

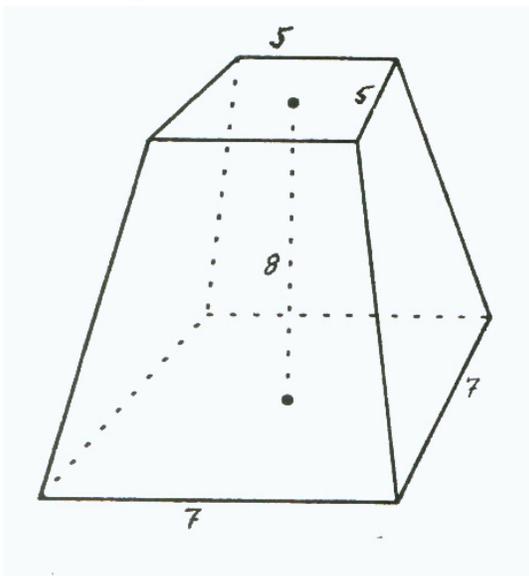
The Greatest Egyptian Pyramid Activity

In about 1850 BCE, a scribe in ancient Egypt wrote 25 mathematical problems in a papyrus scroll. One of the problems, Problem #14, gave the following instructions (Dunham, 3) for computing the volume of the truncated square pyramid shown in Figure 1 (Dunham, 4). A truncated pyramid also is called a *frustum* of a pyramid.

Problem #14: If you are told: A truncated pyramid of 6 for the vertical height by 4 on the base by 2 on the top. You are to square this 4, result 16. You are to double 4, result 8. You are to square 2, result 4. You are to add the 16, the 8, and the 4, result 28. You are to take a third of 6, result 2. You are to take 28 twice, result 56. See, it is 56. You will find it right.

1. Work through the scribe's instructions, making sure the final result of your calculations is 56.

Figure 2



2. Use the scribe's instructions to calculate the volume of the truncated pyramid of height 8 and with square bases of side lengths 5 and 7 shown in Figure 2 (diagram adapted from Dunham, 4). Point out any instructions that seem unclear, or that could be interpreted in more than one way.

3. Use the scribe's instructions to calculate the volume of the truncated pyramid of height h and with square bases of side lengths a and b shown in Figure 3 (diagram adapted from Dunham, 4). That is, use the scribe's instructions to write a general formula for the volume of a truncated square pyramid.

Figure 1

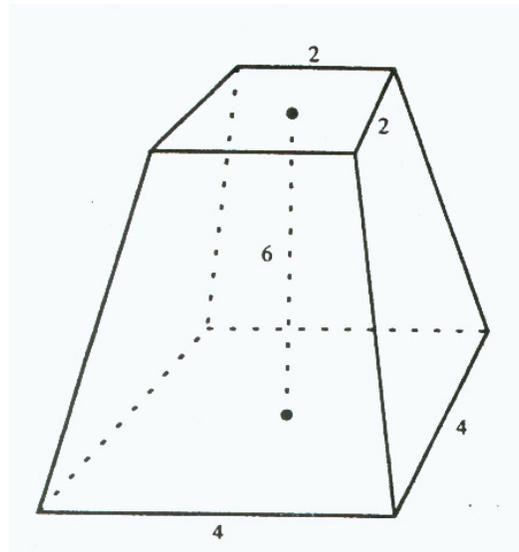
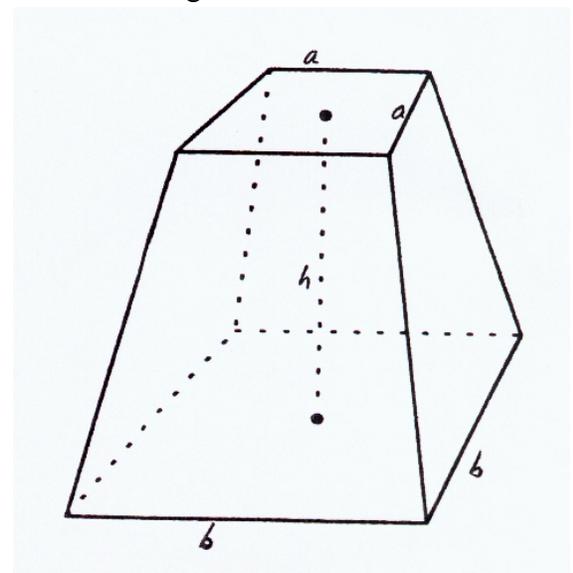


Figure 3.



At this point, your instructor may ask you to compare your results with those of other students. S/he may reveal a correct formula for the volume of a truncated square pyramid or she may ask you to derive one!

