Questions of Measurement for Desegregation Accountability

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Questions of Measurement for Desegregation Accountability

Prior to the Civil Rights Act of 1964, there was no standard measure of impermissible racial segregation in the schools (Dunn, 1967; also see R. Farley, 1975; Farley & Taeuber, 1974), and not surprisingly, no progress toward dismantling the dual school systems outlawed in the *Brown* decisions (1954, 1955). Without standards, progress could not be documented and desegregation orders could not be enforced. Social scientists and public officials have developed a number of segregation indices aimed at holding school officials accountable for desegregation (see, e.g., Dziuban & Esler, 1983; James & Taeuber, 1985; Mitchell & Mitchell, 2010; Reardon & Firebaugh, 2002; U.S. Civil Rights Commission, 1967; White, 1986; Zoloth, 1976). However, these diverse segregation measures are more or less incompatible with one another and can be difficult to interpret (see, e.g., Johnston, Poulsen, & Forrest, 2007; Massey & Denton, 1988; Massey, White, & Phua, 1996). Confusion comes not from unreliable data but from unsolved problems in how segregation is defined and how to interpret mathematical formulae for its assessment, *especially for multiethnic populations.* This study conceptually examines the merits and limits of these alternative measures. We demonstrate that different measures provide different assessments of school desegregation but cannot singly provide sufficient information to monitor or guide school desegregation.

Framework

The ethnoracial group patterns of enrollment across schools have demanded and continue to demand the attention of policy makers because enrollment patterns are important equity indicators; changing enrollment patterns indicate whether a just and proper balance between public and private needs is being struck in the provision of public education (see

Brown v. Board of Education of Topeka, 1955, p. 300; hereafter, *Brown II*). There are at least five motivations for desegregating public schools and continued monitoring of school segregation. First, as established in *Brown v. Board of Education of Topeka* (1954; hereafter, *Brown I*; also see Horowitz, 1968; Vose, 1968; Wilson, 2003), desegregation is a means by which to protect vulnerable minorities from mistreatment or neglect by institutions and/or their staff members.

> [Educational] opportunity, where the state has undertaken to provide it, is a right which must be made available to all on equal terms…. [However,] "segregation with the sanction of law … has a tendency to [retard] the educational and mental development of negro children and to deprive them of some of the benefits they would receive in a racial[ly] integrated school system." … Separate educational facilities are inherently unequal. (*Brown I,* pp. 493-495)

The Court thus characterizes segregation as a social arrangement that allows institutional and professional racism to have pernicious effects. "Discriminations based on race alone are *obviously irrelevant and invidious* (emphasis added, *Steele v. Louisville & N.R. Co.,* 1944, p. 203; importantly, also see *Goss v. Board of Education,* 1963, p. 687). And as articulated in the *Steele* (1944) decision pertaining to how unions are required to represent their full membership in collective bargaining, school districts must similarly serve all students "without hostile discrimination, fairly, impartially, and in good faith" (*Steele v. Louisville & N.R. Co.,* 1944, p. 204). Separate treatment denies equal protection under the law.

A second motivation for having the ability to measure and assess the extent of school segregation is to monitor whether schools organize enrollment in a manner that reproduces stereotypical inter-ethnic relationships. As Iris Marion Young (1990) observed,

> Evaluating patterns of distribution is often an important starting point for questioning about justice. For many issues of social justice, however, what is important is not the particular pattern of distribution at a particular moment, but rather the reproduction of a regular distributive pattern over time. (p. 29)

Well after the *Brown* decisions, the U.S. Supreme Court declared, “the time for mere ‘deliberate speed’ has run out” (*Griffin v. School Board,* 1964, p. 234). The unconstitutional segregative patterns of enrollment observed a decade previously continued to be reproduced. Justice remained elusive.

Additional motivations arise out of beliefs that inter-ethnic exposure is a necessary means to produce three possible social outcomes: status equalization, inter-ethnic understanding, and societal cohesion. Martha Minow (2008) articulated these three facets well.

> Sophisticated people disagree over the implications of the commitment to [sociodemographic] equality. Some hold that it calls for color-blindness (and hence, official indifference to the racial, ethnic, religious, gender, linguistic, and disability characteristics of individual children) to avoid stereotyping and reducing individuals to group traits. Others say that it inspires measures to ensure integration across these lines of difference in order to overcome prejudice and build school communities that prepare students for a multicultural world.

> Disagreements over these visions reach beyond the context of race, where the demographic facts require more complex analysis of multiple racial and ethnic groups than the common focus on two or three categories. Even if there was no disagreement over whether integration remains an attractive and lawful ideal, it is difficult to analyze whether a school bringing together Mexican American, Puerto Rican American, Caribbean American, Chinese Asian, Vietnamese American, Indian American, Pakistani American, African American, and African immigrant students should be deemed "integrated," or if integration requires a palpable presence of white students. The situation is complicated if the white students are themselves primarily recent immigrants from Eastern Europe. Is integration primarily an ideal directed at overcoming the legacies of slavery in this country; overcoming misconceptions about racial, ethnic, and religious differences; or socializing all students to conceptions of citizenship, academic achievement, and career aspirations that have been associated with middle and upper class communities? (p. 23)

Clearly, the extent and character of school desegregation may be defined differently depending on the perspective taken, and adequate measurement and monitoring of school desegregation must be able to assess whether any particular perspective-taker recognizes success or failure in the attainment of desegregation goals.

Thinking About Indices

We turn in this section to the challenge of identifying a scheme for monitoring school (de/re)segregation. Though this is sometimes seen as a highly technical matter best suited to "separating and apportioning blame among the many state and private actors responsible for …

segregative effects" (Bryant, 2001, p. 73), the foregoing discussion makes it clear that this is fundamentally a matter of realizing political values. Value conflicts have to be worked out before any segregation index becomes a tool for monitoring progress toward social goals because the selection of any specific index implies a commonality of purpose that may not exist (Raffel, 1980, p. 202). However, barring complete abandonment of equity by the courts, interested parties are typically invited to participate in shaping the legal-rational decisions (remedies) with which compliance will be required. And, approval of specific remedies implies an ability to model what desegregation means in the local context. The choice of a specific segregation index for monitoring compliance operationalizes the definition of segregation implicit in the index, even if adopters do not always recognize the model being applied (Kelly & Miller, 1989). The most important challenges and considerations informing appropriate index selection arise from clearly articulating how each index models segregation in ways that represent valued outcomes, as well as hold school officials responsible for kinds of remedial action.

Before addressing the major issues, we must reiterate that index choice(s) should serve to orient our attention to desegregation goals. Our sense of the required extent of desegregation and the benefits of satisfying that requirement cannot be equally represented by the available indices (Zoloth, 1976). Each index implies a specific relationship between the observed student ethnoracial diversity and the numerical value obtained for an index. What French President Nicolas Sarkozy noted about economic indicators holds true for segregation measures: "our statistics and accounts reflect our aspirations, the values that we assign things.

They are inseparable from our vision of the world and the economy, of society, and our conception of human beings and our interrelations" (Sarkozy, 2009, p. vii).

Similarly, at a technical level, values remain central to index selection. "Justification for a particular index rests more on philosophy than science, since *the primary purpose of the index is to reflect a human value* rather than to capture an important property of state in the functioning of communities" (emphasis added, Sugihara, 1982, p. 564). Hutchens (1991) concurs that "an assessment of [categorical] inequality is inextricably tied to value judgments about the nature of inequality" (p. 51). With this human value judgment criterion in mind we explore why Stearns and Logan (1986), Clotfelter (2004), Orfield and Lee (2007), and others find it necessary to calculate multiple segregation measures with differing meanings and mathematical formulae.

Eight Conceptual Issues Affecting Index Selection

So, how do we go about deciphering the axiomatic mysteries of segregation indices in order to reflect our aspirations and assign the values we intend to represent? We must attend to eight issues that relate to what goes into the construction of a segregated system and, therefore, how we might select a segregation index.

1. Desegregation constructs. Available indices differ in their representation of two legal-policy constructs, namely, racial isolation and racial imbalance (Fiss, 1965; U.S. Civil Rights Commission, 1967). Concern with racial isolation leads observers to measure ethnoracial exposure among different groups; those interested in racial imbalance focus, instead, on evenness in the distribution of groups across classrooms, schools, districts or regions (e.g., Kelly & Miller, 1989; Massey & Denton, 1988). The choice between exposure and evenness in conceptualizing the desegregation problem has a profound effect on which index is appropriate. Federal mandates initially focused on exposure—whether racial groups interact with or are separated from each other—to attack the isolation of groups separated into single race or non-white schools (e.g., *Green v. County Sch. Bd. of New Kent County,* 1968; *Keyes v. School District No. 1,* 1973; *Mendez v. Westminster,* 1946; also see Conger, 2010; Dye, 1968; Edelman, 1973; Valencia, Menchaca, & Donato, 2002; Wollenberg, 1974). Before long, however, evenness of distribution became the focus of attention. Evenness is more important when the primary goal is exposure of everyone to a common educational experience; exposure is more important when the goal is overcoming interpersonal prejudices or developing intergroup cooperation and commitments. An emphasis on balancing student and staff assignments by race—resolving the problem of racial imbalance—meant redefining desegregation as requiring even dispersion of all groups, creating a fierce battle over "quotas" (e.g., *Swann v. Charlotte-Mecklenburg,* 1971; *U.S. v. Montgomery Bd. of Educ.,* 1969; also see, Crespino, 2006; Colton & Uchitelle, 1992; Welner, 2006).

At a mechanical level, exposure is a central tendency or average-like concept while evenness is a dispersion or variance-like concept (e.g., J. Farley, 2005; James & Taeuber, 1985; Kelly & Miller, 1989; Reardon & Firebaugh, 2002). This distinction highlights exposure as an absolute matter of interaction opportunities that is intimately dependent on the ethnoracial mix of the schools; exposure is the central tendency (mean or median, see J. Farley, 2005) of the distribution of interaction opportunities across all schools in the district—the average probability that a member of one group will encounter a member of the same group (isolation) or another group (interaction). In contrast, evenness is a relative matter when it comes to interaction opportunities because it depends on the ethnoracial mix of the district and the degree to which school enrollments deviate from that mix. To be evenly dispersed is to find that each school's ethnoracial group distribution matches that of the district as a whole.

2. Jurisdictional extent. We have to address the question: Are we interested in segregation within classrooms (between seating patterns or instructional groups), within schools (between classrooms or curriculum tracks), within districts (between schools), within metropolitan areas (between districts), etc.? On a technical level, this is important for two reasons. First, the smaller the population of the targeted unit within a jurisdiction is, the greater the sampling error associated with the estimate will be (Carrington & Troske, 1997), which is particularly noteworthy for within-classrooms and within-schools segregation. Second, given the nested (hierarchical) organization of jurisdictional authority, simultaneous attention to "within" and "between" segregation requires an index that permits additive decomposition in order to determine which jurisdictional levels are to be evaluated and held accountable for remediation (e.g., Bischoff, 2008; Mitchell, Batie, & Mitchell, 2010; Reardon, Yun, & Eitle, 2000; for technical discussions of additive decomposition, see Frankel & Volij, 2011; Hutchens, 2004; Mora & Ruiz-Castillo, 2011; Reardon & Firebaugh, 2002).

Further, some discussions of desegregation accountability may be unfair to a particular jurisdictional unit, like a school district, that lacks the authority or the population mix needed to comply with accountability guidelines (see Vose, 1968, p. 144). For example, the *Milliken v. Bradley* (1974) decision effectively removed metropolitan area-wide desegregation remedies from the list of options that may be employed by the district courts (i.e., eliminated court-mandated cross-district population movement; for some exceptions, see Colton & Uchitelle, 1992; Hankins, 1989; Raffel, 1980). This means that, in the absence of a fundamental change in policy or reconceptualization by the Court, we are limited to between schools (within district) desegregation remedies. Though social scientists may examine school segregation throughout a metropolitan area (e.g., Logan, Oakley, & Stowell, 2008; Mitchell, Batie, & Mitchell, 2010; Orfield & Lee, 2007; Reardon, Yun, & Eitle, 2000), or the nation for that matter (e.g., Frankel & Volij, 2011; Mora & Ruiz-Castillo, 2011), neither the courts nor school districts are likely to be making such assessments. Looking at smaller jurisdictional units, however, there have been efforts to address within school (between classrooms or tracks) segregation (e.g., *Hobson v. Hansen,* 1967; *Moses v. Washington Parish School Board,* 1972; also see Harvard Law Review Association, 1989; Hawley et al., 1983, pp. 118-124; Mitchell & Mitchell, 2005; Rossell, 2002; West, 1994).

3. Organizational unit of analysis. Are we interested in identifying individual schools that indicate persistent or reestablished segregation,[1] or are we satisfied with evaluating whether the district, as a whole, has achieved adequate desegregation? This is important because some indices are constructed in a manner that may provide meaningful descriptions of the extent of segregation at the individual school level as well as a district-level average segregation. Other indices only meaningfully evaluate district-level segregation and do not

[1] When it comes to the importance of being able to evaluate ethnoracial isolation at the school level vs. balance throughout the district, we cite *Swann* (1971, p. 26):

> It should be clear that the existence of some small number of one-race, or virtually one-race, schools within a district is not in and of itself the mark of a system that still practices segregation by law…. [However,] where the school authority's proposed plan for conversion from a dual to a unitary system contemplates the continued existence of some schools that are all or predominately of one race, they have the burden of showing that such school assignments are genuinely nondiscriminatory. *The court should scrutinize such schools,* and the burden upon the school authorities will be to satisfy the court that their racial composition is not the result of present or past discriminatory action on their part. (Emphasis added.)

identify interpretable school-level contributions to district-level segregation. That is, we wish to make the distinction between segregation indices that are able to unambiguously identify the contribution of the smallest fundamental unit being considered (e.g., groups within classroom, classrooms or tracks within schools, schools within districts, etc.) and those indices that evaluate segregation at a level that aggregates the smallest fundamental unit (e.g., the segregation of a classroom based upon group-level data, the segregation of a school based upon classroom-level data, the segregation of a district based upon school-level data, etc.).

4. Ethnoracial diversity. Are we attending to the ethnoracial identities of all students and their distribution among schools (a necessarily *multigroup* measurement problem), or are we interested only in such dichotomies as majority vs. minority, white vs. non-white, historically underrepresented vs. historically overrepresented (in, say, college preparatory curriculum, special education, college attendance, etc.), boys vs. girls, U.S.-born vs. foreign-born, etc.? (For a discussion of additional sociodemographic groups that might be considered, see Minow, 2008; Mitchell & Mitchell, 2011.) This is critically important because the behavior and interpretation of some indices change when attending to the distribution of more than two identifiable groups across schools (see Reardon & Firebaugh, 2002). And on a practical level, largely as a consequence of multiple changes in federal immigration law beginning in 1965 (e.g., see Massey, 2007) there are far more multiethnic schools and districts throughout the United States today than there were at the time of the *Brown I* decision in 1954. Though there may still be occasions when dichotomous comparisons are justified, the oft repeated yet always false claim of convenience that there are only two ethnoracial groups that are segregated—black and white—is no longer a sustainable or intellectually honest excuse for choosing an inadequate or inappropriate index.

5. Comparisons across space or time. Are we interested in monitoring (de/re)segregation over time, comparing segregation extent across a variety of placed-based (non-overlapping) jurisdictions, or determining whether a school governance entity has achieved desegregation compliance at a specific time for whatever standard applies at that time and in that place? This is important because ethnoracial group composition is unstable over time, including fundamental contraction or expansion of the number of groups represented. Also, the number of schools open is unstable over time (i.e., responses to population changes, either in number or in density, may include school openings or closures). School sizes have become particularly prone to change with the development and widespread use of portable classrooms. Finally, there is no guarantee that the number and identity of ethnoracial groups represented in one jurisdiction is identical to that in another jurisdiction.

