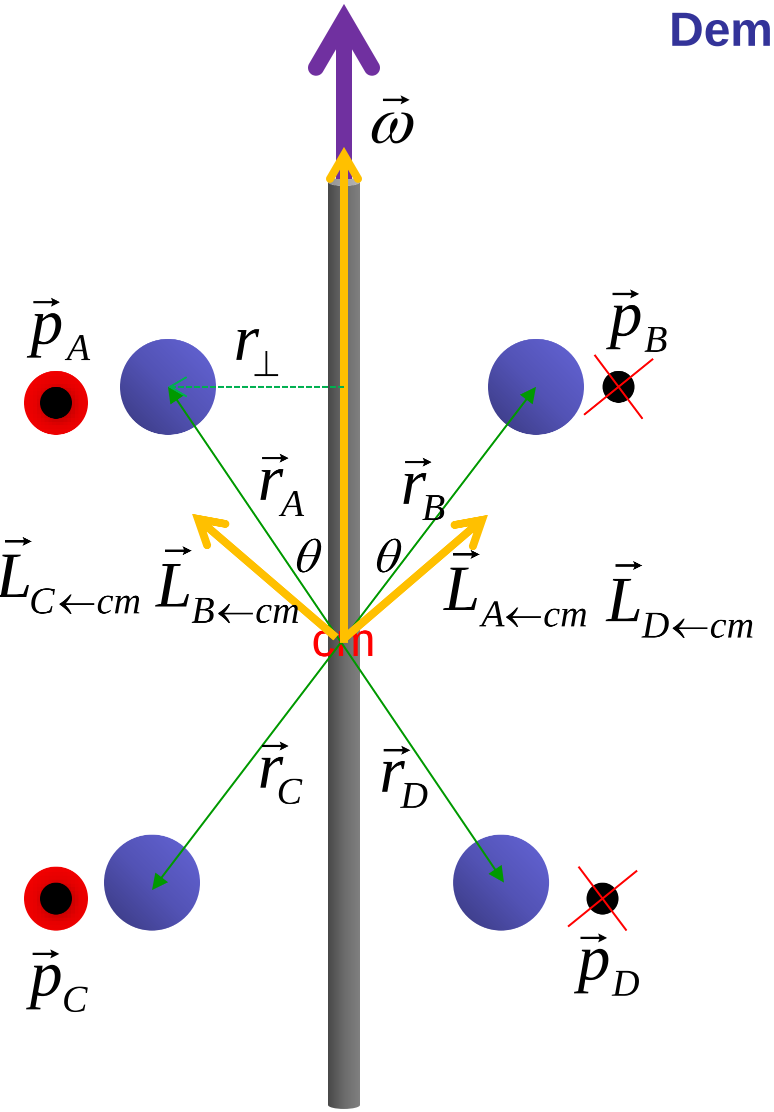
Direction



Orient Right hand so fingers curl from the axis and with motion, then thump points in direction of angular momentum.

If the Masses Don't Lie in a Plane

Demonstrating I_{axis}



$$\vec{L}_{total\leftarrow cm} = \vec{r}_A \times \vec{p}_A + \vec{r}_B \times \vec{p}_B + \vec{r}_C \times \vec{p}_C + \vec{r}_D \times \vec{p}_D$$

Given the symmetry,

$$\vec{L}_{total\leftarrow cm} = 4mr_\perp^2\omega\hat{z} \quad = I_{axis}\vec{\omega}$$

Generally, it's the moment of inertia about the *rotational axis of symmetry* through cm

Rotational Angular Momentum and Rotational Energy

		Analogous to
Recall	$K_{rot} = \frac{1}{2} I\omega^2$	$K = \frac{1}{2} mv^2$
now	$\vec{L}_{rot} = I\vec{\omega}$	$\vec{p} = m\vec{v}$
so	$K_{rot} = \dfrac{L^2}{2I}$	$K = \dfrac{p^2}{2m}$

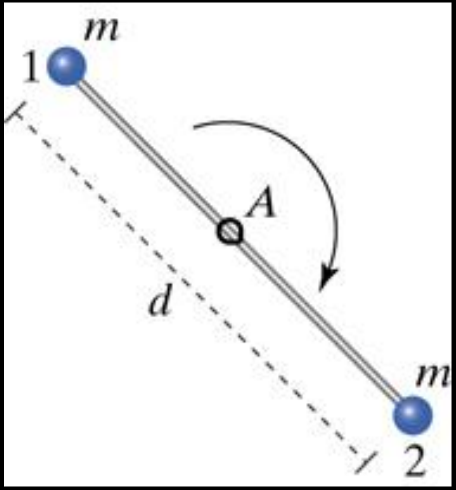
Rotational Angular Momentum and Kinetic Energy
Special case: rigid body

$$\vec{L}_{rot-axis} = I_{axis}\vec{\omega}_{axis}$$

$$K_{rot} = \tfrac{1}{2} I_{axis}\omega_{axis}^2$$

$$I_{axis} = \sum_i m_i r_{i-axis}^2$$

Example: A barbell spins around a pivot at its center at *A*. The barbell consists of two small balls, each with mass 500 grams (0.5 kg), at the ends of a very low mass rod of length 50 cm (0.5 m). The barbell spins clockwise with angular speed ω = 120 radians/s.



a) What is the moment of inertia about A?

b) What is the direction of the angular velocity?

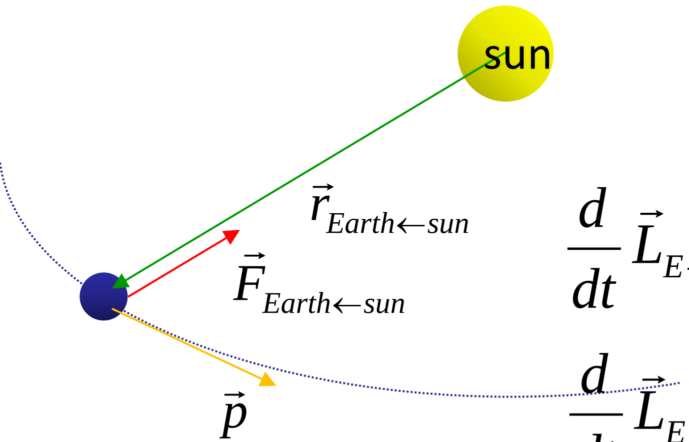
c) What is the *rotational* angular momentum?

d) What is the *total* angular momentum?

e) What is the rotational kinetic energy?