4. Many historians of mathematics point to Problem #14 as evidence that the ancient Egyptians knew how to calculate the volume of a truncated square pyramid of any dimensions---that is, of any height and with square bases of any side length. Some of them even call this calculation “the greatest of the Egyptian pyramids” (Burton, 54), greater even than the pyramids themselves. Does Problem #14 demonstrate that the ancient Egyptians had a general method or formula for computing volumes of frusta of pyramids? Why or why not?
5. Write correct and easy-to-imitate instructions for calculating the volume of the truncated pyramid of height 8 and with square bases of side lengths 5 and 7 from Problem 2. Write your instructions in sentences in the style of the Egyptian scribe.
6. If your instructions from Problem 5 had appeared in an Egyptian mathematical papyrus, would your fellow Egyptians have needed a formula for the volume of a truncated pyramid with a , b , and h in it? Explain. If you could have just one of the two, your instructions from Problem 5 or a formula with a , b , and h in it, which would you rather have available and why?
7. Set $a = 0$ in the formula for the volume of a truncated pyramid, and simplify. Do you recognize the resulting formula? For what solid geometric object does it give the volume?

Instructor Notes

Objective: Students will learn the formula for the volume of a frustum of a pyramid. Perhaps more importantly, they will explore what constitutes an effective example and what qualifies as a mathematical method or formula.

Materials: To help students visualize a truncated pyramid, you might buy or build a wooden, plastic, or Styrofoam model of a pyramid with a removable top. If you wish to have students derive the formula for the volume of a frustum of a pyramid, you might also want to be able to remove side and corner pieces from your pyramid (see the Extension Activity, below).

How to Use: Have students work in groups of 2-4 students each on Problems 1-3, then have groups compare their results. Ideally, students will interpret the scribe’s instructions (that is, imitate his example) in at least three or four different ways, hence will have three or four different values for the volume of the truncated pyramid in Problem 2 and three or four different general formulas for the volume of a truncated pyramid in Problem 3. You may need to let students know that you expect more than one answer. Have groups complete Problems 4-7, then discuss their answers to Problems 4-6 as a class.

Background Information: The instructions for calculating the volume of the truncated pyramid given in this activity appear as Problem #14 of the Egyptian mathematical

papyrus known as the *Moscow Papyrus*. Believed to date from 1850 BCE, this papyrus was purchased in Egypt in 1893 by V.S. Golenishchev, who eventually sold it to the Moscow Museum of Fine Arts. It contains 25 mathematical problems. The most famous of these problems is Problem #14, which some mathematics historians have called “the greatest of the Egyptian pyramids” (Burton, 54). This activity explores the widely accepted belief that Problem #14 provides incontrovertible evidence that the ancient Egyptians knew how to compute the volumes of pyramids and their frusta---that is, that they had general methods or formulas for computing these volumes.

Students may ask if calculations of volumes of pyramids and their frusta appear in other problems from Egyptian papyrus scrolls. The answer is that there are no such surviving problems. Although there are five problems involving the *seked*, or slope of the sides, of a pyramid in the *Rhind Papyrus* (Gillings, 185), the only evidence of ancient Egyptian knowledge of pyramid volume formulas is Problem #14 from the *Moscow Papyrus* and the pyramids themselves.

Solutions: 1. $(4^2 + 4 \cdot 2 + 2^2) \left(\frac{6}{3} \right) = 28 \cdot 2 = 56.$

2. Students should point out in this problem or in Problem 4 that the instruction to “double 4” is not clear: are we to compute 2×7 (that is, double 7), or are we to compute 5×7 ? They also should point out here or in Problem 4 that the instruction to “take 28 twice” is not clear: are we to compute 109×2 (that is, take 109 twice), or are we to compute $109 \times \frac{8}{3}$? As a result of these ambiguities, students might compute the volume in one of the following ways.

$$(7^2 + 7 \cdot 2 + 5^2) \cdot 2 = 176. \text{ (Note that } \frac{1}{3}(8) = \frac{8}{3} \text{ is computed but not used.)}$$

$$(7^2 + 7 \cdot 2 + 5^2) \left(\frac{8}{3} \right) = 234 \frac{2}{3}.$$

$$(7^2 + 7 \cdot 5 + 5^2) \cdot 2 = 218. \text{ (Again, } \frac{8}{3} \text{ is computed but not used.)}$$

$$(7^2 + 7 \cdot 5 + 5^2) \left(\frac{8}{3} \right) = 290 \frac{2}{3}.$$

3. The four computations in Problem 2 suggest the general formulas, $V = 2(a^2 + 2b + b^2)$, $V = \frac{h}{3}(a^2 + 2b + b^2)$, $V = 2(a^2 + ab + b^2)$, and $V = \frac{h}{3}(a^2 + ab + b^2)$, respectively. Only the last of the four formulas is correct.

4. Most historians of mathematics believe that the ancient Egyptians did have a general method or formula for calculating the volume of a truncated square pyramid (see, for example, Gillings, 187). However, most admit that the evidence---essentially just Problem #14 of the *Moscow Papyrus*---is not solid.

Mathematicians (and plenty of other people!) disagree on exactly what constitutes a mathematical method or formula. Many mathematicians would require general applicability, symbolic expression, and mathematical proof of the correctness of the

formula. Students know from experience that a few good examples, or even just one well-chosen and unambiguous example, can convey a general formula. For instance, if students are told that a triangle with base of length 7 and altitude 5 has area $\frac{1}{2} \cdot 7 \cdot 5 = \frac{35}{2}$, they can find the area of any triangle for which they are given the base and altitude and can write the general formula $A = \frac{1}{2}bh$.