We should note, at this point, that comparisons across space are not in reference to geographical (spatial) data. Comparison across space really means comparisons from place to place. That is, using the Massey and Denton (1988) typology, even though districts must analyze the residential *concentration* of ethnic groups, the *clustering* of racially concentrated neighborhoods or ghettos, and, in large cities, ethnic group *centrality* of particular groups in the city center, which complicate desegregation planning, we restrict ourselves to recognizing that school district boundaries are the most important spatial dimension limiting remedies (e.g., see Bischoff, 2008; *Milliken v. Bradley,* 1974). Except for district boundaries, spatial data have had little impact on desegregation supervision and monitoring. The non-incorporation of geographic considerations becomes troubling when districts are very large, where assignment and transportation of students to anywhere within the district may be impractical.

6. Compliance margins. Are we requiring complete desegregation, or allowing a tolerance window around complete desegregation? This is important because most, though not all, index values specify how much deviation remains to be overcome relative to complete desegregation (defined as identical ethnoracial distributions within each school). With politically controversial student movement and significant fiscal issues involved, however, complete desegregation is virtually impossible to attain let alone mandated by any court order (e.g., see Berger, 1984; Giles, 1977; Smith & Mickelson, 2000; Rossell & Armor, 1996; Welner, 2006). The numerical values of available segregation indices do not provide clear answers regarding how much desegregation remains to be accomplished in order to come within compliance bands. That is, all “normalized” indices are designed to assign zero to complete desegregation and one to complete segregation, but the path between these poles can differ quite a bit from one index to another and lead to different definitions of acceptably desegregated enrollment.

7. Interpretability. Interpreting the meaning of a desegregation index has two important dimensions. First is the problem of interpreting the practical meaning of an index's numerical value. Are we interested in an index than immediately lends itself to being read as a directive for action, or are we willing to engage in a translation procedure that guides desegregation decisions? The second interpretation issue is being able to use changes in the index value to determine whether segregation in one district, or at one point in time, can be reliably and accurately related to the index value at some other point in time or some other district's configuration of schools. **This is important because, in the absence of achieving complete desegregation, none of the available indices provides direct guidance for comparison across cases and none offer explicit guidance for action.** Moreover, they all require translation. Nonetheless, some indices help to communicate the magnitude of existing segregation more easily than others (e.g., the Dissimilarity Index can be interpreted as the proportion of students who have to be moved in order to achieve complete desegregation, which specifies the magnitude of the problem for districts facing challenges).

More easily understood interpretations are identified by Dziuban and Esler (1983) as critical because, to be useful for political and administrative purposes, a segregation index must be “easily applied to policy decisions in ways that yield numbers of pupils to be placed in specified schools” (p. 120). However, we are skeptical about whether easy applications exist and raise the possibility that Sarkozy’s (2009) experience with economic indicators is true for the social scientific practice of segregation analysis as well:

> There has indeed been a long-standing problem with what we calculate and the way we use what we find…. We [have known] that our indicators had limitations, but we went on using them as if they didn’t. They made communications easier. (p. xi)

In fact, rather than devote much attention to this point later in the paper, we would like to dismiss the Dziuban and Esler (1983) maxim as a distraction by way of a simple analogy. Consider SAT, GRE, or IQ scores, for example. They are fundamentally arbitrary. If we want to understand what a specific score really means, both for the one scored and for those needing to respond to such scores for a variety of purposes, we have to look at the details of what produced the score (e.g., specific items correct, missed, or skipped, specific performance domain, etc.). Our contention is that interpretability should be understood identically for segregation indices. That is, like a test performance score for an individual, an index score reveals where on the range of desegregation performance a district may be found and, when an index is constructed properly, the score may be compared to other districts for whether the district is more or less segregated than others. Further, by examining the details of what goes into the index score calculation, we find the answers to more detailed questions about, for example, whether specific schools should be targeted to receive, transfer, or exchange students from among those enrolled in the district.

8. Simplicity. It is important to remember that all of the available indices of segregation oversimplify the problem of desegregation that confronts school officials. None of the indices, for example, currently take account of such contingencies as distance to travel, traffic safety issues, built and natural barriers to travel, etc. (e.g., *Keyes v. School Dist. No. 1,* 1973; *Swann v. Charlotte-Mecklenburg,* 1971; also see Grannis, 2005; Reardon & O'Sullivan, 2004). It is also important to note that the simplifications that are imposed on the problem are not the same among the indices that have been used to describe the extent of segregation in most school settings. Some indices can specify exactly how many students would have to be moved to produce complete desegregation, but they cannot tell which students need to be moved. Some indices do a better job of identifying exactly where the students are that need to be moved, but they do not do an intuitively meaningful job of specifying how serious the current level of segregation is. This is important because the burdensomeness of desegregation actions is almost always considered when establishing compliance standards, which means that perfect desegregation is never the required outcome. Though more complex indices may be constructed to directly account for contingencies, they have not been systematically developed. Regardless of whether an index is simple or complex, as pointed out in the discussion of a compliance margin (issue six, above), some sort of tolerance band around perfect desegregation must be defined for any practical evaluation. There is no single numerical solution for achieving desegregation.

Nine Considerations for Index Construction and Selection

With the conceptual territory mapped out, we take up the nine specific axioms or properties that place limits on the acceptable mathematical expression for a segregation index. These axioms and properties serve both technical and conceptual purposes. Here, we wish to focus on the meaning of the axioms and properties that address the conceptual concerns articulated above. We do not want to get bogged down in technicalities, but will introduce some technical considerations as needed.

Hutchens (1991, 2004), Reardon and Firebaugh (2002), and Frankel and Volij (2011), for example, agree that there are five conceptual axioms or properties necessary to both minimally define a segregation index and have it be adequate for measuring school segregation. They are referred to, here, as *exchange* (i.e., reciprocal transfer between units within a jurisdiction), *transfer* (i.e., non-reciprocal transfer from one unit to another within a jurisdiction—similar to Pigou-Dalton transfer), *organizational equivalence, size invariance,* and *additive decomposability.* Additionally, the aforementioned authors acknowledge that there are four axioms that have been considered beyond these five—namely, *composition invariance* (also known as scale invariance), *group equivalence* (or group division), *symmetry,* and *range*—though there is not agreement about their necessity. The reasons for of this lack of agreement will be addressed in the discussion that follows. Unless otherwise noted, we employ the terminology and definitions given by Reardon and Firebaugh (2002, pp. 37-38).

Since desegregation requires student reassignment in order to eliminate ethnoracial isolation and move toward ethnoracial balance, we start with the axioms that pertain to moving students from school to school and then discuss the others in the order listed. Keep in mind that this discussion is in reference to ethnoracial enrollment data organized in a two-way contingency table (or matrix) in which each mutually exclusive ethnoracial category is a column and each mutually exclusive school (or smallest student organizing unit) is a row.[2] Also, though it may go without saying, the nature of the data is such that ethnoracial category membership is treated as fixed (i.e., a person cannot be transferred—change—from one ethnic group to another) while school membership is not (i.e., a person can be transferred from one school to another). This means that the number of ethnoracial categories (columns), and the number of students identified in each category (column marginal totals), does not change through participation in a desegregative act. These numbers only change when students enter or leave the jurisdiction that defines the two-way table. In contrast, the number of schools (rows), and the number of students in each school (row marginal totals), may change as a consequence of desegregation without anyone having to enter or leave the jurisdiction.

Axiom #1. Exchange. The calculated value of a segregation index must be reduced when students are reciprocally transferred between schools if the student in ethnoracial group *j*

[2] Jargowsky and Kim (2009) have developed an information-theoretic approach that permits non-exclusive categorization of ethnoracial identity. For example, the oft seen "two or more races" category into which respondents are classified because they have checked two or identifying categories on a survey can be treated for its full information rather than losing that information by answer aggregation or by treating all the multiple-response combinations as their own distinct ethnoracial categories. They do this by assigning partial-person weights. This weighting scheme makes it possible to include the same person across multiple categories. However, we will not take up this recent development in this paper.

coming from school h moves from higher group representation[3] in school h to lower group representation in the receiving school i ($\pi_{hj} > \pi_{ij}$) while, at the same time, the student from a different ethnoracial group k in school i moves from higher group representation in school i to lower group representation in receiving school h ($\pi_{ik} > \pi_{hk}$). Note that the exchange must be for whole persons, so the smallest exchange is 1 person from each group; fraction exchanges are not allowed. For example, a segregation index that satisfies this axiom will have a smaller value following *Exchange 1*, where one student from Group 1 and one student from Group 2 are exchanged between Unit 1 and Unit 2,

Exchange 1 shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{exchange} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 51 & 99 \\ 49 & 1 \\ 100 & 0 \end{bmatrix} \end{array}$$

Exchange 1 shown as row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{exchange} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.34 & 0.66 \\ 0.98 & 0.02 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

but not following *Exchange 2*, where, again, one student from each Group is exchanged between Unit 1 and Unit 2,

Exchange 2 shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 77 & 73 \\ 23 & 27 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{exchange} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 78 & 72 \\ 22 & 28 \\ 100 & 0 \end{bmatrix} \end{array}$$

[3] Group representation is defined as the group's proportion of the total school enrollment. Given the number of students in ethnoracial category *j* enrolled in school *i*, where + means summed across the columns (or rows, as appropriate), the group's proportion of the total school enrollment is expressed as follows: $\pi_{ij} = n_{ij}/n_{i+}$.

Exchange 2 shown as row proportions (π_{ij})

$$\begin{array}{lcc} & Grp\ 1 & Grp\ 2 \\ Unit\ 1 & 0.51 & 0.49 \\ Unit\ 2 & 0.46 & 0.54 \\ Unit\ 3 & 1.00 & 0.00 \end{array} \xrightarrow{exchange} \begin{array}{lcc} & Grp\ 1 & Grp\ 2 \\ Unit\ 1 & 0.52 & 0.48 \\ Unit\ 2 & 0.44 & 0.56 \\ Unit\ 3 & 1.00 & 0.00 \end{array}$$

In other words, a segregation index should reflect desegregation (have a lower value) when students are exchanged between schools so that their proportional group representation becomes more equal in both the sending and receiving schools. However, since school sizes must remain constant under exchange—all marginal frequencies remain unchanged in exchange—this is a necessary but not sufficient condition to impose on a segregation index. As a practical matter, school sizes (row marginal frequencies) do not remain constant.

Axiom #2. Transfer. A calculated segregation index value must be reduced by the non-reciprocal transfer of a student in, say, ethnoracial group i from one school (j) to another (k) if the move is from higher group representation to lower group representation ($\pi_{ij} > \pi_{ik}$). That is, a segregation index value should become lower after moving a student from one school to another so that the proportional ethnoracial group representation is more equal across the two schools after the transfer than it was before the transfer.[4] Similar to exchange, whole persons must be transferred from one unit to the next—no fractional transfers are permitted. For example, a segregation index that satisfies this axiom will have a smaller value following *Transfer 1a*, where one student in Group 2 is transferred from Unit 1 to Unit 2,

[4] We note that Frankel and Volij (2011, p. 7) express skepticism that this Pigou-Dalton-like transfer makes sense when discussing segregation for cases where there are more than two ethnoracial groups. Instead, they propose a composite axiom, which they call the School Division Property, which captures both organizational equivalence and transfer. On a practical level, their complaint against the Reardon and Firebaugh (2002) extension of the transfer principle to the multigroup case makes no substantive difference, but their discussion of the challenges to interpreting transfer in a multigroup situation is worth the reader's attention (see Frankel and Volij, 2011, pp. 6-7, including note 10).

Transfer 1a shown in frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{transfer} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 99 \\ 50 & 1 \\ 100 & 0 \end{bmatrix} \end{array}$$

Transfer 1a shown in row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{transfer} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.34 & 0.66 \\ 0.98 & 0.02 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

and *Transfer 1b*, where, one student in Group 1 is transferred from Unit 3 to Unit 1,

Transfer 1b shown in frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{transfer} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 51 & 100 \\ 50 & 0 \\ 99 & 0 \end{bmatrix} \end{array}$$

Transfer 1b shown in row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{transfer} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.34 & 0.66 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

But, neither *Transfer 2a*, where one student in Group 1 is transferred from Unit 1 to Unit 2,

Transfer 2a shown in frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 49 & 100 \\ 51 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{transfer} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 48 & 100 \\ 52 & 0 \\ 100 & 0 \end{bmatrix} \end{array}$$

Transfer 2a shown in row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{transfer} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.32 & 0.68 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

nor *Transfer 2b*, where one student in Group 1 is transferred from Unit 3 to Unit 2, would lead to a smaller index value,

Transfer 2b shown in frequency counts (n_{ij})

$$\begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{transfer} \begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 51 & 0 \\ 99 & 0 \end{bmatrix} \end{array}$$

Transfer 2b shown in row proportions (π_{ij})

$$\begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{transfer} \begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

This time, school sizes are not required to remain constant, so we have a necessary improvement on exchange. What is not explicitly incorporated here, however, is the opening or closing of schools. Transfer is undefined in the case of a school opening. We need a formalism that allows us to increase the number of schools while, at the same time, further developing a clear definition of what it means to be more or less segregated.

Axiom #3. Organizational equivalence. A calculated segregation index must not change when we divide a school into smaller schools (or any other organizational unit into smaller units that are otherwise identically defined), if the divided units have proportionally identical ethnoracial group enrollments. Additionally, the segregation index must not change if we consolidate schools with proportionally identical ethnoracial group enrollments into a single school. For example, a segregation index that satisfies this axiom will have the same value following *Organizational Division 1*, where the 50 Group 1 and 100 Group 2 students in Unit 1 were evenly divided among Units 1a and 1b, *Organizational Division 1* shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{c} \\ Unit\ 1a \\ Unit\ 1b \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 25 & 50 \\ 25 & 50 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array}$$

Organizational Division 1 shown as row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{c} \\ Unit\ 1a \\ Unit\ 1b \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

or *Organizational Division 2*, where the 50 Group 1 students in Unit 2 were evenly divided among Units 2a and 2b,

Organizational Division 2 shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2a \\ Unit\ 2b \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 25 & 0 \\ 25 & 0 \\ 100 & 0 \end{bmatrix} \end{array}$$

Organizational Division 2 shown as row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2a \\ Unit\ 2b \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

(Note: School divisions need not be even divisions of student groups between a pair of units, only that the proportions of each group in each resulting unit remain identical.) Reversing each school division would be a school consolidation (i.e., combining Units 1a and 1b back into Unit 1, and Units 2a and 2b back in to Unit 2). Given our three-unit starting configuration, following the organizational equivalence axiom, we may also produce *Organizational Consolidation 1* by combining Unit 2 and Unit 3 into Unit 2′ since Units 2 and 3 are already identically composed:

Organizational Consolidation 1 shown as frequency counts (n_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{organizational\ consolidation} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2' \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 150 & 0 \end{bmatrix} \end{array}$$

Organizational Consolidation 1 shown as row proportions (π_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{organizational\ consolidation} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2' \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

This is the formalism that allows us to open or close schools. By school division, we can open a school, and by school consolidation, we can close a school. The fact that this axiom only addresses schools of identical ethnoracial composition is not a problem because, from the previous two axioms, we know the segregative effects of the exchanges and transfers that might be required to level enrollment across newly divided units or reconfigure enrollments for consolidation of units.[5]

Axiom #4. Size invariance. A calculated segregation index must not change when every ethnoracial group in every school is proportionally increased or decreased in size identically. Mechanically, this says that when the joint frequencies in the cells of the two-way enrollment table are multiplied by the same number that segregation is unchanged. Whether the size of