Interaction (not) Changing Angular Momentum relative to source of a radial force

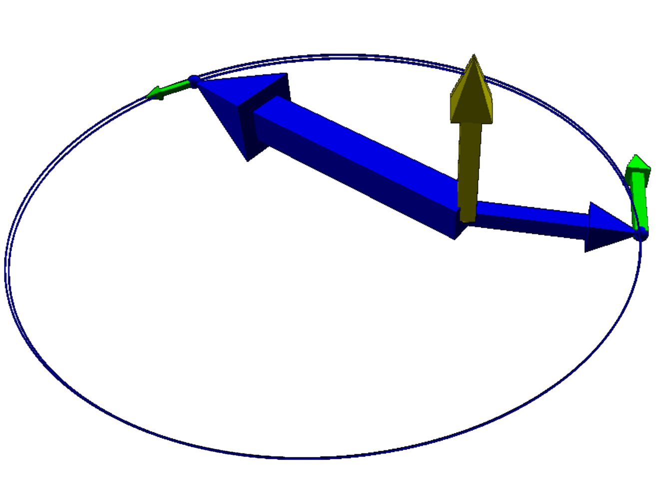
(like gravity or electric)



$$\frac{d}{dt}\vec{L}_{E-S} = \frac{d}{dt}\left(\vec{r}_{E-S}\times\vec{p}_E\right) = \frac{d\vec{r}_{E-S}}{dt}\times\vec{p}_E + \vec{r}_{E-S}\times\frac{d\vec{p}_E}{dt}$$

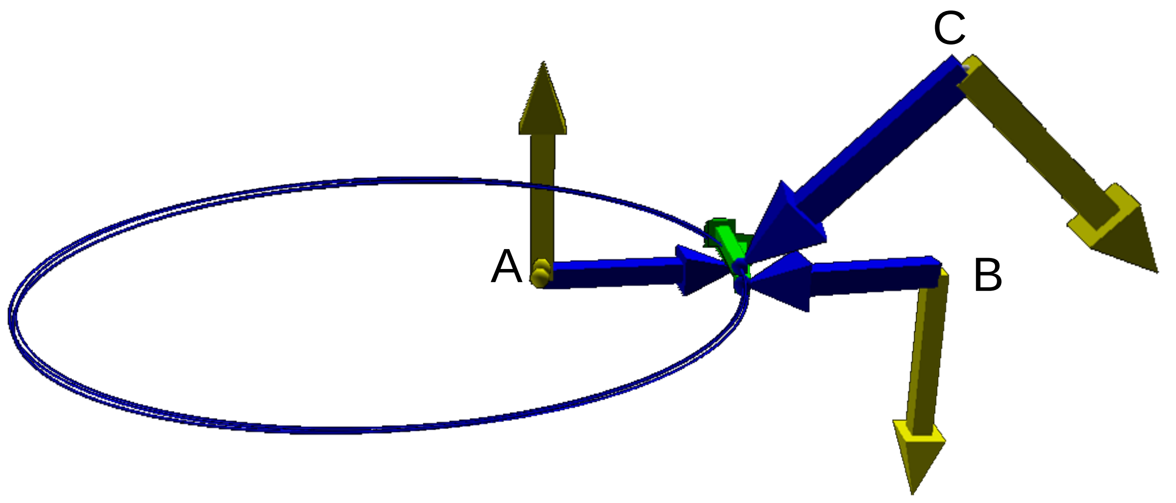
$$\frac{d}{dt}\vec{L}_{E-S} = \underbrace{\cancel{\vec{v}_E\times\vec{p}_E}}_{\text{Parallel}} + \underbrace{\cancel{\vec{r}_{E-S}\times\vec{F}_{Earth\leftarrow sun}}}_{\text{Parallel}} = 0$$

$$\vec{L}_{E-S} = \text{constant}$$

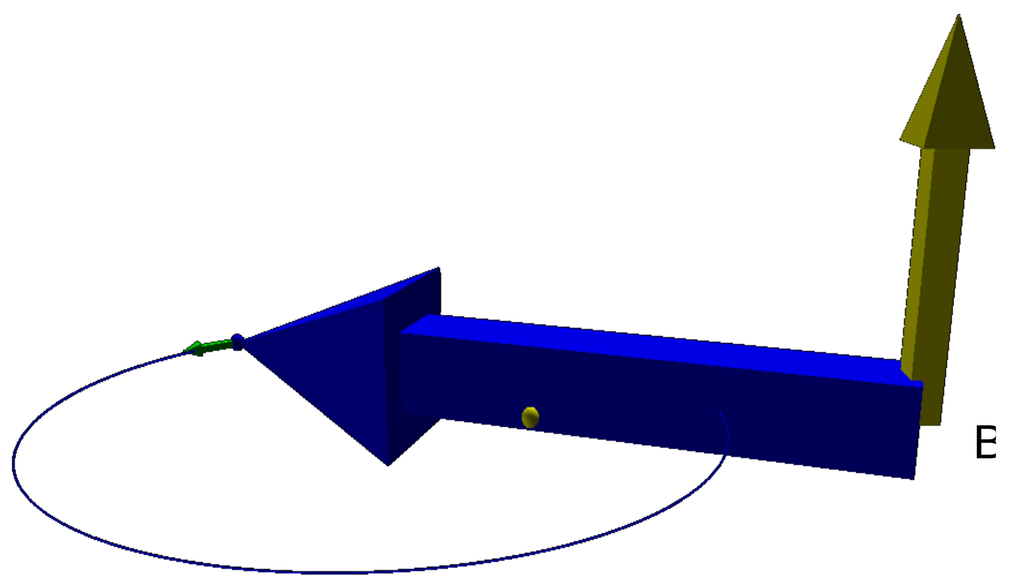


Orbit noncircular.py

Interaction *Changing* Angular Momentum relative to a *different* point

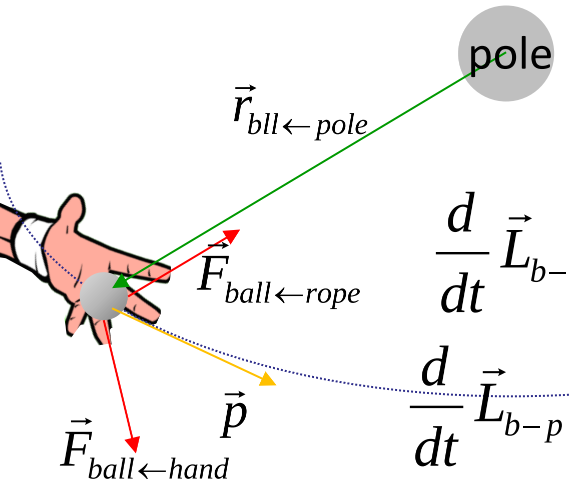


Generally, *changing* Angular Momenta



Orbit Non-circular L.py

Interaction *Changing* Angular Momentum relative to a point *other than* source of force