Students may argue that the ambiguities in the instructions given in Problem #14 of the *Moscow Papyrus* (see the solution to Problem 2 given above) make them impossible to apply to truncated pyramids of other dimensions, hence that the ancient Egyptians may not have known how to compute volumes of truncated pyramids of various dimensions correctly. Students may object that just one example, even an unambiguous example, is not the same as having a general formula or method. They may argue that a formula must have algebraic or other symbols in it. They even may argue that a formula must be established by mathematical proof, and no proof is given in Problem #14 of the *Moscow Papyrus*.

On the other hand, students may argue that Problem #14, if read carefully, does show that the ancient Egyptians had a general method for finding the volume of a truncated pyramid. Indeed, if one remembers that the Egyptians multiplied numbers together by repeated doubling, one might interpret an instruction to double a bit more generally. Finally, the translation of Problem #14 given in the instructions has become standard in mathematics history texts. However, a more literal translation might read (van der Waerden 1971, Plate 5), “Add together this 16 with this 8 and this 4. You get 28. Compute $\frac{1}{3}$ of 6. You get 2. Multiply 28 by 2. You get 56. Behold: it is 56. You have found right.” These instructions omit calculation of the ab term, and certainly clear up any ambiguity in multiplication by the $\frac{h}{3}$ term.

5. Problem #14: If you are told: A truncated pyramid of 8 for the vertical height by 7 on the base by 5 on the top. You are to square this 7, result 49. You are to multiply 7 by 5, result 35. You are to square 5, result 25. You are to add the 49, the 35, and the 25, result 109. You are to take a third of 8, result $\frac{8}{3}$. You are to multiply 109 by $\frac{8}{3}$, result $290\frac{2}{3}$. See, it is $290\frac{2}{3}$. You will find it right.

6. Students may show that their instructions from Problem 5 are easy to imitate by applying them to another example or by using them to write the correct general formula. Students’ preferences for examples or general formulas may lead to interesting class discussion. During class discussion of Problems 4-6, be sure to ask students how they would define a general mathematical method or formula. You might have them consult a dictionary for help with their definitions. Ask them if a formula has to be expressed symbolically. Finally, ask them if an unambiguous example “counts as” a formula.

7. Setting $a = 0$ in the formula $V = \frac{h}{3}(a^2 + ab + b^2)$ results in the formula $V = \frac{1}{3}b^2h$ for the volume of a pyramid of height h and with a square base of side length b .

Extension Activity: Have students derive the formula $V = \frac{h}{3}(a^2 + ab + b^2)$ for the volume V of a truncated pyramid of height h and with square bases of side lengths a and b . If at all possible, use a solid model of a pyramid with removable top, side, and corner pieces. Here are two ways to compute the volume.

First, if we think of forming the truncated pyramid by slicing a smaller pyramid from the top of a larger pyramid, then we may compute the volume of the truncated pyramid by subtracting the volume of the smaller pyramid from the volume of the original, larger pyramid. Note that the side length of the square base of the smaller pyramid is a while the side length of the square base of the larger pyramid is b . If the height of the large pyramid is H , then the height of the small pyramid is $H - h$, and, by similar triangles, $\frac{H}{b} = \frac{H - h}{a} = \frac{h}{b - a}$. This yields $H = \frac{bh}{b - a}$ and $H - h = \frac{ah}{b - a}$, so that

$$V = \frac{1}{3}b^2H - \frac{1}{3}a^2(H - h) = \frac{1}{3}\left(b^2 \frac{bh}{b - a} - a^2 \frac{ah}{b - a}\right) = \frac{h}{3}\left(\frac{b^3 - a^3}{b - a}\right) = \frac{h}{3}(a^2 + ab + b^2).$$

If we focus just on the truncated pyramid itself, then we may dissect it into a rectangular solid, four prisms (side pieces), and four pyramids (corner pieces). If these pieces are cut perpendicularly to the base, then their volumes are a^2h (center),

$\frac{1}{2}a \frac{b - a}{2}h = \frac{1}{4}a(b - a)h$ (side), and $\frac{1}{3}\left(\frac{b - a}{2}\right)^2 h$ (corner), respectively. The volume of

the truncated pyramid of height h and base lengths a and b is then

$$V = a^2h + 4 \cdot \frac{1}{4}a(b - a)h + 4 \cdot \frac{1}{3}\left(\frac{b - a}{2}\right)^2 h = \frac{h}{3}(a^2 + ab + b^2).$$

For several more ideas as to how the Egyptians may have found the volume of a pyramid and of a truncated pyramid by dissection and rearrangement, see Richard Gillings' *Mathematics in the Time of the Pharaohs*, pages 189-193.

References: Activity from *Lengths, Areas, and Volumes*, by J. Beery, C. Dolezal, A. Sauk, and L. Shuey, in *Historical Modules for the Teaching and Learning of Secondary Mathematics*, Mathematical Association of America, Washington, D.C., 2003.

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