[5] Consolidating units that cannot first be made proportionally identical in composition is not part of the axiomatic triple, but this would only arise when we are confronted with the indivisibility of persons such that identical composition is not mathematically possible. This is sure to happen when school consolidation leaves a district with a single school, but then the question of segregation is moot. The indivisibility of persons may arise in other configurations, but these will be very rare multiethnic configurations (e.g., where there are three or more groups represented, two ethnoracial groups having only one member each throughout the district would create an indivisible person problem).

the table is proportionally scaled up, or scaled down, segregation must not change. For example, a segregation index that satisfies this axiom will have the same value following *Size Scaling 1*, where the number of students in each Group and Unit is multiplied by 2,

Size Scaling 1 shown as frequency counts (n_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{size\ scaling} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 100 & 200 \\ 100 & 0 \\ 200 & 0 \end{bmatrix} \end{array}$$

Size Scaling 1 shown as row proportions (π_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{size\ scaling} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

and *Size Scaling 2*, where the number of students in each Group and Unit is multiplied by 0.5 (divided by 2),[6]

Size Scaling 2 shown as frequency counts (n_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{size\ scaling} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 25 & 50 \\ 25 & 0 \\ 50 & 0 \end{bmatrix} \end{array}$$

Size Scaling 2 shown as row proportions (π_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{size\ scaling} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

This augments our constraints on a segregation index while, at the same time, allowing us to compare segregation index values among jurisdictions (e.g., school districts). Prior to this point, our requirements only ensured that we could measure the effects of desegregative

[6] Again, the indivisibility of persons is important because multiplying by a non-integer, particularly a number between one and zero (i.e., division), is likely to result in a fractional person, which is not allowed.

moves for a single jurisdiction of a given size. Now, because we are explicitly working with proportions rather than frequencies to construct a segregation index, we can compare jurisdictions of any total enrollment. In other words, with the four axioms described, we have the ability to make some important *comparisons across space or time.*

Axiom #5. Additive decomposability. (Also see Frankel & Volij, 2011.) When units or groups within a jurisdiction are aggregated as clusters (i.e., super-units or super-groups), the total calculated segregation index across all units (or groups) must be the sum of segregation *between* all clusters (super-units/groups) and the weighted average segregation *within* clusters (super-units/groups). (See technical discussion of the *symmetry* axiom, below.) For example, when clusters of schools within a metropolitan area are aggregated into districts, the segregation for the complete two-way table of ethnoracial group enrollments across all schools in the metropolitan area can be expressed as a sum of the segregation between districts and a weighted average of the segregation between schools within districts. To provide an artificial illustration, consider a metropolitan area with four schools and four ethnoracial groups that is divided into two districts with two schools in each district, which gives us:

Aggregation 1 shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} G\ 1 & G\ 2 & G\ 3 & G\ 4 \end{array} \\ \begin{bmatrix} 33 & 67 & 0 & 0 \\ 25 & 25 & 25 & 25 \\ 0 & 0 & 30 & 70 \\ 50 & 0 & 40 & 10 \end{bmatrix} \end{array} \xrightarrow{aggregation} \begin{array}{c} \\ Dist\ 1 \\ Dist\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} G\ 1 & G\ 2 & G\ 3 & G\ 4 \end{array} \\ \begin{bmatrix} 58 & 92 & 25 & 25 \\ 50 & 0 & 70 & 80 \end{bmatrix} \end{array}$$

$$+\,Weight\ 1\left(\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} G\ 1 & G\ 2 & G\ 3 & G\ 4 \end{array} \\ \begin{bmatrix} 33 & 67 & 0 & 0 \\ 25 & 25 & 25 & 25 \end{bmatrix} \end{array}\right)$$

$$+\,Weight\ 2\left(\begin{array}{c} \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} G\ 1 & G\ 2 & G\ 3 & G\ 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 30 & 70 \\ 50 & 0 & 40 & 10 \end{bmatrix} \end{array}\right)$$

Aggregation 1 shown as row proportions (π_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} Gr\ 1 & Gr\ 2 & Gr\ 3 & Gr\ 4 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 & 0.00 & 0.00 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.00 & 0.20 & 0.30 & 0.70 \\ 0.50 & 0.00 & 0.40 & 0.10 \end{bmatrix} \end{array} \xrightarrow{aggregation} \begin{array}{r} \\ Dist\ 1 \\ Dist\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} Grp\ 1 & Grp\ 2 & Grp\ 3 & Grp\ 4 \end{array} \\ \begin{bmatrix} 0.290 & 0.460 & 0.125 & 0.125 \\ 0.250 & 0.000 & 0.350 & 0.400 \end{bmatrix} \end{array}$$

$$+ Weight\ 1 \left(\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \end{array}\begin{array}{c} \begin{array}{cccc} Gr\ 1 & Gr\ 2 & Gr\ 3 & Gr\ 4 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 & 0.00 & 0.00 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{bmatrix} \end{array} \right)$$

$$+ Weight\ 2 \left(\begin{array}{r} \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} Gr\ 1 & Gr\ 2 & Gr\ 3 & Gr\ 4 \end{array} \\ \begin{bmatrix} 0.00 & 0.20 & 0.30 & 0.70 \\ 0.50 & 0.00 & 0.40 & 0.10 \end{bmatrix} \end{array} \right)$$

Similarly, aggregation can be across the columns. This second type of aggregation occurs, for example, when all non-white ethnoracial groups are aggregated into a single super-group, the segregation for the complete two-way table of ethnoracial group enrollments across all schools in the metropolitan area can be expressed as a sum of the segregation between whites and non-whites and a weighted average of the segregation between ethnoracial groups within the non-white super-group. Starting with the same artificial illustration of a metropolitan area with four schools and four ethnoracial groups, this time, we designate Group 1 as the white group, and aggregate Groups 2, 3, and 4 to be the non-white super-group, which gives us:

Aggregation 2 shown as frequency counts (n_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} W & G\ 2 & G\ 3 & G\ 4 \end{array} \\ \begin{bmatrix} 33 & 67 & 0 & 0 \\ 25 & 25 & 25 & 25 \\ 0 & 0 & 30 & 70 \\ 50 & 0 & 40 & 10 \end{bmatrix} \end{array} \xrightarrow{aggregation} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{cc} W & NW \end{array} \\ \begin{bmatrix} 33 & 67 \\ 25 & 75 \\ 0 & 100 \\ 50 & 50 \end{bmatrix} \end{array}$$

$$+ Weight\ 1 \left(\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} W \\ \begin{bmatrix} 33 \\ 25 \\ 0 \\ 50 \end{bmatrix} \end{array} \right) + Weight\ 2 \left(\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} G\ 2 & G\ 3 & G\ 4 \end{array} \\ \begin{bmatrix} 67 & 0 & 0 \\ 25 & 25 & 25 \\ 0 & 30 & 70 \\ 0 & 40 & 10 \end{bmatrix} \end{array} \right)$$

Aggregation 2 shown as row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{cccc} W & Gr\ 2 & Gr\ 3 & Gr\ 4 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 & 0.00 & 0.00 \\ 0.25 & 0.25 & 0.25 & 0.25 \\ 0.00 & 0.20 & 0.30 & 0.70 \\ 0.50 & 0.00 & 0.40 & 0.10 \end{bmatrix} \end{array} \xrightarrow{aggregation} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{cc} W & NW \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 0.25 & 0.75 \\ 0.00 & 1.00 \\ 0.50 & 0.50 \end{bmatrix} \end{array}$$

$$+\,Weight\ 1\left(\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} W \\ \begin{bmatrix} 1.00 \\ 1.00 \\ 1.00 \\ 1.00 \end{bmatrix} \end{array}\right) + Weight\ 2\left(\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} G\ 2 & G\ 3 & G\ 4 \end{array} \\ \begin{bmatrix} 1.00 & 0.00 & 0.00 \\ 0.33 & 0.33 & 0.33 \\ 0.00 & 0.30 & 0.70 \\ 0.00 & 0.80 & 0.20 \end{bmatrix} \end{array}\right)$$

Here, for the first time, we see that the convention of referring to the row (unit) proportions—the ethnoracial composition of each unit—incorporates a particular bias in perspective about how to mathematize segregation. Two other strategies are possible: column proportions[7] and total table proportions.[8] If we were to choose column proportions then this illustration of group aggregation would become directly analogous to the previous illustration of unit aggregation (see discussion of *symmetry* axiom further along in this section of the paper); it would be as though we transposed the rows and columns. However, this merely transposes the bias in perspective. Namely, our attention would be focused on the enrollment distribution of each ethnoracial group across units; we would be privileging the ethnoracial group for attention rather than the unit in which one or more ethnoracial groups may be enrolled. For example, let us look at:

[7] Column proportions are symbolized as $\wp_{ij} = n_{ij}/n_{+j}$.

[8] Total table proportions are symbolized as $\rho_{ij} = n_{ij}/n_{++}$.

Aggregation 2 shown as column proportions ($\wp_{ij}$)

$$\begin{array}{c|cccc|} & W & Gr\,2 & Gr\,3 & Gr\,4 \\ Unit\ 1 & 0.31 & 0.73 & 0.00 & 0.00 \\ Unit\ 2 & 0.23 & 0.27 & 0.26 & 0.24 \\ Unit\ 3 & 0.00 & 0.00 & 0.32 & 0.67 \\ Unit\ 4 & 0.46 & 0.00 & 0.42 & 0.09 \end{array} \xrightarrow{aggregation} \begin{array}{c|cc|} & W & NW \\ Unit\ 1 & 0.31 & 0.23 \\ Unit\ 2 & 0.23 & 0.26 \\ Unit\ 3 & 0.00 & 0.34 \\ Unit\ 4 & 0.46 & 0.17 \end{array}$$

$$+\,Weight\ 1' \left(\begin{array}{c|c|} & W \\ Unit\ 1 & 0.31 \\ Unit\ 2 & 0.23 \\ Unit\ 3 & 0.00 \\ Unit\ 4 & 0.46 \end{array} \right) +\,Weight\ 2' \left(\begin{array}{c|ccc|} & G\ 2 & G\ 3 & G\ 4 \\ Unit\ 1 & 0.73 & 0.00 & 0.00 \\ Unit\ 2 & 0.72 & 0.26 & 0.24 \\ Unit\ 3 & 0.00 & 0.32 & 0.67 \\ Unit\ 4 & 0.00 & 0.42 & 0.09 \end{array} \right)$$

The bias is easily removed by working with total table proportions—and all index calculations will be unaffected by making this change—but the discussion becomes all the more abstract. We will address the matter of unit vs. group (row vs. column) bias further in the next section, which is grounded in the discussion of specific segregation indices.

Though it is agreed that additive decomposability is not required to define a segregation measure (e.g., James & Taeuber, 1985; Reardon & Firebaugh, 2002; Frankel & Volij, 2011), matters of *jurisdictional extent* would have no formalism without additive decomposability. That is, when it comes to the clustered (and hierarchical) organization of school governance, which is a unit-based form of organization, not being able to partition segregation into between and within components would be a serious shortcoming.

Axiom #6. Composition invariance. A calculated segregation index must not change when a given ethnoracial group is proportionally increased or decreased in size identically in every school. Mechanically, this says that when the joint frequencies in a single column of the two-way enrollment table are multiplied by the same number that segregation is unchanged. Whether the size of the group is proportionally scaled up, or scaled down, segregation must not change. For example, a segregation index that satisfies this axiom will have the same value following *Group Scaling 1*, where the number of students in Group 1 is multiplied by 10,

Group Scaling 1 shown as frequency counts (n_{ij})

$$\begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{group\ scaling} \begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 500 & 100 \\ 500 & 0 \\ 1000 & 0 \end{bmatrix} \end{array}$$

Group Scaling 1 shown as row proportions (π_{ij})

$$\begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{group\ scaling} \begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.83 & 0.17 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

or *Group Scaling 2*, where the number of students in Group 2 is multiplied by 5,

Group Scaling 2 shown as frequency counts (n_{ij})

$$\begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{group\ scaling} \begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 50 & 500 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array}$$

Group Scaling 2 shown as row proportions (π_{ij})

$$\begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{group\ scaling} \begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.09 & 0.91 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

Unlike size invariance (i.e., multiplying the whole table), a requirement for composition invariance remains disputed. Largely, this dispute revolves around the construct of exposure versus evenness. For example, if an ethnoracial group goes from being a very small share of the population—which occurs to Group 2 in *Group Scaling 1*—to a very large share of the population—which occurs to Group 2 in *Group Scaling 2*—then this group would go from high random probability of interaction with other groups in the population and low random probability of interaction with same-group peers (low isolation) to low probability of interaction with others and high probability of interaction with same-group peers (high isolation). Since composition invariance requires an index to remain unchanged under the circumstances illustrated above, group exposure changes are completely ignored.

On a technical level, composition invariance only makes sense if the analytic reference is the column (group) proportions, not the row (unit) proportions, because column proportions are invariant to the operation described in this section. For example, this is illustrated by *Group Scaling 1*, or *Group Scaling 2* for that matter, since both are identically represented using column proportions:

Group Scaling 1 shown as column proportions ($\wp_{ij}$)

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.25 & 1.00 \\ 0.25 & 0.00 \\ 0.50 & 0.00 \end{bmatrix} \end{array} \xrightarrow{group\ scaling} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.25 & 1.00 \\ 0.25 & 0.00 \\ 0.50 & 0.00 \end{bmatrix} \end{array}$$

Composition invariance requires that segregation remain unchanged when it is obvious that both isolation and imbalance have changed appreciably. In the case of *Group Scaling 1*, Group 2 has become a small fraction of the total enrollment, causing absolute imbalance to be diminished, and Group 1 is isolated throughout. In the case of *Group Scaling 2*, Group 2 has become the clear majority of the total enrollment, causing Group 2 to be strongly isolated in Unit 1, and the overall imbalance across units to be increased. These notable changes in isolation and imbalance make it difficult to accept the idea that the value of a segregation index should remain unchanged following the group scaling operation; composition invariance requires a reconceptualization of the meaning of segregation inconsistent with school desegregation measurement requirements.[9] In particular, since school desegregation is supposed to eliminate the condition that renders a school racially identifiable as a consequence of state action (*Swann v. Charlotte-Mecklenburg,* 1971), composition invariance is an axiom that does not facilitate attending to racially identifiable schools.[10]

Axiom #7. Group equivalence. (See Frankel & Volij, 2011.) A calculated segregation index must not change if we divide an ethnoracial group into smaller groups, each with proportionally identical enrollments across schools, nor does the calculated segregation index value change if we consolidate groups with proportionally identical school enrollments into a single ethnoracial group. For example, a segregation index that satisfies this axiom will have the same value following *Group Division 1*, where the 50 Unit 1, 50 Unit 2, and 100 Unit 3 students in Group 1 were evenly divided among Groups 1a and 1b,

Group Division 1 shown as frequency counts (n_{ij})

$$\begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{group\ division} \begin{array}{r c} & \begin{array}{ccc} Grp\ 1a & Grp\ 1b & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 25 & 25 & 100 \\ 25 & 25 & 0 \\ 50 & 50 & 0 \end{bmatrix} \end{array}$$

Group Division 1 shown as row proportions (π_{ij})

$$\begin{array}{r c} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{group\ division} \begin{array}{r c} & \begin{array}{ccc} Grp\ 1a & Grp\ 1b & Grp\ 2 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.16 & 0.16 & 0.67 \\ 0.50 & 0.50 & 0.00 \\ 0.50 & 0.50 & 0.00 \end{bmatrix} \end{array}$$

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[9] Composition invariance is an asymmetric requirement, which permits ethnoracial group membership to predict the school in which these students are enrolled, but not vice versa (Frankel & Volij, 2011, p. 8).