$$\frac{d}{dt}\vec{L}_{b-p} = \frac{d}{dt}\left(\vec{r}_{b-p}\times\vec{p}_b\right) = \frac{d\vec{r}_{b-p}}{dt}\times\vec{p}_b + \vec{r}_{b-p}\times\frac{d\vec{p}_b}{dt}$$

$$\frac{d}{dt}\vec{L}_{b-p} = \underbrace{\cancel{\vec{v}_b\times\vec{p}_b}}_{\text{Parallel}} + \vec{r}_{b-p}\times\sum_{all.\,forces}\vec{F}_{ball\leftarrow} = \underbrace{\cancel{\vec{r}_{b-p}\times\vec{F}_{ball\leftarrow rope}}}_{\text{Parallel}} + \vec{r}_{b-p}\times\vec{F}_{ball\leftarrow hand}$$

Angular Momentum Principle

$$\frac{d}{dt}\vec{L}_{(about)A} = \sum_{net}\vec{\tau}_{(about)A}$$

Torque

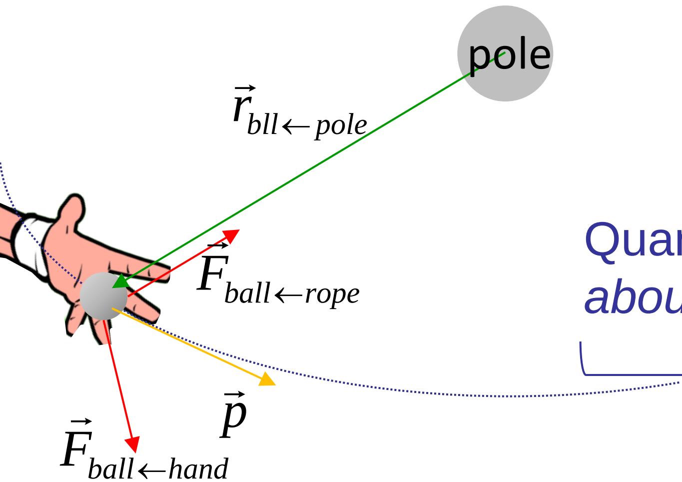
where, $\vec{\tau}_{(about)A} \equiv \vec{r}_{(from)A}\times\vec{F}$

analogous to

Momentum Principle

$$\frac{d}{dt}\vec{p} = \sum_{net}\vec{F}$$

Angular Momentum Principle



$$\frac{d}{dt}\vec{L}_{(about)A} = \sum_{net} \vec{\tau}_{(about)A}$$

Torque

where, $\vec{\tau}_{(about)A} \equiv \vec{r}_{(from)A} \times \vec{F}$

Quantifies motion *about* a point

Interaction that changes motion *about* a point

analogous to

Momentum Principle

$$\frac{d}{dt}\vec{p} = \sum_{net} \vec{F}$$

Quantifies motion

Interaction that changes motion

Example:

At t = 15 s, a particle has angular momentum <6, 8, -5> kg · m^2/s relative to location *A*. A constant torque <13, -14, 18> N·m relative to location *A* acts on the particle. At t = 15.2 s, what is the angular momentum of the particle relative to location *A*?

Angular momentum update relation

$$\frac{d}{dt}\vec{L}_A = \sum_{net}\vec{\tau}_A \quad \text{or} \quad \Delta\vec{L}_A = \left(\sum_{net}\vec{\tau}_A\right)_{ave}\Delta t \quad \text{so} \quad \vec{L}_{A.f} = \vec{L}_{A.i} + \left(\sum_{net}\vec{\tau}_A\right)_{ave}\Delta t$$

$$\vec{L}_{A.f} = \langle 6,8,-5\rangle \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s} + \left(\langle 13,-14,18\rangle \mathrm{Nm}\right)\left(15.2\mathrm{s} - 15\mathrm{s}\right)$$

$$\vec{L}_{A.f} = \langle 6,8,-5\rangle \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s} + \left(\langle 13,-14,18\rangle \mathrm{Nm}\right)\left(0.2\mathrm{s}\right)$$

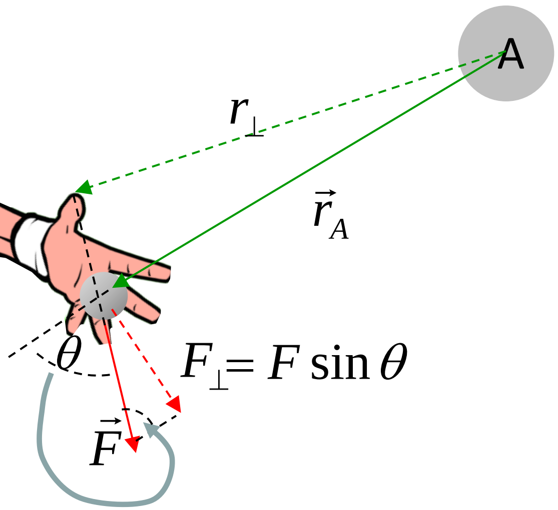
$$\vec{L}_{A.f} = \langle 6,8,-5\rangle \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s} + \langle 2.6,-2.8,3.6\rangle \mathrm{Nms}$$

$$\mathrm{Nms} = \left(\mathrm{kg}\cdot\mathrm{m/s}^2\right)\cdot\mathrm{m}\cdot\mathrm{s}$$

$$\vec{L}_{A.f} = \langle 8.6,5.2,-1.4\rangle \mathrm{kg}\cdot\mathrm{m}^2/\mathrm{s}$$

