Calculus I: Derivatives as Limits

Step A. For each function *f* and each value $x = a$, compute the derivative $f'(a)$ accurate to 4 places after the decimal point by making successive approximations to $f'(a)$ using the formula $f'(a) \approx \dfrac{f(a+h)-f(a-h)}{2h}$ for $h \approx 0$. You are required to list at least five estimates, including those for $h = 1$, $h = 0.1$, $h = 0.01$, $h = 0.001$, and $h = 0.0001$, <u>and</u> to display at least three estimates in a row that are the same to 4 places after the decimal point. Record all estimates to at least 4 places after the decimal point.

Step B. Sketch the graph of *f* on the specified interval. For example, in Exercise 1, sketch the graph of $f(x) = x^3$ on the interval $-2 \le x \le 3$. This means your *x*-axis should run from $x = -2$ to $x = 3$. Then sketch two lines: a secant line from the point with *x*-coordinate $x = a - 1$ to the point with *x*-coordinate $x = a + 1$ on the graph of *f*, and a line tangent to the graph of *f* at the point $(a, f(a))$. Label each of these lines with its slope.

1. $f(x) = x^3$, $x = 2$, $-2 \le x \le 3$.
   [Answer to A: $f'(2) = 12.0000$ with estimates 13.0000 resulting from setting $h = 1$, 12.0100 resulting from $h = 0.1$, 12.0001 from $h = 0.01$, and 12.0000 from $h = 0.001$, $h = 0.0001$, and $h = 0.00001$. Answer to B: The secant line through the points $(1,1)$ and $(3,27)$ has slope 13. The tangent line at the point $(2,8)$ has slope 12.0000]

2. $f(x) = \cos(x)$, $x = -1$ and $x = 2$, $-\pi \le x \le \pi$. [Answers: 0.8415, –0.9093]

3. $f(x) = \ln(x)$, $x = 3$, $0 < x \le 5$. [Answer: 0.3333]

4. $f(x) = e^x$, $x = 1$, $-1 \le x \le 2$. [Answer: 2.7183]

5. $f(x) = x^2$, $x = 3$, $-1 \le x \le 4$. [Answer: 6.0000]
   Challenge questions: For $f(x) = x^2$ and $a = 3$, does $\dfrac{f(a+h)-f(a-h)}{2h} = 6.0000$ no matter what *h* is? For $f(x) = x^2$ and any fixed value of *a,* does $\dfrac{f(a+h)-f(a-h)}{2h}$ always have the same value? What about for $f(x) = 5x^2$? For $f(x) = 5x^2 + 4x - 7$? For $f(x) = bx^2 + cx + d$?

6. $f(x) = 3^x$, $x = 2$, $-1 \le x \le 3$. [Answer: 9.8875]

7. $f(x) = x^x$, $x = 2$, $0 < x \le 3$. To see the shape of the graph better, graph *f* from $x = 0$ to $x = 1$ and then from $x = 1$ to $x = 3$. [Answer: 6.7726]

8. $f(x)=|x|$, $x=-2$ and $x=0$, $-3 \le x \le 3$. [Answers: $f'(-2) = -1.0000$
Although the limit of the estimates is 0.0000, the derivative does not exist at $x = 0$.]

Terminology to remember:

Ways to phrase Step A:	The estimate $\dfrac{f(a+h)-f(a-h)}{2h}$ is …
Compute the **derivative** $f'(a)$.	… called a **difference quotient**.
Compute the **slope of the graph** of f at the point $(a, f(a))$.	… the **slope of the secant line** from the point $(a-h, f(a-h))$ to the point $(a+h, f(a+h))$ on the graph of f.
Compute the **slope of the line tangent to the graph** of f at the point $(a, f(a))$.	… the **slope of the secant line** from the point $(a-h, f(a-h))$ to the point $(a+h, f(a+h))$ on the graph of f.
Compute the **instantaneous rate of change** of f at $x = a$.	… the **average rate of change** from $x = a + h$ to $x = a - h$.