[10] A rarely invoked axiom, which is similar to Composition Invariance, is Enrollment Invariance (also called Organizational Invariance). This axiom requires that a single unit (row) may be multiplied by some constant so that the unit composition is proportionally identical, but its size has increased or decreased. Like Composition Invariance, Enrollment Invariance runs afoul of a number of considerations, but we will not address these details in this paper.

or *Group Division 2*, where the 100 Unit 1 students in Group 2 were evenly divided among Units 2a and 2b,

Group Division 2 shown as frequency counts (n_{ij})

$$\begin{array}{cc} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{group\ division} \begin{array}{cc} & \begin{array}{ccc} Grp\ 1 & Grp\ 2a & Grp\ 2b \end{array} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 50 & 50 & 50 \\ 50 & 0 & 0 \\ 100 & 0 & 0 \end{bmatrix} \end{array}$$

Group Division 2 shown as row proportions (π_{ij})

$$\begin{array}{cc} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{group\ division} \begin{array}{cc} & \begin{array}{ccc} Grp\ 1 & Grp\ 2a & Grp\ 2b \end{array} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.33 & 0.33 & 0.33 \\ 1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \end{bmatrix} \end{array}$$

(Note: Group divisions need not be even divisions of units between a pair of groups, only that the proportions of each unit in each resulting group remain identical.) Reversing each group division would be a group consolidation (i.e., combining Groups 1a and 1b back into Group 1, and Groups 2a and 2b back in to Group 2). Given our unequally distributed two-group starting configuration, we cannot illustrate a group consolidation following the group equivalence axiom, but we suspect that, at this point, an example would be unnecessary. Instead, we present *Group Division 1* and *Group Division 2* in terms of column (group) proportions, so that it is possible to see how group equivalence, like group invariance, requires an analytic reference to the column proportions. We have:

Group Division 1 shown as column proportions ($\wp_{ij}$)

$$\begin{array}{cc} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.25 & 1.00 \\ 0.25 & 0.00 \\ 0.50 & 0.00 \end{bmatrix} \end{array} \xrightarrow{group\ division} \begin{array}{cc} & \begin{array}{ccc} Grp\ 1a & Grp\ 1b & Grp\ 2 \end{array} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.25 & 0.25 & 1.00 \\ 0.25 & 0.25 & 0.00 \\ 0.50 & 0.50 & 0.00 \end{bmatrix} \end{array}$$

and

Group Division 2 shown as column proportions ($\wp_{ij}$)

$$\begin{array}{cc} & \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.25 & 1.00 \\ 0.25 & 0.00 \\ 0.50 & 0.00 \end{bmatrix} \end{array} \xrightarrow{group\ division} \begin{array}{cc} & \begin{array}{ccc} Grp\ 1 & Grp\ 2a & Grp\ 2b \end{array} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.25 & 1.00 & 1.00 \\ 0.25 & 0.00 & 0.00 \\ 0.50 & 0.00 & 0.00 \end{bmatrix} \end{array}$$

Group equivalence is problematic, however, because neither exchange nor transfer applies to groups. Ethnoracial identity may be arbitrary in that it is a social construction, but it is thoroughly real (reified) when it comes to segregation. White children do not suddenly become black children and vice versa (exchange), nor do Chinese children suddenly become Mexican children (transfer).[11] As noted previously, organizational equivalence is not similarly burdened because the school in which a child enrolls is fundamentally arbitrary and the child can be exchanged or transferred between schools; organizational identity changes are legitimate, but group identity changes are not.

Because there is no group-to-group equivalent of exchange and transfer, group equivalence cannot sensibly serve as the function that promotes division of consolidation, except by rare and fortuitous accident. Group equivalence is incapable of revealing any underlying heterogeneity within a collective ethnoracial group designation by group division; for example, Hispanics cannot be separated into Mexicans and other Latin Americans, unless the two more narrowly defined groups were genuinely present in direct proportion to each other across all schools. Further, this formalism only allows us to create ethnoracial heterogeneity where it did not exist (i.e., mixed super-group from distinct groups), by group

[11] This is not exactly true in the politics of desegregation, however, because parents presenting their children for enrollment would sometimes declare a different ethnoracial identity than previously declared so that their child would not be turned away and sent to the overflow school due group enrollment caps would be exceeded otherwise. (It was personally communicated to us that several years ago, in San Francisco, African American parents would identify their children as Native American in order to keep their child in their new neighborhood school rather be bussed elsewhere.)

consolidation, but only if two or more groups were truly present in direct proportion to one another across all schools. In other words, generally speaking, this axiom betrays the assumption of a mutually exclusive and exhaustive list of ethnoracial categories. Even though group equivalence would provide complementary symmetry with organizational equivalence, its invocation fundamentally threatens the integrity of any ethnoracial group designation system and invalidates the whole idea of segregation analysis as we presently understand it.

Axiom #8. Symmetry. Ethnoracial group labels and school labels are arbitrary and without any necessary order, so any interchange of rows or columns in the two-way contingency table of enrollment data must result in no change in the calculated segregation index value. For example, a segregation index that satisfies this axiom will have the same value following *Interchange 1*, where Unit 1 becomes the second row, and Unit 2 becomes the first row,

Interchange 1 shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 100 \\ 50 & 0 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{interchange} \begin{array}{c} \\ Unit\ 2 \\ Unit\ 1 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 50 & 0 \\ 50 & 100 \\ 100 & 0 \end{bmatrix} \end{array}$$

Interchange 1 shown as row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.33 & 0.67 \\ 1.00 & 0.00 \\ 1.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{interchange} \begin{array}{c} \\ Unit\ 2 \\ Unit\ 1 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 1.00 & 0.00 \\ 0.33 & 0.67 \\ 1.00 & 0.00 \end{bmatrix} \end{array}$$

Since this operation is straightforward, we do not illustrate the interchange of Groups 1 and 2. For both unit and group interchange, the column proportions remain the same, as did the row proportions in the interchange just illustrated.

In addition to these interchange operations, symmetry implies that any aggregative operation on the rows (e.g., clustering schools into districts) may be similarly applied to the columns (e.g., clustering the white and Asian ethnoracial groups into an overrepresented-in-higher-education super-group and clustering the Hispanic, American Indian, and African American ethnoracial groups into an underrepresented-in-higher-education super-group). Also, as noted in the related discussion of composition invariance, interest in a specific ethnoracial group by its identity is not possible if symmetry is required (i.e., focusing on a specific group means the label is no longer arbitrary; it has specific meaning or significance). As discussed by Lieberson and Carter (1982) and Massey and Denton (1988), for example, exposure defined as intergroup interaction is not equivalent for any pairing of groups that are not of equal size—the interaction of whites with non-whites (${}_WP_N^*$) is not that same as interaction of non-whites with whites (${}_NP_W^*$). In other words, the symmetry axiom is what sets apart the two *desegregation constructs,* namely, evenness from exposure, because repeating the calculation of a P^*-type exposure index following an interchanging of columns would generate one of two conceptually different results: 1) the oppositely referenced interaction index described in the previous sentence; or 2) the isolation of a group different from the one calculated prior to interchange. Exposure is not a symmetric construct.

Axiom #9. Range. The numerical values of the segregation index must range from zero to one, inclusive. (Typically, zero is no segregation, and one is complete segregation.) Though not strictly required, many analysts demand the *range* axiom to set commonly recognizable bounds on the segregation index values. This range normalization helps to identify whether fully 100% (i.e., maximum value of 1 × 100%) of the possible desegregation has been achieved for any given distribution of ethnoracial-group-identified students across units, but it obscures the absolute sense of how many students have to be reassigned in order achieve maximum possible desegregation. For example, in a district that is 98% Hispanic there just are not that many students who would have to participate in desegregative moves to eliminate racial imbalance even if nearly all of them are concentrated in a single school. This was the case in 2006 for California’s Coachella Valley Unified School District elementary schools, where Mountain Vista Elementary had only 9.7% of the district’s total enrollment of 9,690 elementary students, but 41.7% of the district’s 187 non-Hispanic elementary students. The range axiom is particularly important when it comes to *interpretability.*

A Summary of Index Concepts and Considerations

To summarize, the agreed upon axioms of exchange, transfer, organizational equivalence, and size invariance are particularly important to permitting comparisons across space or time. The property of additive decomposability remains compelling because it makes possible the incorporation of jurisdictional extent into a desegregation analysis. However, we argue that the contested axioms of composition invariance and group equivalence are, indeed, inappropriate constraints on the construction and selection of a segregation index. The symmetry axiom is one that distinguishes between evenness and exposure, which means that its appropriateness depends on the sort of segregation index we wish to construct. And, the range axiom remains important, but we must remain aware of the specific interpretation that accompanies its use.

Assessment of Established Segregation Indices

A wide variety of segregation indices have appeared in the social sciences literature (see, e.g., Frankel & Volij, 2011; Johnston, Poulsen, & Forrest, 2007; Massey & Denton, 1988; Massey, White, & Phua, 1996; Mora & Ruiz-Castillo, 2011; Reardon & O'Sullivan, 2004). Many of them have been compared, their various characteristics highlighted, and their virtues declared. However, not all of them have been assessed both in terms of the eight conceptual issues and the nine axioms or properties identified here, and nowhere has this evaluation been brought together in a single treatment. In the following analysis, we provide a full assessment of established and recently developed evenness and exposure segregation indices within the school desegregation framework articulated above.

Here, we consider eleven segregation indices for their value in communicating the extent of ethnoracial segregation in the schools and monitoring desegregation progress. Unique to this review, we illustrate what can be learned from them in terms of whether specific schools should be targeted for their contribution to segregation within a district—schools deserving special scrutiny—not just whether a district is segregated or making overall progress toward desegregation.

Before specifying each index, we need to define our mathematical notation for index calculation. The two-way contingency table of assigned units (e.g., groups, tracks, classrooms, schools, etc.) by ethnoracial groups enrollment data has u units (rows) and g groups (columns). The joint frequencies in the table are given by n_{ij}, where i identifies the rows (numbering from 1 to u) and j identifies the columns (numbering from 1 to g). The row and column marginal totals are given by $n_{i+} = \sum_{j=1}^{g} n_{ij}$ and $n_{+j} = \sum_{i=1}^{u} n_{ij}$, respectively, and the total frequency by

$n_{++} = \sum_{i=1}^{u} \sum_{j=1}^{g} n_{ij}$. The joint proportions are given by $\rho_{ij} = \frac{n_{ij}}{n_{++}}$, the row proportions by $\rho_{i+} = \frac{n_{i+}}{n_{++}}$, and the column proportions by $\rho_{+j} = \frac{n_{+j}}{n_{++}}$.

Evenness Indices

We begin with indices that are expressions of deviations from a reference distribution. Typically, the reference distribution is the observed group composition and unit enrollment distribution of the set of units (classrooms, schools, districts, etc.) being considered. In other words, the reference distribution is the joint frequencies expected based upon the observed row and column marginal totals of the two-way contingency table. Deviations of observed frequencies from expected frequencies are most frequently modeled as absolute differences, squared differences (or variance ratios), and logarithmic differences (or information differences). As discussed below, with varying levels of subtlety and sophistication, nearly all of the indices presented in this subsection address the *desegregation construct* labeled evenness. The notion of evenness, however, is not limited to continuous and probabilistic models. As noted in the discussion of desegregation constructs and compliance margins, above, the literature identifies a threshold-based approach to defining evenness, which is labeled racial (im)balance when modeled this way. In particular, depending on the circumstances, balance has been defined as a unit (e.g., a school) with an ethnoracial group enrollment percentage that is within 10 (or 15, or 20) percentage points of the overall ethnoracial group enrollment for the collection of units (e.g., a district).

Absolute difference indices. The Dissimilarity Index (D) and the Gini Index (G) take distinctive absolute differences as their definition for what kind of deviations add up to segregation. They can be found in both normalized and unnormalized (unadjusted) forms in the literature (e.g., Agresti, 2007; Frankel & Volij, 2011; Reardon & Firebaugh, 2002). They are normalized by the Simpson Index (I)[12] because the Simpson Index defines the maximum aggregate ethnoracial diversity from which any enrollment distribution may differ (e.g., if the units are schools, the Simpson Index is a measure of the district's ethnoracial diversity and defines the maximum amount from which perfectly segregated schools may differ from completely desegregated schools). The unadjusted forms are expressed here in terms of both frequencies and proportions:

Dissimilarity Index, unadjusted

$$D_{unadj} = \frac{1}{2n_{++}}\sum_{j=1}^{g}\sum_{i=1}^{u}\left|n_{ij} - \mu_{ij}\right| = \frac{1}{2}\sum_{j=1}^{g}\sum_{i=1}^{u}\left|\left(\rho_{ij} - \rho_{i+}\rho_{+j}\right)\right|$$

Gini Index, unadjusted

$$G_{unadj} = \frac{1}{2n_{++}^{2}}\sum_{j=1}^{g}\sum_{i=1}^{u}\sum_{k=1}^{u}\left|n_{k+}n_{ij} - n_{i+}n_{kj}\right| = \frac{1}{2}\sum_{j=1}^{g}\sum_{i=1}^{u}\sum_{k=1}^{u}\left|\rho_{k+}\rho_{ij} - \rho_{i+}\rho_{kj}\right|$$

where k indexes the first of two row sums, and $\mu_{ij} = \frac{n_{i+}n_{+j}}{n_{++}}$. For brevity's sake, the normalized forms, which satisfy the *range* axiom, are expressed in terms of proportions only:

Dissimilarity Index

$$D = \frac{1}{2I}\sum_{j=1}^{g}\sum_{i=1}^{u}\left|\left(\rho_{ij} - \rho_{i+}\rho_{+j}\right)\right|$$

Gini Index

$$G = \frac{1}{2I}\sum_{j=1}^{g}\sum_{i=1}^{u}\sum_{k=1}^{u}\left|\rho_{k+}\rho_{ij} - \rho_{i+}\rho_{kj}\right|$$

where $I = \sum_{j=1}^{g}\rho_{+j}\left(1 - \rho_{+j}\right)$.

[12] The Simpson Index is also a popular measure of diversity (Patil & Taillie, 1982), which is often expressed as $I = 1 - \sum_{j=1}^{g}\rho_{+j}^{2}$, and it is equivalent to the form provided in the text above.

The absolute differences that define the Dissimilarity Index add up to how many individuals have to be moved to match the reference distribution (i.e., the sum of the absolute deviations from the expected proportion in every cell, using row and column proportions to calculate the expected value). More accurately, the result of the sum, after dividing by two, is the number students who have to be exchanged between units in order to balance—completely desegregate—enrollments across all units. (When following through with the full specification and dividing by the total enrollment, n_{++}, D_{unadj} provides the proportion of the total enrollment that much be exchanged to achieve complete desegregation; and when further dividing by Simpson's Index, I, D provides the proportion all possible exchanges that must be made to completely desegregate.) This has a wonderful *simplicity* and *interpretability* to it, but practical desegregation is about falling within *compliance margins,* which establish a different set of targets from the singular target of complete desegregation. Compliance margins not only provide a range of target values within which the index must fall, they also define a set of possible enrollment configurations too numerous to list (Mitchell & Mitchell, 2010).

Another challenge faced by the Dissimilarity index is that such a sum of unweighted absolute differences is insensitive to concentrated imbalances.[13] Since the *exchange* axiom specifies how segregation becomes lower depending on the relative proportions of the ethnoracial groups in each of the units between which exchange occurs, we must attend to how concentrated (how large or small the proportion of) each ethnoracial group is. Insensitivity to concentration is why the Dissimilarity Index fails to meet the requirement of the *exchange* axiom. In other words, "a [segregative] movement will not always cause this index to indicate

[13] Mathematically, the Dissimilarity Index's insensitivity to concentration arises because it is composed of linear addition of deviation frequencies, regardless of where the deviations occur or how large they are.

more [segregation]" (Hutchens, 1991, p. 47). Nor will a desegregative movement always cause the index to indicate more desegregation.