Torque

$$\vec{\tau}_{(about)A} \equiv \vec{r}_{(from)A} \times \vec{F}$$

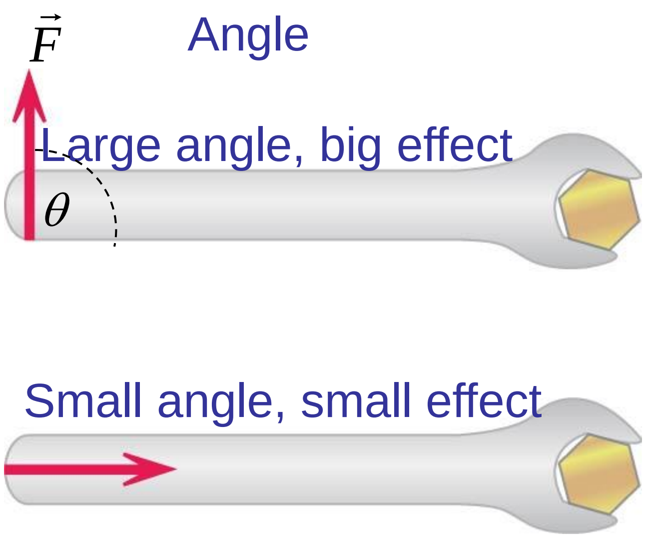


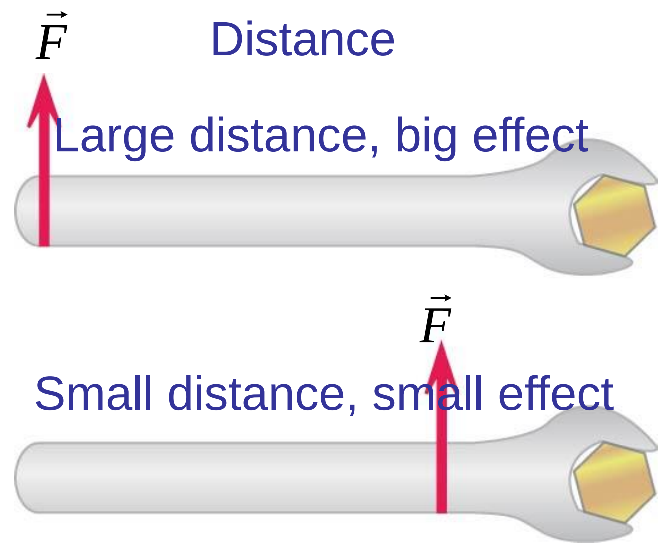
Magnitude
(yet another cross product)

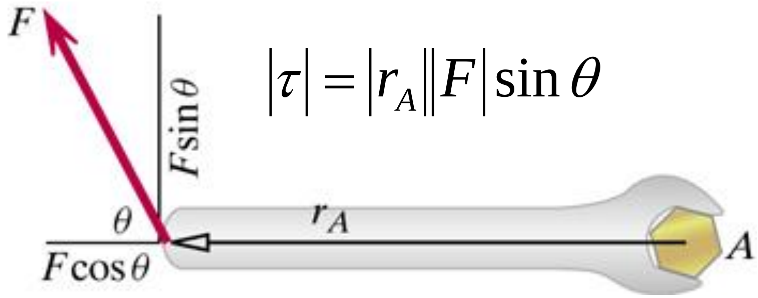
$$|\tau_A| = |r_A||F_{\perp}| \quad = |r_{A\perp}||F|$$

$$|\tau_A| = |r_A|\left(|F|\sin\theta\right) = \left(|r_A|\sin\theta\right)|F| \quad = |r_A||F|\sin\theta$$

Making sense of the
factors and cross-product







$$|\tau| = |r_A||F|\sin\theta$$

Example:

a) If r_A = 4 m, F = 8 N, and θ = 73°, what is the magnitude of the torque about location A, including units?

$$|\tau| = (4\mathrm{m})(8\mathrm{N})\sin(73°) = 30.6\mathrm{Nm}$$

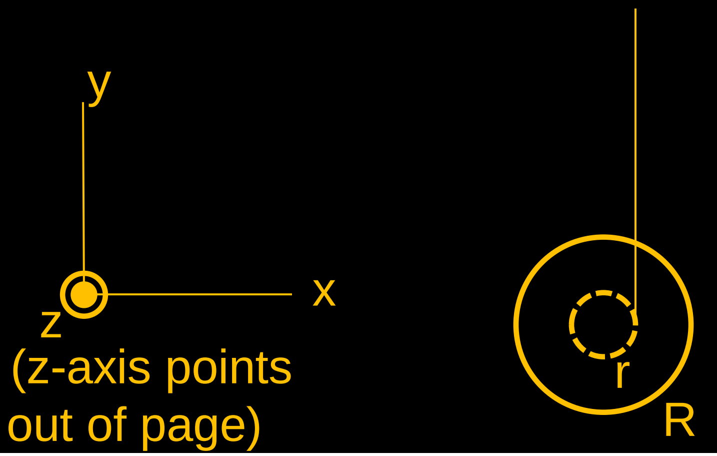
(b) If the force were perpendicular to $\vec{r}_A$ but gave the same torque as in the preceding question, what would its magnitude be?

$$|F| = \frac{|\tau|}{|r_A|\sin\theta}$$

$$|F| = \frac{30.6\mathrm{Nm}}{(4\mathrm{m})(1)} = 7.65\mathrm{N}$$

A yo-yo is in the x-y plane. You pull up on the string with a force of magnitude 0.6 N. What is the magnitude of the torque (about its center) you exert on the yo-yo?

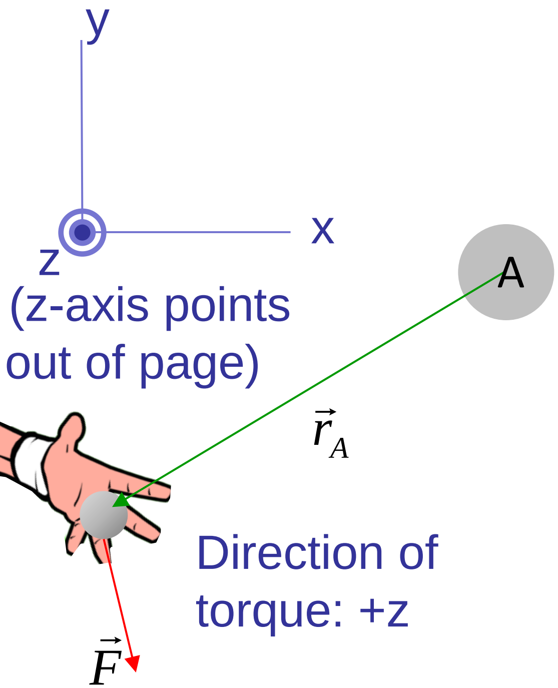
$r = 0.005$ m, $R = 0.035$ m



a) 0.005 N· m
b) 0.003 N· m
c) 0.021 N· m
d) 0.035 N· m
e) 0.6 N· m
f) cannot be determined without knowing the length of the string

Torque

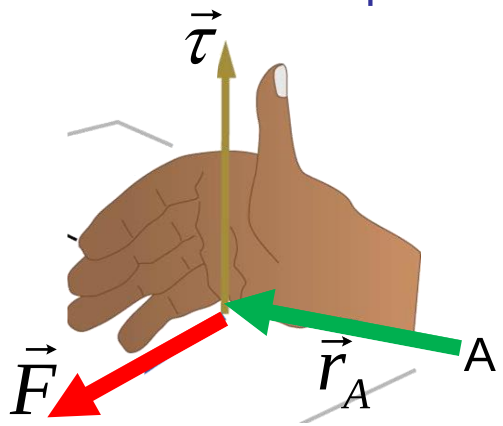
$$\vec{\tau}_{(about)A} \equiv \vec{r}_{(from)A} \times \vec{F}$$