Let's consider two exchanges where the Dissimilarity Index fails to register (de)segregation so that this point is clear. First, any desegregative exchange starting from the configuration on the left (see *Exchange 3*, below), where exchange increases the proportion of Group 1 in Unit 1 (by transfer of a Group 1 member of Unit 2, where the proportion is higher, into Unit 1; $\pi_{21} > \pi_{11}$) and increases the proportion of Group 2 in Unit 2 (by reciprocal transfer of a Group 2 member of Unit 1, where the proportion is higher, into Unit 2; $\pi_{12} > \pi_{22}$), leaves the Dissimilarity Index value unchanged. Consider

Exchange 3 shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 67 & 83 \\ 33 & 17 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{exchange} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 68 & 82 \\ 32 & 18 \\ 100 & 0 \end{bmatrix} \end{array}$$

Exchange 3 shown as row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.447 & 0.553 \\ 0.660 & 0.340 \\ 1.000 & 0.000 \end{bmatrix} \end{array} \xrightarrow{exchange} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.453 & 0.547 \\ 0.640 & 0.360 \\ 1.000 & 0.000 \end{bmatrix} \end{array}$$

Before and after exchange, $D = 0.500$. Second, any segregative exchange starting from the configuration on the left (see *Exchange 4*, below), where the exchange increases the proportion of Group 1 in Unit 1 (by transfer of a Group 1 member of Unit 2, where the proportion is lower, into Unit 1; $\pi_{21} \not> \pi_{11}$, but instead $\pi_{21} < \pi_{11}$) and increases the proportion of Group 2 in Unit 2 (by reciprocal transfer of a Group 2 member of Unit 1, where the proportion is lower, into Unit 2; $\pi_{12} \ngtr \pi_{22}$, but instead $\pi_{12} < \pi_{22}$), leaves the Dissimilarity Index value unchanged. Consider

Exchange 4 shown as frequency counts (n_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 76 & 74 \\ 24 & 26 \\ 100 & 0 \end{bmatrix} \end{array} \xrightarrow{exchange} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 77 & 73 \\ 23 & 27 \\ 100 & 0 \end{bmatrix} \end{array}$$

Exchange 4 shown as row proportions (π_{ij})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.507 & 0.493 \\ 0.480 & 0.520 \\ 1.000 & 0.000 \end{bmatrix} \end{array} \xrightarrow{exchange} \begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{cc} Grp\ 1 & Grp\ 2 \end{array} \\ \begin{bmatrix} 0.513 & 0.487 \\ 0.460 & 0.540 \\ 1.000 & 0.000 \end{bmatrix} \end{array}$$

Again, before and after exchange, $D = 0.500$. These results are in contrast to the behavior of the Gini Index, and the other evenness indices to follow.

The Gini Index is sensitive to how concentrated imbalances are from one unit to the next; it sums weighted differences for ethnoracial groups between units (i.e., a sum of the absolute values of the differences between all pairs of cells when each cell's proportion is multiplied by the row proportion of the comparison cell).[14] Weighting (multiplying) by the size of the enrollment in a given unit makes the calculation sensitive to whether ethnoracial imbalances are concentrated in particular units or more evenly distributed across all units. This is what it takes to satisfy the *exchange* axiom. In the desegregative case of *Exchange 3*, above, before exchange, $G = 0.580$, and after exchange, $G = 0.570$, as required. Similarly, in the

[14] Because the weights are unit (row) totals rather than joint frequencies, the differences are not precisely differences of squared terms, but this weighting scheme does cause the differences to be in units of persons-squared and not linear in persons. This is important because non-linearity creates sensitivity to concentration. For example, this is most simply seen with the Concentration Index, or Hirschman-Herfindahl Index (Theil, 1972, p. 42), which is simply the sum of squared frequencies scaled to the population size (i.e., it is actually written in terms of proportions, not frequencies: $C = \sum_{i=1}^{n} \rho_i^2$).

segregative case of $Exchange\ 4$, before exchange, $G = 0.510$, and after exchange, $G = 0.520$, also as required.

Neither the Dissimilarity Index nor the Gini Index satisfies the *transfer* axiom (Reardon & Firebaugh, 2002). One exception is that the Gini Index satisfies the transfer axiom in the case of two ethnoracial groups, but this is not adequate for measuring segregation where there is additional *ethnoracial diversity* (i.e., three or more ethnoracial groups). This is a rather devastating critique because so many of the school districts throughout the United States are multiethnic (i.e., three or more ethnoracial groups represented). This failure to generally satisfy the transfer axiom means that both indices are inadequate for *comparisons across space or time*. Using these indices in a multigroup context, we will neither reliably assess whether a district is more or less segregated on separate occasions, when school-to-school enrollments are sure to have changed, nor accurately compare two districts that do not share proportionally identical enrollments across the same number of schools.

Additionally, neither index satisfies the *additive decomposability* property (Reardon & Firebaugh, 2002). This makes them both inadequate for evaluating questions of *jurisdictional extent.* Segregation cannot be accurately apportioned between that which is due to imbalances across districts, for example, and that which is due to imbalances within districts. At a practical (administrative) level, with two exceptions, this is not such a devastating critique. The first exception is where metropolitan desegregation remedies are legally required because these are fundamentally multi-jurisdictional. The second exception is found in large urban and county districts where sub-districts or other quasi-independent administrative structures govern geographical sub-regions within the larger district. Regardless, when it comes to the politics of desegregation, questions of jurisdictional extent may be revisited, at least rhetorically, if not relitigated. As a consequence, desegregation policies and politics require that there be available a segregation index that satisfies additive decomposability.

Given that racially isolated schools are subject to judicial scrutiny regardless of the net balance of enrollment across all schools in a district (see *Swann v. Charlotte-Mecklenburg,* 1971; also note 1), we have to ask whether the Dissimilarity Index or the Gini Index can assess both the district and an individual school as the *organizational unit of analysis.* Unit-by-unit contributions to the total sum for the Dissimilarity Index can be isolated for a description of the extent of imbalance in each school (unit), but these imbalances do not necessarily highlight single-race isolation, nor do they qualitatively agree with the unit-by-unit assessment that would be obtained from the Gini Index in all cases. For example, consider *Multiple Exchanges 1*, a three-school district with three ethnoracial groups in which multiple exchanges take place, specifically, 50 students from Group 1 are exchanged with 50 from Group 2 between Unit 1 and Unit 2:

Multiple Exchanges 1 shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 50 & 100 & 50 \\ 50 & 0 & 0 \\ 100 & 0 & 0 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 100 & 50 & 50 \\ 0 & 50 & 0 \\ 100 & 0 & 0 \end{bmatrix} \end{array}$$

Multiple Exchanges 1 shown as row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ 1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 0.50 & 0.25 & 0.25 \\ 0.00 & 1.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \end{bmatrix} \end{array}$$

Now, let's look at the components of D and G for each school (Unit indexed by i = 1 ... 3):

Multiple Exchanges 1 showing Dissimilarity Index components (D_{i+})

$$\begin{array}{c c} & D_{i+}^{before} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.321 \\ 0.107 \\ 0.214 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{c c} & D_{i+}^{after} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.107 \\ 0.179 \\ 0.214 \end{bmatrix} \end{array}$$

Multiple Exchanges 1 showing Gini Index components (G_{i+})

$$\begin{array}{c c} & G_{i+}^{before} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.321 \\ 0.107 \\ 0.214 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{c c} & G_{i+}^{after} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.250 \\ 0.179 \\ 0.214 \end{bmatrix} \end{array}$$

The initial (before) assessment is the same for D and G. Both D and G draw our attention to the first school (Unit 1) as most out of balance—making the largest contribution to segregation. However, after *Multiple Exchange 1*, D and G draw our attention to different schools (units) for where the largest contribution to segregation is coming from.

These results are noteworthy for three reasons. First, for both D and G, the initial assessment identifies the school with the greatest ethnoracial diversity (Unit 1) as making the greatest contribution to segregation, not either of the racially isolated schools (i.e., where π_{21} = π_{31} = 1.00). Second, D and G do not agree on which school contributes most to observed segregation following *Multiple Exchanges 1*. In this latter situation, where there are still two racially isolated schools (i.e., where π_{22} = π_{31} = 1.00), D identifies the third school (Unit 3), a racially isolated school, as contributing most to segregation, while G continues to identify the first school (Unit 1) as contributing more to segregation than either of the racially isolated schools. Third, an alternative interpretation about the meaning of racial isolation might be appropriate. Namely, G consistently identifies the first school (Unit 1) as contributing the most to segregation because all of the members of Group 2 and Group 3 are packed into this one school rather than distributed among any of the other two schools.

In order to sort out whether G is pointing us to a better interpretation of what constitutes a racially isolated school, two examples, in addition to the one above, are necessary. For our second example, consider *Multiple Transfers 1*, which has the same starting enrollment as *Multiple Exchanges 1*, but this time 50 members of Group 2 are transferred from Unit 1 to Unit 2 without a reciprocal transfer Group 1 members.

Multiple Transfers 1 shown as frequency counts (n_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 50 & 100 & 50 \\ 50 & 0 & 0 \\ 100 & 0 & 0 \end{bmatrix} \end{array} \xrightarrow{multiple\ tranfers} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 50 & 50 & 50 \\ 50 & 50 & 0 \\ 100 & 0 & 0 \end{bmatrix} \end{array}$$

Multiple Transfers 1 shown as row proportions (π_{ij})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ 1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{multiple\ tranfers} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 0.33 & 0.33 & 0.33 \\ 0.50 & 0.50 & 0.00 \\ 1.00 & 0.00 & 0.00 \end{bmatrix} \end{array}$$

Now, let's look at the components of D and G for each school (Unit indexed by i = 1 … 3):

Multiple Transfers 1 showing Dissimilarity Index components (D_{i+})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} D_{i+}^{before} \\ \begin{bmatrix} 0.321 \\ 0.107 \\ 0.214 \end{bmatrix} \end{array} \xrightarrow{multiple\ tranfers} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} D_{i+}^{after} \\ \begin{bmatrix} 0.179 \\ 0.107 \\ 0.214 \end{bmatrix} \end{array}$$

Multiple Transfers 1 showing Gini Index components (G_{i+})

$$\begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} G_{i+}^{before} \\ \begin{bmatrix} 0.321 \\ 0.107 \\ 0.214 \end{bmatrix} \end{array} \xrightarrow{multiple\ tranfers} \begin{array}{c} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} \begin{array}{c} G_{i+}^{after} \\ \begin{bmatrix} 0.214 \\ 0.143 \\ 0.214 \end{bmatrix} \end{array}$$

After *Multiple Transfers 1*, D and G both draw our attention to the third school (Unit 3) for where the largest contribution to segregation is coming from, which is now the only schools with an isolated ethnoracial group, but G still draws our attention to the first school (Unit 1) as contributing just as much to segregation as the third school. This time, it is a less plausible to see this behavior of G as due to a better alternative understanding of racial isolation because the members of Group 2 are now spread across two schools and, though all of the members of Group 3 are still enrolled in a single school (Unit 1), 100 members of Group 1 remain completely isolated from all other groups in the third school (Unit 3).

For our third example, consider *Organizational Division 3*, in which the first school (Unit 1) is split into two equal-sized and identically composed schools (Units 1a and 1b) that are half the size of the original school.

Organizational Division 3 shown as frequency counts (n_{ij})

$$\begin{array}{r c} & \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 50 & 100 & 50 \\ 50 & 0 & 0 \\ 100 & 0 & 0 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{r c} & \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{array}{r} Unit\ 1a \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 1b \end{array} & \begin{bmatrix} 25 & 50 & 25 \\ 50 & 0 & 0 \\ 100 & 0 & 0 \\ 25 & 50 & 25 \end{bmatrix} \end{array}$$

Organizational Division 3 shown as row proportions (π_{ij})

$$\begin{array}{r c} & \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ 1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{r c} & \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{array}{r} Unit\ 1a \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 1b \end{array} & \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ 1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 0.25 & 0.50 & 0.25 \end{bmatrix} \end{array}$$

Now, let's look at the components of D and G for each school (Unit indexed by i = 1 … 3):

Organizational Division 3 showing Dissimilarity Index components (D_{i+})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} D_{i+}^{before} \\ \begin{bmatrix} 0.321 \\ 0.107 \\ 0.214 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{r} \\ Unit\ 1a \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 1b \end{array}\begin{array}{c} D_{i+}^{after} \\ \begin{bmatrix} 0.161 \\ 0.107 \\ 0.214 \\ 0.161 \end{bmatrix} \end{array}$$

Organizational Division 3 showing Gini Index components (G_{i+})

$$\begin{array}{r} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} G_{i+}^{before} \\ \begin{bmatrix} 0.321 \\ 0.107 \\ 0.214 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{r} \\ Unit\ 1a \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 1b \end{array}\begin{array}{c} G_{i+}^{after} \\ \begin{bmatrix} 0.161 \\ 0.107 \\ 0.214 \\ 0.161 \end{bmatrix} \end{array}$$

After *Organizational Division 3*, D and G both draw our attention to the third school (Unit 3) for where the largest contribution to segregation is coming from. This is the larger of the two racially isolated schools (the other is Unit 2). For this third example, both d and g unequivocally identify the same racially isolated school as contributing the most to the observed segregation. What we learn from the three examples, considered together, is that neither of these indices can be used reliably to draw the attention of the courts to racially isolated schools, where judicial scrutiny is required, because we have a basic inconsistency. Clearly, the unit-by-unit components of segregation that sum to give the index values of D and G are not reliable for identifying racially isolated schools (units).

Finally, for the Dissimilarity Index and the Gini Index, as well as nearly all of the indices to follow, there has been no systematic investigation of how to define *compliance margins* around complete desegregation for any definition of minimally acceptable desegregation. Mitchell and Mitchell (2010) demonstrated a Monte Carlo simulation approach for understanding the relationship between compliance margins and index values, but their study was limited to the behavior of Theil's H. Moreover, their study did not explore what it would mean to use a value of Theil's H itself as a boundary for compliance. Certainly, the limits of desegregation accountability would be more readily defined if work such as that by Mitchell and Mitchell were extended and applied to more segregation indices. This and all of the foregoing discussion of the Dissimilarity Index and Gini Index is summarized in Table 1.

[INSERT TABLE 1 ABOUT HERE.]

Variance-ratio-based indices. The three established variance-ratio-based or chi-squared-like indices are the square of Cramer's V (hereafter Cramer's V^2, or simply V^2), the Goodman-Kruskal $\tau_{y|x}$ (or simply $\tau_{y|x}$, or τ_{ij} when row and column indices are required), and the Normalized Exposure Index (P). (E.g., see Conover, 1999; Goodman & Kruskal, 1979; Reardon & Firebaugh, 2002.) These are normalized indices by construction (i.e., satisfy the *range* axiom by having possible values from 0 to 1). The expressions for these indices in terms of both frequencies and proportions are as follows (assuming the number of units, u, is greater than or equal to the number of ethnoracial groups, g, for V^2; otherwise, instead of g−1, the denominator for V^2 should have u−1).

Cramer's V^2

$$V^2 = \frac{\chi^2}{n_{++}(g-1)} = \frac{1}{n_{++}(g-1)} \sum_{j=1}^{g} \sum_{i=1}^{u} \frac{\left(n_{ij} - \mu_{ij}\right)^2}{\mu_{ij}} = \frac{1}{(g-1)} \sum_{j=1}^{g} \sum_{i=1}^{u} \frac{\left(\rho_{ij} - \rho_{i+}\rho_{+j}\right)^2}{\rho_{i+}\rho_{+j}}$$

Goodman-Kruskal $\tau_{y|x}$

$$\tau_{y|x} = \frac{1}{n_{++}I} \sum_{j=1}^{g} \frac{n_{+j}}{n_{++}} \sum_{i=1}^{u} \frac{\left(n_{ij} - \mu_{ij}\right)^2}{\mu_{ij}} = \frac{1}{I} \sum_{j=1}^{g} \rho_{+j} \sum_{i=1}^{u} \frac{\left(\rho_{ij} - \rho_{i+}\rho_{+j}\right)^2}{\rho_{i+}\rho_{+j}}$$

Normalized Exposure Index

$$P = \frac{1}{n_{++}} \sum_{j=1}^{g} \left(\frac{n_{+j}}{n_{++} - n_{+j}} \right) \sum_{i=1}^{u} \frac{\left(n_{ij} - \mu_{ij}\right)^2}{\mu_{ij}} = \sum_{j=1}^{g} \left(\frac{\rho_{+j}}{1 - \rho_{+j}} \right) \sum_{i=1}^{u} \frac{\left(\rho_{ij} - \rho_{i+}\rho_{+j}\right)^2}{\rho_{i+}\rho_{+j}}$$

Where $\chi^2 = \sum_{j=1}^{c} \sum_{i=1}^{r} \frac{\left(n_{ij} - \mu_{ij}\right)^2}{\mu_{ij}}$ (Pearson's chi-squared).

There are two important differences among these three indices. First, each index places a different importance (weight) on the contribution of small ethnoracial groups to the degree of overall segregation—the evenness of dispersion for a group with a small share of the total enrollment is not equally important for each index. Cramer's V^2 has no group weight, so it does not discount small groups in any way. The relative deviation from the expected cell frequency for each ethnoracial group is equally valued. However, the Goodman-Kruskal $\tau_{y|x}$ weights contributions in direct proportion to the group's representation in the district (ρ_{+j}), which means it discounts small ethnoracial groups in direct proportion to their representation (i.e., the smaller the group is, the smaller its contribution to overall segregation). Finally, the Normalized Exposure Index weights contributions in proportion to the odds of group representation in the district ($\rho_{+j}/(1 - \rho_{+j})$), which substantially discounts small ethnoracial groups relative to any group in the plurality—plurality ethnoracial groups dominate the calculation of this index. For example, if the proportion of the plurality group is 0.30 of the total (say, among four fairly but certainly not perfectly evenly distributed ethnoracial groups) then its weight is 0.43, but if the plurality group proportion is 0.40 of the total then its weight is 0.67 (i.e., a 33% increase in the plurality results in a 56% increase in the weight). Further, if the plurality proportion is 0.60 (a solid majority), the weight becomes 1.50, which is 3.5 times the weight for a group only half the size. That is, the magnitude of the Normalized Exposure Index is least affected by the unevenness in the distribution of a small group across the schools in a district—small groups are strongly discounted.

The second major difference has to do with index *additive decomposability.* All three exhibit additive organizational decomposability—within and between units (e.g., schools) segregation—for two ethnoracial groups (i.e., for cases with severely limited *ethnoracial diversity*). However, only the Goodman-Kruskal $\tau_{y|x}$ is additively decomposable for any number of groups. None of the three indices, as given above, can be decomposed into an expression for within and between ethnoracial groups segregation, which is called additive group decomposability (Reardon & Firebaugh, 2002). However, transposing rows and columns (units and groups) would give a Goodman-Kruskal $\tau_{x|y}$ that would have the additive group decomposability property,[15] but then it would no longer allow for additive organizational decomposition.

Unfortunately, the additive decomposability of these indices, even in the most limited cases, does not make up for their faults. None of the variance-ratio-based indices satisfies the *transfer* axiom for more than two ethnoracial groups (Reardon & Firebaugh, 2002), which makes them inadequate for measuring multigroup segregation (i.e., where there is genuine *ethnoracial diversity*). Because the transfer axiom is not generally satisfied, the indices are inadequate for *comparisons across space or time*. Further, because additive decomposability is not a property of all of these indices, only $\tau_{y|x}$ would be adequate, and only in the most ethnoracially limited situations, for evaluating questions of *jurisdictional extent.*

[15] Notice that the subscripts are reversed because this measure is directional; talking about distributions within rows gives one form of τ, where as transposing the discussion into one about distributions within columns gives this new, reversed subscripts, version.

We may still ask whether the variance-ratio-based indices can assess both the district and an individual school as the *organizational unit of analysis.* Unit-by-unit contributions to the total sum for each of these indices can be isolated for a description of the extent of imbalance in each school (unit). For example, consider the previously introduced *Multiple Exchanges 1*, for each of these variance-ratio-based indices by looking at the components of V^2, $\tau_{y|x}$, and P for each school (Unit indexed by i = 1 ... 3):

Multiple Exchanges 1 showing Cramer's V^2 components (V_{i+}^2)

$$\begin{array}{cc} & V_{i+}^{2\,(before)} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.121 \\ 0.054 \\ 0.107 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{cc} & V_{i+}^{2\,(after)} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.027 \\ 0.179 \\ 0.107 \end{bmatrix} \end{array}$$

Multiple Exchanges 1 showing Goodman-Kruskal $\tau_{y|x}$ components (τ_{i+})

$$\begin{array}{cc} & \tau_{i+}^{before} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.161 \\ 0.071 \\ 0.143 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{cc} & \tau_{i+}^{after} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.018 \\ 0.214 \\ 0.143 \end{bmatrix} \end{array}$$

Multiple Exchanges 1 showing Normalized Exposure Index components (P_{i+})

$$\begin{array}{cc} & P_{i+}^{before} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.182 \\ 0.081 \\ 0.162 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{cc} & P_{i+}^{after} \\ \begin{array}{c} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.015 \\ 0.214 \\ 0.162 \end{bmatrix} \end{array}$$

The initial (before) assessment is the same for all three indices; they draw our attention to the first school (Unit 1) as most out of balance—making the largest contribution to segregation. However, after *Multiple Exchanges 1*, they draw our attention to the second school (Unit 2) for where the largest contribution to segregation is coming from.

These results are noteworthy for two reasons. First, as with the Dissimilarity Index and the Gini Index before them, the initial assessment by the three variance-ratio-based indices identifies the school with the greatest ethnoracial diversity (Unit 1) as making the greatest contribution to segregation, not either of the racially isolated schools. Second, unlike the absolute difference indices, the variance-ratio-based indices agree on which school contributes most to observed segregation following *Multiple Exchanges 1*. In this latter situation, where there are again two racially isolated schools, all three identify the second school (Unit 2), a racially isolated school, as contributing most to segregation. Since both Groups 1 and 2 are distributed across two schools, rather than only Group 1 being distributed across all three schools and Groups 2 and 3 enrolled only in the first school (Unit 1), it certainly makes sense that the first school is no longer the largest contributor to observed segregation. However, closer inspection of the expected frequencies is required to understand why the second rather than the third school would be identified as the greater contributor to observed segregation. This happens because the expected frequency of Group 2 in Unit 2 is very small and its contribution is inversely proportional to the expected cell frequency (i.e., it is made very large). As a consequence, the smaller rather than the larger racially isolated school makes the larger contribution to the observed segregation.

For the other two examples used to illustrate D and G, namely, *Multiple Transfers 1* and *Organizational Division 3*, the larger (or only) racially isolated school is consistently identified as the making the largest contribution to the observed segregation for V^2, $\tau_{y|x}$, and P. What we learn from the three examples, considered together, is that none of these indices can be used reliably to draw the attention of the courts to racially isolated schools, where judicial scrutiny is required, because we have a basic inconsistency. However, the three variance-ratio-based indices are less likely to draw our attention to other schools than the absolute difference indices.

Finally, as noted in the discussion of the absolute difference indices, there has been no systematic investigation of how to define *compliance margins* around complete desegregation for any definition of minimally acceptable desegregation using the three variance-ratio-based indices. This and all of the foregoing discussion is summarized in Table 1.

Information theory indices. (Also see Frankel & Volij, 2011; Mora & Ruiz-Castillo, 2011.) Substantial recent attention has been given to the information theory indices presented or clearly related to those developed by Henri Theil (e.g., Theil, 1972), namely, Theil's H, the Mora-Ruiz-Castillo H*, and Theil's M (e.g., Frankel & Volij, 2011; Mora & Ruiz-Castillo, 2011; Reardon & Firebaugh, 2002). The M index is not normalized. The H and H* indices are specific normalizations of M, dividing M by the ethnoracial group composition (column) entropy and assigned unit distribution (row) entropy, respectively[16] (thereby satisfying the *range* axiom). Before providing the expression for M, we first define the joint entropy ($E^{(u \cdot g)}$), the ethnoracial group entropy ($E^{(g)}$), and the assigned unit entropy ($E^{(u)}$):

$$E^{(u \cdot g)} = \sum_{i=1}^{u} \sum_{j=1}^{g} \frac{n_{ij}}{n_{++}} \ln\left(\frac{n_{++}}{n_{ij}}\right) = \sum_{i=1}^{u} \sum_{j=1}^{g} \rho_{ij} \ln\left(\frac{1}{\rho_{ij}}\right)$$

$$E^{(g)} = \sum_{j=1}^{g} \frac{n_{+j}}{n_{++}} \ln\left(\frac{n_{++}}{n_{+j}}\right) = \sum_{j=1}^{g} \rho_{+j} \ln\left(\frac{1}{\rho_{+j}}\right)$$

$$E^{(u)} = \sum_{i=1}^{u} \frac{n_{i+}}{n_{++}} \ln\left(\frac{n_{++}}{n_{i+}}\right) = \sum_{i=1}^{u} \rho_{i+} \ln\left(\frac{1}{\rho_{i+}}\right)$$

[16] The Uncertainty Coefficient, (e.g., see SAS Institute, 2008), depending on whether one chooses the column or row variable as the dependent variable, is identical to H or H*, respectively.

where $\ln(x)$ is the natural logarithm of x (logarithm of x to the base e).

Now, we can provide a succinct expression for Theil's M:

$$M = E^{(u)} + E^{(g)} - E^{(u \cdot g)} \,.$$

Similarly succinct expressions for Theil's H and the Mora-Ruiz-Castillo H^* can be given as simply normalizations of M:

*Theil's H and the Mora-Ruiz-Castillo H**

$$H = \frac{M}{E^{(g)}};\; H^* = \frac{M}{E^{(u)}} \,.$$

Further, we note that Theil (1972) identified M (originally symbolized as $J(X, Y)$; see p. 126) as the "expected mutual information" and provided the following expression for its calculation (using the notation defined in this paper):

Theil's M

$$M = \sum_{i=1}^{u} \sum_{j=1}^{g} \rho_{ij} \ln\left(\frac{\rho_{ij}}{\rho_{i+}\rho_{+j}}\right).$$

This alternative expression helps to identify how the family of information theory indices is defined in reference to a set of expected values, as has been the case for all indices discussed thus far. That is, if we rewrite the natural logarithm term, we get an expression that clearly represents a deviation between the observed and the expected joint proportions: $\ln\left(\frac{\rho_{ij}}{\rho_{i+}\rho_{+j}}\right) = \ln(\rho_{ij}) - \ln(\rho_{i+}\rho_{+j})$.

There are three important and specifically technical sorts of differences among these information theory indices. We draw heavily upon the work of Mora and Ruiz-Castillo (2011) for this exposition. First, though each satisfies *additive decomposability,* Theil's H and the Mora-Ruiz-Castillo H^* do not satisfy the same types of decomposability equally well, while Theil's M

satisfies both types. Theil's H satisfies "weak" additive organizational decomposability because the within-term is weighted by the ethnoracial diversity of the district, not just by the total population share of the super-unit ($H = H^{Between} + \sum_{h=1}^{s} \frac{n_{h+}}{n_{++}} \frac{E_h^{(g)}}{E^{(g)}} H_h^{Within}$, where h indexes 1 ... s super-units [typically districts]), but not "weak" additive group decomposability because both the between- and within-terms of the decomposition depend on ethnoracial diversity ($H = \frac{E^{(u)}}{E^{(g)}} H^{Between} + \sum_{k=1}^{s'} \frac{n_{k+}}{n_{++}} \frac{E_k^{(g)}}{E^{(g)}} H_k^{Within}$, where k indexes 1 ... s' super-groups). In contrast, the Mora-Ruiz-Castillo H^* satisfies "weak" additive group decomposability because the within-term is weighted by the enrollment diversity of the district ($H^* = H^{*\,(Between)} + \sum_{k=1}^{s'} \frac{n_{k+}}{n_{++}} \frac{E_k^{(u)}}{E^{(u)}} H_k^{*\,(Within)}$), but not "weak" additive group decomposability because both the between- and within-terms depend on enrollment diversity ($H^* = \frac{E^{(g)}}{E^{(u)}} H^{*\,(Between)} + \sum_{h=1}^{s} \frac{n_{h+}}{n_{++}} \frac{E_h^{(u)}}{E^{(u)}} H_h^{*\,(Within)}$). Theil's M satisfies both types of additive decomposability, and does so in the "strong" sense that the within-terms are weighted only by super-group or super-unit proportions and not their relative diversity as well (e.g., $M = M^{Between} + \sum_{h=1}^{s} \frac{n_{h+}}{n_{++}} M_h^{Within}$). At the same time, satisfying either the "strong" or "weak" forms of additive decomposability allows for full consideration of *jurisdictional extent*, which means that all three information theory indices are candidates to be used, within their limits of applicability, to evaluate changes in segregation due to enrollment changes across jurisdictional boundaries (e.g., in- or out-migration of families due to changes in residence or enrollment).

Second, while Theil's H represents the *desegregation construct* of evenness well, Theil's M does not. Theil's M simultaneously indicates evenness and "representativeness" (also see

Frankel & Volij, 2011), which means that it is a mixed construct index. This is because the construction of Theil's M requires no distinction between the two forms of categorical variation that constitute a segregation index, namely, categorically differentiated individuals (here, designated as a member of one ethnoracial group among several) and organizational units among which the individuals are distributed (here, multiple classrooms, schools, districts, etc.).[17] Previous artificial examples help to highlight how this dual nature of Theil's M causes it to mask an increase in evenness when there is a concomitant change in the number of ethnoracial groups defining the distribution. Let's return to $Group\ Division\ 1$ and $Group\ Division\ 2$ in the previous section for data from which index values may be calculated. In both cases, a two-group district with three schools becomes a three-group district with three schools by the *group equivalence* operation of group division—splitting one ethnoracial group into two identically distributed ethnoracial groups of equal (and half of the) size. Theil's M remains constant for these group division operations, but segregation decreases. Decreased segregation is not because the expected mutual information changed (M = 0.318 in both cases), nor that the assigned unit (school enrollment) entropy changed ($E^{(u)}$ = 1.011 in both cases)—school enrollments remained constant—but because ethnoracial group entropy increased. The increase in group entropy is due to two characteristics: 1) more even dispersion of the population across the three groups; and 2) greater ethnoracial entropy (*ethnoracial diversity*) by having three instead of two groups. (Specifically, $E^{(g)}$ went from 0.637 to 1.099 in case 1, and to 0.868 in case 2.) The ratio of M to $E^{(g)}$, which is Theil's H, is now smaller in both cases (H

[17] Theil's M can be thought of as the information theory equivalent of the mean square contingency (ϕ^2) obtained by scaling Pearson's chi-squared (χ^2) by the population size (n_{++})— $\varphi^2 = \chi^2/n_{++}$—since Theil's M can be obtained by scaling the likelihood-ratio (Goodman's) chi-squared (G^2) by twice the population size ($2n_{++}$). That is, $M = G^2/2n_{++}$ is an alternative non-directional measure of association for a pair of categorical variables.