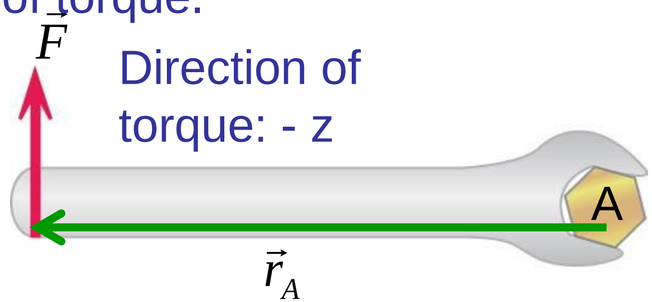
Direction of torque: +z

Direction

(yet another cross product)

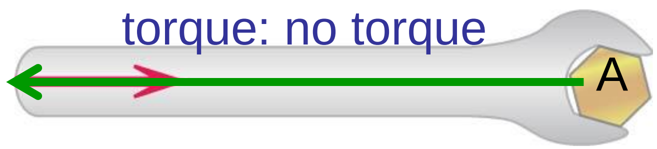


Orient Right hand so fingers start in direction from axis A to point of force’s application, then curl in direction of force. The thump points in direction of torque.



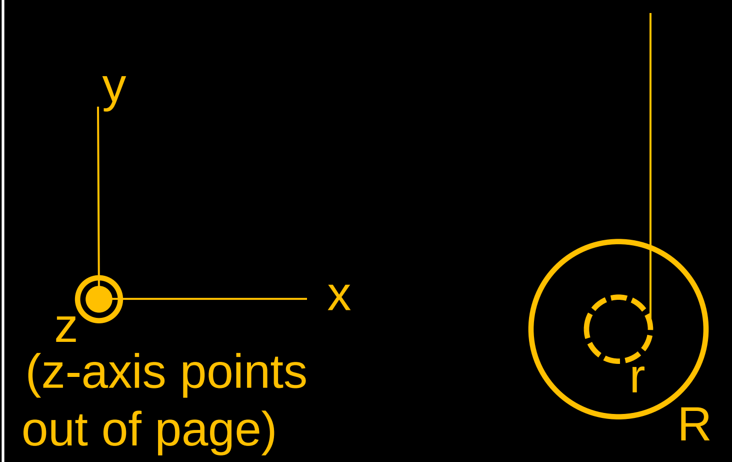
Direction of torque: - z

Direction of torque: no torque



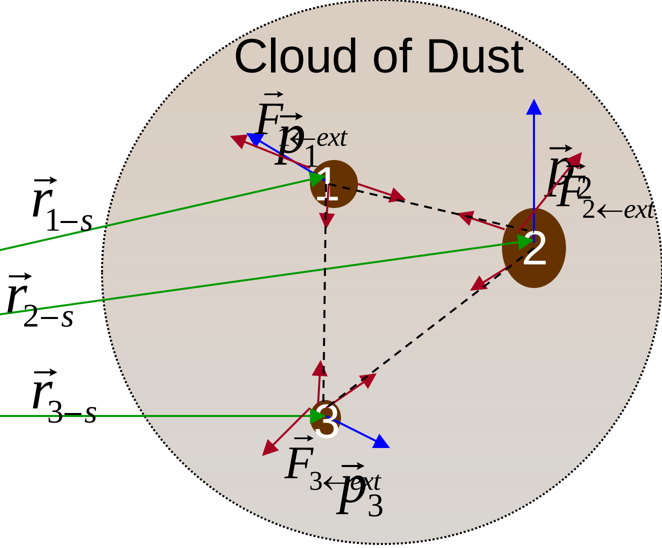
A yo-yo is in the x-y plane. You pull up on the string with a force of magnitude 0.6 N. What is the direction of the torque (about its center) you exert on the yo-yo?

r = 0.005 m, R= 0.035 m



1) +x
2) –x
3) +y
4) –y
5) +z
6) –z
7) zero magnitude

Multi-Particle Angular Momentum Principle:



Cloud of Dust about Star

$$\vec{L}_{c-s} = \vec{L}_{1-s} + \vec{L}_{2-s} + \vec{L}_{3-s} + \ldots$$

$$\vec{L}_{c-s} = \left(\vec{r}_{1-s} \times \vec{p}_1\right) + \left(\vec{r}_{2-s} \times \vec{p}_2\right) + \left(\vec{r}_{3-s} \times \vec{p}_2\right) + \ldots$$

$$\frac{d}{dt}\vec{L}_{c-s} = \frac{d}{dt}\left(\vec{r}_{1-s} \times \vec{p}_1\right) + \frac{d}{dt}\left(\vec{r}_{2-s} \times \vec{p}_2\right) + \frac{d}{dt}\left(\vec{r}_{3-s} \times \vec{p}_3\right) + \ldots$$

focus on one particle

$$\frac{d}{dt}\left(\vec{r}_{1-S} \times \vec{p}_1\right) = \frac{d\vec{r}_{1-S}}{dt} \times \vec{p}_1 + \vec{r}_{1-S} \times \frac{d\vec{p}_1}{dt} = \underbrace{\cancel{\vec{v}_1 \times \vec{p}_1}}_{\text{Parallel}} + \vec{r}_{1-S} \times \vec{F}_{1.net}$$

ditto for the others

$$\frac{d}{dt}\vec{L}_{c-s} = \vec{r}_{1-s} \times \vec{F}_{1.net} + \vec{r}_{2-s} \times \vec{F}_{2.net} + \vec{r}_{3-s} \times \vec{F}_{3.net} + \ldots$$

where — by reciprocity

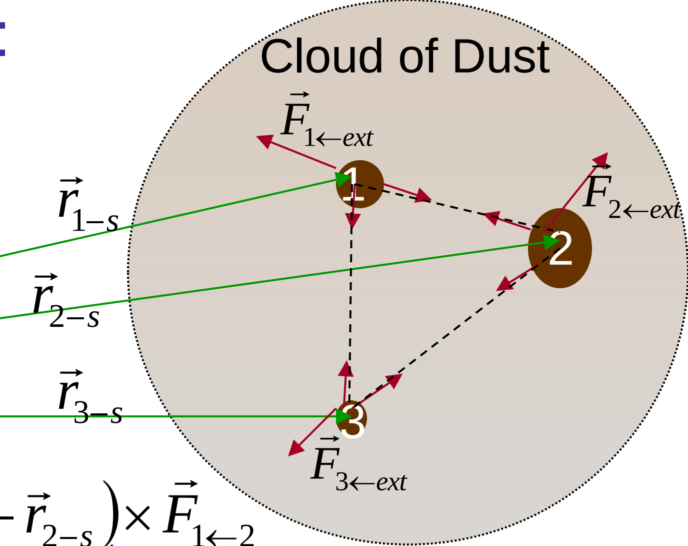
$$\vec{r}_{1-s} \times \vec{F}_{1\leftarrow net} = \vec{r}_{1-s} \times \left(\vec{F}_{1\leftarrow ext} + \vec{F}_{1\leftarrow 2} + \vec{F}_{1\leftarrow 3}\right) = \vec{r}_{1-s} \times \left(\vec{F}_{1\leftarrow ext} + \vec{F}_{1\leftarrow 2} + \vec{F}_{1\leftarrow 3}\right)$$

$$\vec{r}_{2-s} \times \vec{F}_{2\leftarrow net} = \vec{r}_{2-s} \times \left(\vec{F}_{2\leftarrow ext} + \vec{F}_{2\leftarrow 1} + \vec{F}_{2\leftarrow 3}\right) = \vec{r}_{2-s} \times \left(\vec{F}_{2\leftarrow ext} - \vec{F}_{1\leftarrow 2} + \vec{F}_{2\leftarrow 3}\right)$$

$$\vec{r}_{3-s} \times \vec{F}_{3\leftarrow net} = \vec{r}_{3-s} \times \left(\vec{F}_{3\leftarrow ext} + \vec{F}_{3\leftarrow 1} - \vec{F}_{3\leftarrow 2}\right) = \vec{r}_{3-s} \times \left(\vec{F}_{3\leftarrow ext} - \vec{F}_{1\leftarrow 3} - \vec{F}_{2\leftarrow 3}\right)$$

$$\frac{d\vec{L}_{c-s}}{dt} = \left(\vec{r}_{1-s} \times \vec{F}_{1\leftarrow ext} + \vec{r}_{2-s} \times \vec{F}_{2\leftarrow ext} + \vec{r}_{3-s} \times \vec{F}_{3\leftarrow ext}\right) + \left(\vec{r}_{1-s} - \vec{r}_{2-s}\right) \times \vec{F}_{1\leftarrow 2} + \left(\vec{r}_{1-s} - \vec{r}_{3-s}\right) \times \vec{F}_{1\leftarrow 3} + \left(\vec{r}_{2-s} - \vec{r}_{3-s}\right) \times \vec{F}_{2\leftarrow 3}$$

Multi-Particle Angular Momentum Principle:



Cloud of Dust about Star

$$\frac{d}{dt}\vec{L}_{c-s} = \vec{r}_{1-s}\times\vec{F}_{1.net} + \vec{r}_{2-s}\times\vec{F}_{2.net} + \vec{r}_{3-s}\times\vec{F}_{3.net} + \ldots$$

$$\frac{d\vec{L}_{c-s}}{dt} = \left(\underbrace{\vec{r}_{1-s}\times\vec{F}_{1\leftarrow ext}} + \underbrace{\vec{r}_{2-s}\times\vec{F}_{2\leftarrow ext}} + \underbrace{\vec{r}_{3-s}\times\vec{F}_{3\leftarrow ext}}\right) + \underbrace{\left(\vec{r}_{1-s}-\vec{r}_{2-s}\right)}\times\vec{F}_{1\leftarrow 2}$$

$$+\underbrace{\left(\vec{r}_{1-s}-\vec{r}_{3-s}\right)}\times\vec{F}_{1\leftarrow 3}$$

$$+\underbrace{\left(\vec{r}_{2-s}-\vec{r}_{3-s}\right)}\times\vec{F}_{2\leftarrow 3}$$

$$\frac{d\vec{L}_{c-s}}{dt} = \left(\vec{\tau}_{1\leftarrow ext} + \vec{\tau}_{2\leftarrow ext} + \vec{\tau}_{3\leftarrow ext}\right) + \cancel{\vec{r}_{1-2}\times\vec{F}_{1\leftarrow 2}} + \cancel{\vec{r}_{1-3}\times\vec{F}_{1\leftarrow 3}} + \cancel{\vec{r}_{2-3}\times\vec{F}_{2\leftarrow 3}}$$

For central forces (electric, gravitation) $\vec{r}_{1-2}\parallel\vec{F}_{1\leftarrow 2}$,etc. so $\vec{r}_{1-2}\times\vec{F}_{1\leftarrow 2} = 0$, etc.

$$\frac{d\vec{L}_{c-s}}{dt} = \left(\vec{\tau}_{1\leftarrow ext} + \vec{\tau}_{2\leftarrow ext} + \vec{\tau}_{3\leftarrow ext} + \ldots\right) = \vec{\tau}_{net.ext} = \sum_{i}^{all.particles} \vec{r}_{i-s}\times\vec{F}_{i\leftarrow ext}$$

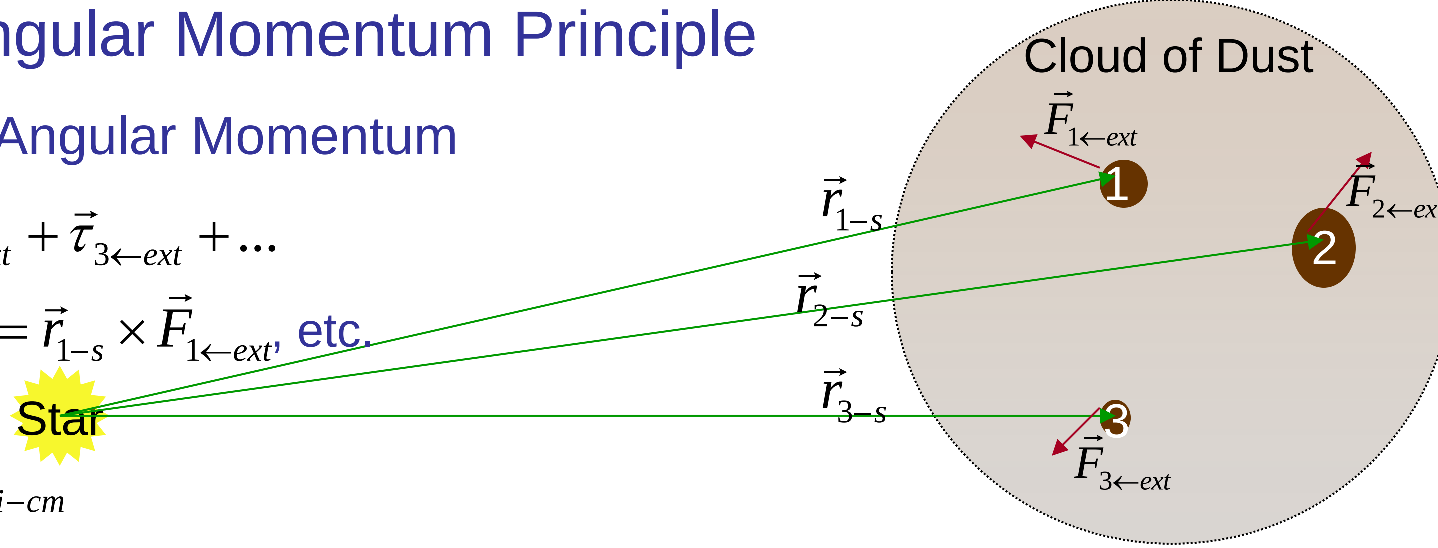
Note: net torque depends on each force and *its* point of application.