= 0.500 before group division and decreases to 0.290 in case 1, and 0.367 in case 2). (Note: the ratio of M to $E^{(u)}$, which is the Mora-Ruiz-Castillo H*, is unchanged; H* = 0.315.)

Third, as may be inferred from the preceding points, the Mora-Ruiz-Castillo H* is not an evenness index; it is a "representativeness" index (Mora & Ruiz-Castillo, 2011), where a value of zero is perfect "representativeness." It might be better to say that the H* index identifies the degree of un-representativeness since larger numerical values, up to a maximum of one, indicate greater deviations from perfect representativeness. Representativeness is about how proportionally distributed group members are among the units in which they are enrolled relative to the total enrollment in each unit. Representativeness goes with *composition invariance*, which we saw with *Group Scaling 1*, because proportionality is referenced to groups (columns), now, instead of units (rows). However, this means that "representativeness" is not the same thing as exposure (see next subsection of paper for exposition of the exposure construct), because exposure is about the proportions of one group relative to another (or itself, in the case of isolation) and not relative to the total enrollment.

We'll use the single-group-against-all-others Isolation Index (${}_jP_j^*$) as a reference for discussing the exposure side of the matter (see the next subsection of this paper for details about the Isolation Index). The reason we do this is because we must turn our attention to the group (not unit) components of H*, which requires that we have some way to refer to the status of an ethnoracial group rather than a school. When we do this for the artificial districts previously illustrated (i.e., the *Multiple Exchanges 1*, *Multiple Transfers 1*, and *Organizational Division 3* examples), we find that H* does not identify the same ethnoracial group for attention as that identified by the Isolation Index. The Mora-Ruiz-Castillo H* identifies Group 2 as the most unrepresentative (largest contribution to un-representativeness) in the "before" enrollment configuration and the "after" configurations, except in the *Multiple Transfers 1* case, where Group 3 is identified. In contrast, the Isolation Index always identifies Group 1 as the most isolated (most exposed to its own members). Moreover, Group 1 makes the greatest group-based contribution to all of the absolute difference indices and the Normalized Exposure Index, the latter being the multigroup extension of the Isolation Index (James, 1986; Reardon & Firebaugh, 2002). However, other variance-ratio-based indices stand between the Isolation Index and the Mora-Ruiz-Castillo $H*$ assessment. That is, Cramer's V^2 and the Goodman-Kruskal $\tau_{x|y}$ [18] make the same initial assessments about which group makes the greatest contribution to the segregation index value as the Isolation Index for the "before" enrollment configuration and the "after" configuration for *Organizational Division 3*, but they agree with $H*$ for the "after" enrollment configurations for *Multiple Exchanges 1* and *Multiple Transfers 1*. This novel behavior of the Mora-Ruiz-Castillo $H*$ suggests that there may be yet another way to discuss school segregation (i.e., representativeness may be another desegregation construct), but it is not at all clear that $H*$ is directly responsive to more than a half-century's political struggle to define and achieve school desegregation.

Now, we are in a different position with the information theory indices than we were with the previous indices. This time, the indices satisfy the *transfer* axiom for more than two ethnoracial groups (Frankel & Volij, 2011; Reardon & Firebaugh, 2002), which provides an adequate set for measuring segregation where there is additional or multigroup *ethnoracial diversity*. Moreover, because the information theory indices satisfy the transfer axiom, they are

[18] The subscripts are reversed because this measure is directional: $\tau_{y|x}$ pairs with H; and $\tau_{x|y}$ pairs with $H*$.

adequate for *comparisons across space or time*. And, as noted earlier, because they satisfy additive decomposability, the information theory indices allow for analyses that include the matter of *jurisdictional extent*. However, for the purpose of working with evenness as a *desegregation construct,* only Theil's H is satisfactory. For a segregation index that addresses the district as the organizational unit of analysis, Theil's H is fully capable of meeting our desegregation measurement needs.

What remains is to determine how well Theil's H serves to identify racially isolated schools. Let us return to our earlier examples, this time providing the school (unit) components of Theil's H to determine where our attention would be drawn first when it comes to unevenness (racial isolation) in schools:

Multiple Exchanges 1 showing Theil's H components (H_{i+})

$$\begin{array}{rl} & H_{i+}^{before} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.127 \\ 0.084 \\ 0.167 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{rl} & H_{i+}^{after} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.024 \\ 0.187 \\ 0.167 \end{bmatrix} \end{array}$$

Multiple Transfers 1 showing Theil's H components (H_{i+})

$$\begin{array}{rl} & H_{i+}^{before} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.127 \\ 0.084 \\ 0.167 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{rl} & H_{i+}^{after} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.069 \\ 0.064 \\ 0.167 \end{bmatrix} \end{array}$$

Organizational Division 3 showing Theil's H components (H_{i+})

$$\begin{array}{rl} & H_{i+}^{before} \\ \begin{array}{r} Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array} & \begin{bmatrix} 0.127 \\ 0.084 \\ 0.167 \end{bmatrix} \end{array} \xrightarrow{multiple\ exchanges} \begin{array}{rl} & H_{i+}^{after} \\ \begin{array}{r} Unit\ 1a \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 1b \end{array} & \begin{bmatrix} 0.061 \\ 0.080 \\ 0.160 \\ 0.061 \end{bmatrix} \end{array}$$

Unlike the absolute difference and variance-ratio-based indices, the initial (before) assessment draws our attention to the third school (Unit 3) as most out of balance—making the largest contribution to segregation. This is contrast to the first school (Unit 1) having been the largest contributor, initially. In all cases the final (after) assessment identifies one of the racially isolated schools as the largest contributor to Theil's H. The results are identical to those obtained for the variance-ratio-based indices in the sense that the largest contribution to segregation is coming from one of the two racially isolated schools and, again, it is the second and smaller racially isolated school (Unit 2) that has the largest contribution in the case of *Multiple Exchanges 1*.

These results are noteworthy because Theil's H appears to consistently draw our attention to racially isolated schools as the largest contributors to desegregation—these are the schools that need "fixing" if we are to achieve acceptable desegregation. However, Theil's H returns us to the issue we first confronted with the Gini Index, namely, it is possible that the biggest "problem" (largest component) comes from a sort of warehousing of all other groups in a multiethnic setting at a single school and not the isolation of a particular group in other schools. The following example, *Organizational Division 4*, which splits Unit 3 into two equal- and half-sized Units 3a and 3b, reveals how to distort enrollment severely enough to make warehousing—the failure to enroll some groups in any other schools—the greatest segregation problem requiring attention:

Organizational Division 4 shown as frequency counts (n_{ij})

$$\begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 50 & 100 & 50 \\ 50 & 0 & 0 \\ 100 & 0 & 0 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3a \\ Unit\ 3b \end{array}\begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 50 & 100 & 50 \\ 50 & 0 & 0 \\ 50 & 0 & 0 \\ 50 & 0 & 0 \end{bmatrix} \end{array}$$

Organizational Division 4 shown as row proportions (π_{ij})

$$\begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ 1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{l} \\ Unit\ 1a \\ Unit\ 2 \\ Unit\ 3 \\ Unit\ 1b \end{array}\begin{array}{c} \begin{array}{ccc} Grp\ 1 & Grp\ 2 & Grp\ 3 \end{array} \\ \begin{bmatrix} 0.25 & 0.50 & 0.25 \\ 1.00 & 0.00 & 0.00 \\ 1.00 & 0.00 & 0.00 \\ 1.00 & 1.00 & 0.00 \end{bmatrix} \end{array}$$

Now, let's look at the components of H for each school (Unit indexed by i = 1 ... 3):

Organizational Division 4 showing Theil's H components (H_{i+})

$$\begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3 \end{array}\begin{array}{c} H_{i+}^{before} \\ \begin{bmatrix} 0.127 \\ 0.084 \\ 0.167 \end{bmatrix} \end{array} \xrightarrow{organizational\ division} \begin{array}{l} \\ Unit\ 1 \\ Unit\ 2 \\ Unit\ 3a \\ Unit\ 3b \end{array}\begin{array}{c} H_{i+}^{after} \\ \begin{bmatrix} 0.127 \\ 0.084 \\ 0.084 \\ 0.084 \end{bmatrix} \end{array}$$

This time, the first school (Unit 1) has the largest component of H, not any of the three much smaller racially isolated schools. For this scenario, the concentration of Groups 2 and 3 in Unit 1 now stands out as the greatest contributor to the observed segregation over any of the other individual schools.

Though this artificial enrollment scenario, *Organizational Division 4*, may seem a rather extreme situation, where the school with mixed enrollment is four times the size of each of the other three, school administrators have at their disposal at least two means by which to significantly increase the capacity of an already large physical facility (especially in conjunction with under-utilization of smaller facilities). Specifically, a school may add portable classrooms to "empty" playgrounds, athletic fields, or parking lots, for example (i.e., create more physical classroom space), and transition to a multi-track year-round calendar (i.e., create space by staggering attendance throughout the year rather than have all students in attendance at the same time). In other words, the idea of warehousing can be realized in practice and, at some point, Theil's H becomes more sensitive to this phenomenon than that of racial isolation. What this means is that, among all of the evenness indices, Theil's H most reliably identifies racially isolated schools, but the specific nature of enrollment imbalances can shift the emphasis from racial isolation in schools to warehousing a subset of the ethnoracial groups in specific schools.

Finally, as noted in the discussion of the absolute difference indices, except for a restricted exploration of Theil's H using Monte Carlo simulations, there has been no systematic investigation of how to define *compliance margins* around complete desegregation for any definition of minimally acceptable desegregation using any of the aforementioned indices. Nonetheless, the functional form of the information theory indices has important implications for modeling desegregation in the schools. The logarithmic function declines rapidly as the enrollment configuration changes from complete segregation toward complete desegregation, but the values become quite small well before complete desegregation is achieved and, therefore, the approach to zero slows substantially. In the language of economics, there are diminishing marginal returns to desegregation (Zoloth, 1976). Conceptually, this fits well with the practical reality that complete desegregation is unattainable, but we don't know how small is small enough that we can identify the threshold at which the marginal return is insufficient to demand further progress. Again, we suggest that work such as that reported by Mitchell and Mitchell (2010) be pursued further. This and all of the foregoing discussion of information theory indices is summarized in Table 1.

Threshold-based indices. As noted at the beginning of this subsection on evenness indices, the idea of imbalance can be defined like a speed limit or an emissions standard, namely, by setting thresholds beyond which violations are noted. The thresholds define the *compliance margins*—threshold-based indices are all about observing compliance. Also, threshold-based indices fundamentally focus on the school as the *organizational unit of analysis*, and a district-level assessment is constructed from the set of individual school assessments for the schools within the district. In the case of imbalance, like the other evenness indices, individual school assessments are made relative to the district as a whole. In contrast to the other indices, the approach is to specifically evaluate schools and score the district based upon school-level results, not to produce an index that evaluates the district while, at the same time, has school level components.

A racially imbalanced school index is calculated by identifying the total number of schools that are racially imbalanced and dividing that number by the total number of schools in the district (see Appendix A). The imbalance criterion is the compliance margin set around the overall district enrollment proportions. For example, a 15%-rule for compliance would mean that each school would have to be within ± 0.15 of the district proportion for each ethnoracial group, or it would be identified as imbalanced.

Due to their definitions being threshold based, the racially imbalanced schools indices all fail to be register the existence of either an *exchange* or a *transfer* for all possible student moves; only moves that bring all students within the margins are register an index change. As a consequence, these indices will always be specific to the school enrollment context of a particular place at a given time. Since neither of the two fundamental axioms of student movement is satisfied, there is no point in further discussing the remaining attributes of racially imbalanced schools indices. (Please see summary in Table 1.)

Basically, racially imbalanced schools indices are good for what you have now and where you are now, further limited by whether your current circumstances affect moves around the compliance margins. That is, the *interpretability* of these indices is severely limited and limiting. Their utility in repeated discussions about the status of desegregation is realized only if one were to say, "I don't care who has come or gone, or what you did last time or are doing now, I only care that I don't find anymore and prefer to find fewer racially imbalanced schools than the last time we talked." There is no way to get credit for progress toward desegregation that has yet to pass a threshold, nor any way to mark a celebration for further desegregation beyond attaining enrollments within the compliance margins. These are crude indices.

Summary for evenness indices. As always, our notions of what segregation requires our attention and whether we wish to monitor progress toward desegregation as a consequence of our attention have been gotten should guide our selection of an index. However, prior to the current review there was insufficient clarity about how our notions truly relate to our choice of index. Using the summary presented in Table 1, we begin to recognize which indices help us accomplish which measurement purpose. The first somewhat startling discovery is that most indices are inadequate for the purpose of making accurate comparisons across space or time. Only the information theory indices obey the transfer axiom in the presence of multigroup ethnoracial diversity. At best, the others do so only when ethnoracial diversity can be reduced to a dichotomous characterization of the situation. Second, beyond the information theory indices, only the Goodman-Kruskal $\tau_{y|x}$ can be used to evaluate matters of jurisdictional extent because it possesses the property of additive decomposability for multiple ethnoracial groups (the other two variance-ratio-based indices do so for just two groups). However, only the racially imbalanced schools indices unequivocally allow us to identify the school (single unit) as our organizational unit of analysis when it comes to whether compliance margins are satisfied or special scrutiny is required. This is important because none of the other evenness indices have been employed for setting compliance margins. Unfortunately, racially imbalance schools indices do not meet the requirements for comparisons across space or time, or assist in matters of jurisdictional extent. Nonetheless, we have presented the possibility that Theil's H could be a candidate replacement for the role played by the racially imbalanced school indices, but further work is required.

Exposure Indices

We finish our detailed examination of segregation index calculation with two indices that are expressions of compositional character. As discussed below, with different levels of subtlety and sophistication, these are indices that address the *desegregation construct* labeled exposure. Remember that exposure is a dual concept in that one face is a group's isolation (e.g., probability of encounter with its own members) and the other is a group's interaction with others (e.g., probability of encounter with other groups' members). (E.g., see Lieberson & Carter, 1982; Massey & Denton, 1988; U.S. Civil Rights Commission, 1967.) In this subsection, we deal strictly with isolation defined in a dichotomous manner, namely, the isolation of a single ethnoracial group in the presence of all other groups, or the isolation of a super-group (e.g., non-white students or underrepresented minority students) in the presence of a single ethnoracial group or super-group (e.g., white students or the overrepresented-group students, respectively). The notion of isolation, however, is not limited to a continuous and probabilistic model. The literature identifies a threshold-based approach to defining isolation as well. Depending on the author and era, isolation has been defined as a unit with more than 50% enrollment of a minority group, or as a unit at or exceeding 80% or 90% of a single ethnoracial group (e.g., see Dye, 1968; Farley & Taeuber, 1974).

Interaction is not addressed because, in the dichotomous case, the interaction between the two (super-)groups has a definite and simple relationship to isolation. Namely, the value of the interaction index (${}_jP_k^*$) can be obtained by subtracting the value of the isolation index (e.g., ${}_jP_j^*$) from one (i.e., ${}_kP_j^*$ = 1 – ${}_kP_k^*$; ${}_jP_k^*$ = 1 – ${}_jP_j^*$). Further, in the multigroup case, the only established exposure index (the Normalized Exposure Index discussed in the previous subsection) takes on a variance-ratio form. That is, conceptualizing the degree of isolation or interaction in a multigroup situation has only been accomplished by simultaneously taking into account the observed distribution of all groups as differences between the observed and expected joint frequencies, which transforms the measurement from one of central tendency to one or dispersion.