Multi-Particle Angular Momentum Principle

With Multi-Particle Angular Momentum

$$\frac{d\vec{L}_{c-s}}{dt} = \vec{\tau}_{1\leftarrow ext} + \vec{\tau}_{2\leftarrow ext} + \vec{\tau}_{3\leftarrow ext} + \ldots$$

where $\vec{\tau}_{1\leftarrow ext} = \vec{r}_{1-s} \times \vec{F}_{1\leftarrow ext}$, etc.



$$\vec{L}_{c-s} = \vec{L}_{cm-s} + \sum_i \vec{L}_{i-cm}$$

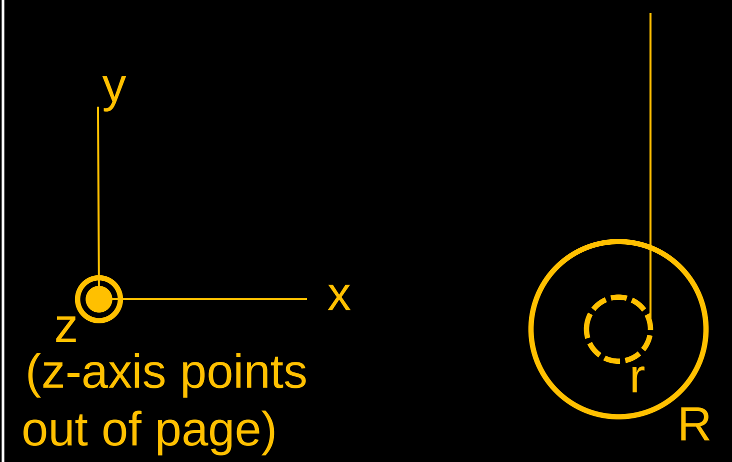
$$\vec{L}_{c-s} = \vec{L}_{trans-s} + \vec{L}_{rot-cm}$$

for rigid objects

$$\vec{L}_{rot-cm} = I_{cm}\vec{\omega}_{cm}$$

A 0.1 kg yo-yo is in the x-y plane. You pull up on the string with a force of magnitude 0.6 N for 0.5 s. If it was initially *not* rotating, what's its angular speed after the pull? $I = {}^{1}/_{2}mR^2$

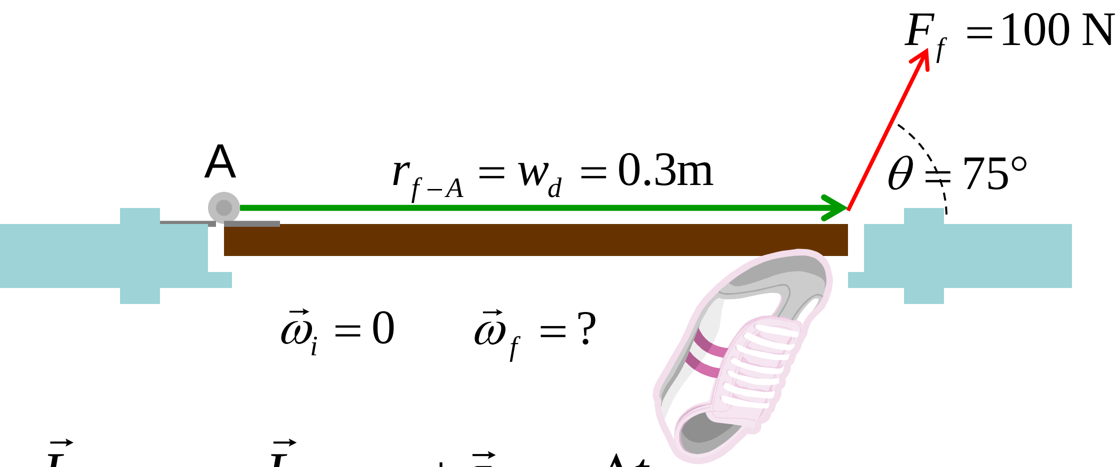
r = 0.005 m, R= 0.035 m, M = 0.1 kg



Multi-Particle Angular Momentum Principle

With Multi-Particle Angular Momentum

Example: opening a door. Your hands are full so you give the door a quick push with your foot near the edge. Say you apply a 100 N force for 0.5 s at 75°. The door's 15kg and 0.3m wide; what's its rate of rotation at the end of your push?



System: door

Active environment: your foot, hinge (but applied at axis-no torque)

Approximations: negligible frictional torque at hinge

Axis: hinge

$$\vec{L}_{door.A.f} = \vec{L}_{door.A.i} + \vec{\tau}_{foot-A}\Delta t$$

$$I_{door.A}\vec{\omega}_f = \cancel{I_{door.A}\vec{\omega}_i} + \vec{\tau}_{foot-A}\Delta t$$

Direction: torque & angular velocity are both "out of the board" (z)

Reasonable?

$$\omega_f = \frac{\tau_{foot-A}\Delta t}{I_{door.A}} = \frac{\left(r_{f-A}F_{foot}\sin\theta\right)\Delta t}{I_{door.A}} = \frac{\left(w_d F_{foot}\sin\theta\right)\Delta t}{\frac{1}{3}m_d w_d^2} = \frac{\left(F_{foot}\sin\theta\right)\Delta t}{\frac{1}{3}m_d w_d}$$

rod of negligible radius, about an end

$$I_{door.A} \approx \tfrac{1}{12}m_d w_d^2 + \tfrac{1}{4}m_d w_d^2 = \tfrac{1}{3}m_d w_d^2$$

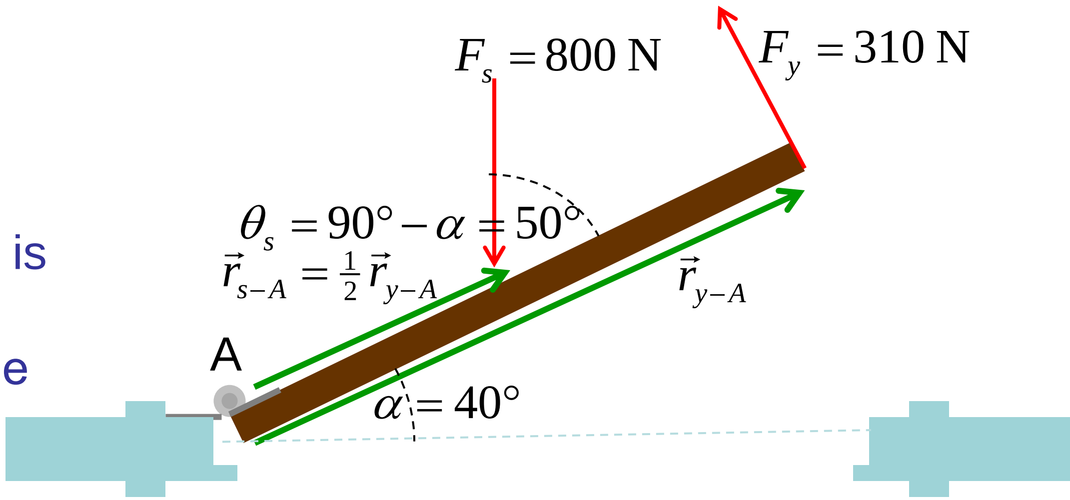
$$= \frac{100\mathrm{N}\cdot\sin(75°)\cdot 0.5\mathrm{s}}{\frac{1}{3}15\mathrm{kg}\cdot 0.3\mathrm{m}}$$

$$= 32\ {}^{\mathrm{rad}}\!/_{\mathrm{s}}$$

units?

Multiple Torques

You come home for the holidays to find that your younger sibling has annexed your room. When you try to push the door open, your nearest & dearest tries pushing it closed. Your younger sibling is stronger than you, but you've got 'physics-knowledge' on your side. While the kid leans hard (800 N) against the *middle* of the door and straight 'out' of the room, you push against the edge (310 N) and perpendicular to the door. Say the door's open 40°. Who wins?



$$\frac{d}{dt}\vec{L}_A = \sum_{net}\vec{\tau}_A = \vec{\tau}_y + \vec{\tau}_s = \vec{r}_{y-A}\times\vec{F}_y + \vec{r}_{s-A}\times\vec{F}_s$$

Your torque is in the +z direction, your sibling's is in the –z direction; if the answer is in +z you win!

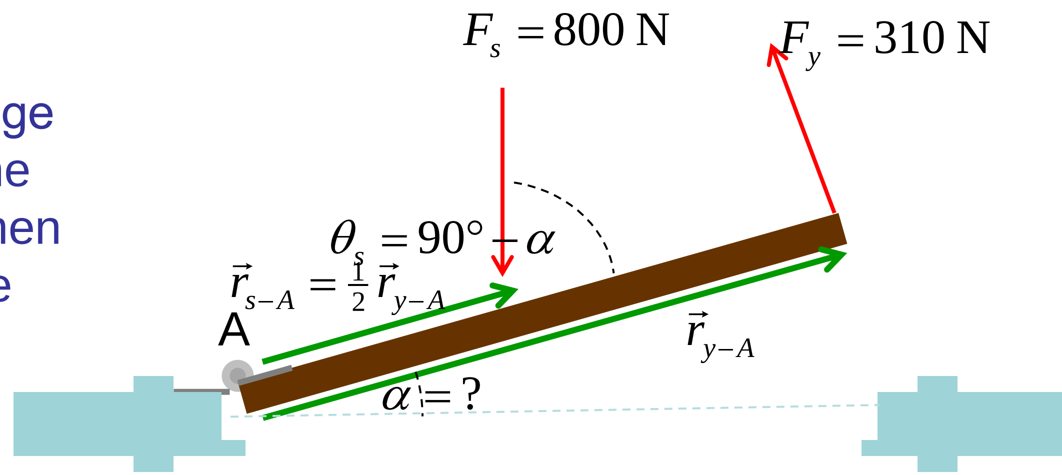
$$\hat{z}: \frac{dL}{dt} = r_{y-A}F_y \sin\theta_y - r_{s-A}F_s \sin\theta_s = r_{y-A}F_y - r_{s-A}F_s \sin\theta_s$$

$$= r_{y-A}\cdot 310\text{N} - \tfrac{1}{2} r_{y-A}\cdot 800\text{N}\cdot\sin(50°) = r_{y-A}(310\text{N} - 306\text{N}) = r_{y-A}(4\text{N}) > 0$$

you win!

Multiple Torques - equilibrium

Clearly, as you push the door further closed, you'll lose some of the advantage you had by pushing perpendicular to the door. At what angle will the door be when you and your sibling are tied / when the door's in equilibrium?



$$\frac{d}{dt}\vec{L}_A = \sum_{net}\vec{\tau}_A = \vec{\tau}_y + \vec{\tau}_s = \vec{r}_{y-A}\times\vec{F}_y + \vec{r}_{s-A}\times\vec{F}_s$$

In equilibrium

$$\hat{z}: 0 = r_{y-A}F_y\sin\theta_y - r_{s-A}F_s\sin\theta_s = r_{y-A}F_y - r_{s-A}F_s\sin\theta_s$$

$$0 = r_{y-A}F_y - \tfrac{1}{2}r_{y-A}F_s\sin\theta_s$$

$$2\frac{F_y}{F_s} = \sin\theta_s = \sin(90° - \alpha) = \cos(\alpha)$$

$$\alpha = \cos^{-1}\left(2\frac{F_y}{F_s}\right) = \cos^{-1}\left(2\frac{310N}{800N}\right) = 39.2°$$

Mon. Tues. Wed. Lab Fri.	11.4-.6, (.13) Angular Momentum Principle & Torque 11.7 - .9, (.11) Motion With & Without Torque L11 Rotation Course Evals 11.10 Quantization, Quiz 11	RE 11.c EP11 RE 11.d RE 11.e
Mon.	Review for Final (1-11)	HW11: Pr's 39, 57, 64, 74, 78
Sat. 9 a.m. Final Exam (Ch. 1-11)		