Isolation index. As noted at the beginning of this subsection on exposure indices, we are addressing the dichotomous index defined as follows:

$$ {}_jP_j^* = \frac{1}{n_{+j}} \sum_{i=1}^{u} \frac{{n_{ij}}^2}{n_{i+}} = \frac{1}{\rho_{+j}} \sum_{i=1}^{u} \frac{{\rho_{ij}}^2}{\rho_{i+}} = \sum_{i=1}^{u} \frac{{\rho_{ij}}^2}{\rho_{i+}\rho_{+j}} $$

The Isolation Index is the two-group case of the Normalized Exposure Index (i.e., when there are only two ethnoracial groups, a dichotomous condition, the Normalized Exposure

Index reduces to the Isolation Index). Therefore, all of the properties of the Normalized Exposure Index previously identified apply to the Isolation Index. Namely, *exchange* is satisfied, but *transfer* and *additive group decompos*ability are not for more than two groups. This is important to reiterate because even though the Isolation Index is constructed as single-group-against-all-others dichotomous index, rather than a multigroup index, the underlying enrollment configuration is still a multigroup situation. For example, when a desegregative transfer involves a group different from the one indexed by *j*, that transfer may, nonetheless, increase the isolation for members of group *j* (i.e., such a desegregative transfer would result in an increase in the index value rather than the required decrease to satisfy the transfer axiom). This is illustrated by *Multiple Transfers 1* (see Absolute differences indices section under Evenness Indices), in which the transfers are between Units 1 and 2 for Group 2, but we are interested in the isolation of Group 3. In the "before" enrollment configuration, ${}_3P_3^* = 0.250$, but in the "after" configuration, ${}_3P_3^* = 0.333$. (Note: As would be expected, this is also reflected in the components of the Normalized Exposure Index, for which $P_{+3}^{before} = 0.018$, and $P_{+3}^{after} = 0.032$.)

When it comes to identifying racially isolated schools, the dichotomous Isolation Index tells us in which school the focal group is most isolated, but it does not identify which school is the most racially isolating across all ethnoracial groups. The exception to this statement is when the focal group has the privileged distinction of the being the indicator for racial isolation. For example, if we were to consider the privileged distinction between white and non-white students that recurs in federal decisions (e.g., *Cumming v. Board of Education,* 1899; *Gong Lum v. Rice,* 1927; *Keyes v. School District No. 1,* 1973; *Mendez v. Westminster,* 1946; *Swann v.*

Charlotte-Mecklenburg Bd. of Educ., 1971; see Mitchell & Mitchell, 2011, pp. 188-192; van Geel, 1980, p. 62), then the components of a non-white Isolation Index would identify which are the most racially isolated schools.[19]

Finally, as was repeatedly observed for evenness indices in the previous subsection, continuous measure segregation indices neither have a history of use in the assessment of whether desegregation has fallen within required compliance margins, nor have they been studied in a manner that makes clear how compliance margins would be set using these indices. This holds true for the Isolation Index as well. Please see a summary of the foregoing in Table 1.

Threshold-based indices. The idea of isolation, like that of imbalance in the previous subsection, can be defined by setting thresholds beyond which violations are noted. The thresholds define the *compliance margins*. Because the formal mathematical expression for a racially isolated schools index is more distracting than revealing, the expression is found in Appendix A. Again, as with racially imbalanced schools indices, due to the definitions being threshold based, the racially isolated schools indices all fail to register the existence of either an *exchange* or a *transfer* for all possible student moves. As a consequence, these indices will always be specific to the school enrollment context of a particular place at a given time. Since neither of the two fundamental axioms of student movement is satisfied, there is no point in

[19] However, this pushes us against the debate over color-blind vs. race-conscious judicial intervention and remedies imposed by the U.S. Supreme Court upon politics and governance of local schools systems that animated the various opinions in *Parents Involved in Community Schools v. Seattle School District No. 1* (2007). Simpson (2007), in a study of residential segregation in England and Wales, helpfully articulates one view of how the ideology of a just society—the basis for a justice's opinion—is expressed in our definition and choice of a segregation index: "To use indices of exposure that are based on the proportion of non-white groups in an area [school or district] in a normative or evaluative manner seems to be prejudicial to areas [schools or districts] on the basis of their colour" (p. 421). That is, indices reflect political values and serve to define the problem stream affecting the political agenda for school desegregation.

belaboring the remaining attributes of racially imbalanced schools indices. (Please see summary in Table 1.)

However, before leaving the topic of racial isolation measurement, we should point out that the racially isolated schools indices can be generalized to the entire district ethnoracial composition in the same manner as the racially imbalanced schools indices. That is, instead of selecting a focal ethnoracial group, evaluate each school on the basis of whether any of the particular groups are exceptionally highly represented in a particular school (see Appendix A). This more general index has no other additional redeeming features than that it provides a systematic approach to defining a racially isolated school and then identifying the share of the district's schools that are racially isolated without requiring that any group have preeminent status in the assessment.

Conclusion

If we were to treat the foregoing discussion as strictly an argument for the best segregation measure, we would share the conclusion of Reardon and Firebaugh (2002) that Theil's H is the measure of evenness that satisfies the broadest range of requirements. However, our argument is also that this review has been an exploration and evaluation of models for how to achieve desegregation. The absence of knowledge about how to impose compliance margins using Theil's H is a non-trivial problem. As we noted previously, Mitchell and Mitchell (2010) have made some important first steps toward improving the interpretability of Theil's H, but we still don't know whether specific values of Theil's H will serve as both sensible and attainable bounds on the extent of allowable segregation. However, this is not a devastating critique. It is merely a straightforward call for the completion of work necessary to more fully and effectively put Theil's H to work for desegregation monitoring and accountability.

Harkening back to Sarkozy (2009, p. xi), since we are accustomed to the ostensibly "easier communications" of the Dissimilarity Index in spite of its "long-standing problems [and] limitations," the field is at a disadvantage when it comes to fully adopting Theil's H as a standard index for monitoring desegregation. Because only the Dissimilarity Index is frequently used, we don't know or have intuition for how much of a change in Theil's H would represent a demonstrable improvement in desegregation, particularly since equal unit changes in population redistribution do not translate into equal unit changes in Theil's H, some research is required to meaningfully anchor the Theil's H measurement scale (e.g., the Fahrenheit temperature scale is anchored at 32° as the freezing point of water and 212° as the boiling point of water).

Finally, we have introduced the idea using the school-by-school (i.e., unit-by-unit) contributions to the total value of Theil's H as indicators for racially isolated or racially warehoused schools. Of course, further inquiry is needed to be certain of how well the components of Theil's H serve this function. Nonetheless, they show great promise for being effectively used to identify schools requiring special scrutiny for their contribution to segregation within a district. And, as we have seen with Theil's H, the problem of racial isolation may be more quickly recognized by the practice of warehousing which, as the term implies, is a consequence of an intent or commitment to further racial segregation.

Administrators, politicians, and policy entrepreneurs don't have to wait for the scientists in the research and development office to finally release their inventions before pursuing and negotiating specific means to achieve policy goals—we don't have to figure out everything about Theil's H, or some yet to be discovered superior index, before engaging desegregation accountability. But, we do need to identify an effective policy tool now and provide a rationale for its use, preferably one that may appeal to a range of ideological perspectives. Otherwise, we risk at least two possible outcomes: 1) there will be continued use of measures that are inadequate to the task they were intended facilitate; or 2) there will be index shopping (i.e., adopting the index that minimizes the burdens of satisfying a desegregation mandate). We make the following recommendations in response to this immediate need.

Recommendations

First, we recommend that Theil's H be used as the monitoring index for tracking progress toward desegregation. Even if a particular district were to begin with just two ethnoracial groups that were separated or isolated from one another, the general increase in and national diffusion of ethnoracial diversity in the schools demands that a multigroup index applicable to comparisons across space or time is selected. Second, until such time as the necessary investigations have been conducted, an alternative and established strategy for setting compliance margins is appropriate. The long-established racially imbalanced schools indices can be used for this purpose, though they are awfully crude. Third, we do have an existing strategy for identifying racially isolated schools, and we have shown how it can be adapted for multigroup application, so this measure should continue to be employed until such time as another index has been shown to be more consistent or superior for achieving this purpose.

We are now at the point where a substantial body of segregation index research can be transformed into administratively and politically sensible ways to measure and monitor school desegregation. We believe it is now possible to be clear about what it means to choose any particular segregation index and what that index can accurately and appropriately indicate about the extent of segregation and progress toward school desegregation. In particular, for any comparative study of segregation from time to time or place to place, whether a scholarly study or a judicial-political inquiry, from among the indices reviewed here, only information theory indices are adequate to the task. With Theil's H, we can evaluate the degree of evenness that has been achieved by efforts to racially balance schools; we can identify which schools contribute most to imbalance, often by identifying the more racially isolated schools from the components of Theil's H; and we can take a larger (or smaller) view of how student assignment structures and policies (e.g., classroom assignments, school catchment areas, district governance boundaries, etc.) by partitioning the total value of Theil's H into between and within contributions to segregation. Finally, this is achieved by matching Theil's H with a fairly ideologically neutral and broadly defensible set of considerations for what an index must be able to accomplish in order to monitor desegregation through all of its organizational transitions.

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Appendix A

Threshold-based Indices

Evenness. Racially imbalanced schools indices are symbolized as Imb_{pct}, where the subscript pct indicates the percentage value used to establish the compliance margins. Here, we define these indices as the proportion of all schools in a district that are imbalanced rather than the number of imbalanced schools:

Racially Imbalanced Schools Index (Imb_{pct})

$$Imb_{pct} = \frac{1}{u}\sum_{i=1}^{u}\delta_i$$

The δ function may take on the values one (1) or zero (0), as defined below. The compliance margin (threshold) to which the pct subscript refers is represented by the capital Greek letter delta (Δ). Any school (indexed by $i = 1 \dots u$) that is within the compliance margins set by Δ for each and every ethnoracial group is racially balanced. The way the Δ criterion functions is that it defines how much a school's enrollment proportion for a specific ethnoracial group (ρ_{ij}, where the group is indexed by $j = 1 \dots g$) may deviate from the district-wide enrollment proportion for that group (ρ_{+j}) and still have the school identified as racially balanced:

Balance Definition

$$\left.\begin{matrix} (\rho_{+j} - \Delta)\cdot\ \rho_{i+} & \text{when } (\rho_{+j} - \Delta)\ \geq 0 \\ 0 & \text{when } (\rho_{+j} - \Delta) < 0 \end{matrix}\right\} \leq \rho_{ij} \leq \left\{\begin{matrix} \rho_{i+} & \text{when } (\rho_{+j} + \Delta)\ > 1 \\ (\rho_{+j} + \Delta)\cdot\ \rho_{i+} & \text{when } (\rho_{+j} + \Delta)\ \leq 1 \end{matrix}\right.$$

δ -function Definition

$$\delta_i = \begin{cases} 0 & \text{when } \rho_{ij} \text{ meets } Balance\ Definition \text{ for all } j = 1 \dots g \\ 1 & \text{otherwise} \end{cases}$$

For example, using a 15%-rule (i.e., $\Delta = 0.15$), district ethnic group proportions with enrollment between 0.15 and 0.85, inclusive, will have compliance margins that are ± 0.15 of the enrollment proportion. But, for group proportions less than 0.15 or greater than 0.85, the full Δ range is truncated when either 0 or 1 is reached (i.e., by definition, no negative values or values greater than one are permissible). Or, to express this example more simply, the Imb_{15} is the total number of schools that are racially imbalanced (violate the 15%-rule) divided by the total number of schools in the district.

Exposure. Racially isolated schools indices are symbolized as Iso_{pct}, where the subscript pct indicates the percentage value used to establish the compliance margins. Here, we define these indices as the proportion of all schools in a district that are racially isolated:

Racially Isolated Schools Index (Iso_{pct})

$$Iso_{pct} = \frac{1}{u}\sum_{i=1}^{u} \delta_i^*$$

The enrollment threshold to which the pct subscript refers is represented by the capital Greek letter theta (Θ).The δ^* function is either one (1) or zero (0) based on the following criterion, where k identifies the group or super-group (column or aggregated columns) that is being observed for its degree of isolation in a specific school (indexed by i = 1 … u):

δ^* -function Definition

$$\delta_i^* = \begin{cases} 1 \quad \text{when } \dfrac{\rho_{ik}}{\rho_{i+}} \geq \Theta \\ 0 \quad \text{when } \dfrac{\rho_{ik}}{\rho_{i+}} < \mathit{\Theta} \end{cases}$$

For example, using a 90%-rule (i.e., $\Theta = 0.90$), when the focal ethnoracial group (or super-group) proportion in a school is or exceeds 0.90 then the school is defined as racially isolated, and the total number of racially isolated schools divided by the total number of schools in the district would be the result of the Iso_{90} calculation.

As noted in the text, racially isolated schools indices can be generalized to the entire district ethnoracial composition by evaluating each school on the basis of whether any of the particular groups are exceptionally highly represented in a particular school. This requires a subtle change in the definition of the δ^* function, namely, changing the index k to j, where j = 1 ... g, and then evaluating whether the threshold, Θ, is surpassed for any group j.

δ^* -function Definition

$$\delta_i^* = \begin{cases} 0 \quad \text{when } \dfrac{\rho_{ij}}{\rho_{i+}} < \Theta \text{ for all } j = 1 \ldots g \\ 1 \qquad\qquad \text{otherwise} \end{cases}$$

For example, using an 80%-rule (i.e., $\Theta = 0.80$), partly because there are now at least three groups under consideration, when any ethnoracial group (or super-group) proportion in a school is or exceeds 0.80 then the school is defined as racially isolated, and the total number of racially isolated schools divided by the total number of schools in the district would be the result of the Iso_{80} calculation.

Table 1. Considerations for the selection of a segregation index to evaluate and monitor school desegregation.

Considerations	Index										
	Dissimilarity	Gini	Cramer's V^2	Goodman-Kruskal τ	Normalized Exposure	Theil's H	Mora-Ruiz-Castillo H^*	Theil's M	Racially Imbalanced Schools	Isolation Index	Racially Isolated Schools
Conceptual Issue											
Desegregation Construct	Evenness	Evenness	Evenness	Evenness	Mixed	Evenness	X	Mixed	Evenness	Exposure	Exposure
Jurisdictional Extent	Single	Single	Single	Multiple	Single	Multiple	Multiple	Multiple	Single	Single	Single
Organizational Unit of Analysis	Aggregate	Aggregate	Aggregate	Aggregate	Aggregate	Aggregate	Aggregate	Aggregate	Single	Aggregate	Single
Ethnoracial Diversity	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	2	2
Comparisons across Space or Time	No	No	No	No	No	Yes	Yes	Yes	No	No	No
Compliance Margins	No	No	No	No	No	No	No	No	Yes	No	Yes
Axiom or Property											
Exchange	No	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	No
Transfer	No	2	2	2	2	Yes	Yes	Yes	No	2	No
Organizational Equivalence	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Size Invariance	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Additive Decomposability	No	No	2	Yes	2	Yes	Yes	Yes	No	2	No
Composition Invariance	2	2	No	No	No	No	No	No	No	No	No
Group Equivalence	No	No	No	No	No	No	Yes	Yes	No	No	No
Symmetry	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	No
Range	Yes	Yes	Yes	Yes	Yes	Yes	Yes	No	Yes	No	Yes

Sources: Frankel and Volij (2011); Reardon and Firebaugh (2002)

Note: For Desegregation Construct, "X" means neither Evenness nor Exposure but "representativeness" (Frankel & Volij, 2011; Mora & Ruiz-Castillo, 2011); for Jurisdictional Extent, "Multiple" means can calculate values for multiple jurisdictions by partitioning, while "Single" means can only calculate value for one jurisdiction at a time; for Ethnoracial Diversity, Transfer, Additive Decomposability, and Composition Invariance, "2" means the consideration holds true only for a two-group definition of the situation and not a more general multigroup definition.