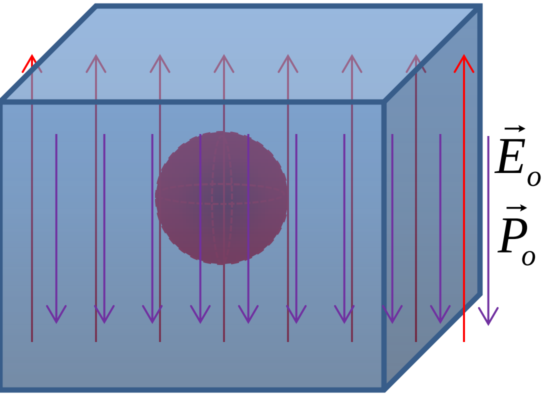
Fri. 10/23	(C14) 4.4.1 Linear Dielectrics (read rest at your discretion)	
Mon.	(C 17) 12.1.1-.1.2, 12.3.1 E to B; 5.1.1-.1.2 Lorentz Force Law: fields and forces	
Wed.		
Thurs.	(C 17) 5.1.3 Lorentz Force Law: currents	HW6
Fri.	(C 17) 5.2 Biot-Savart Law	

Polarization & Electric Displacement

Example

$$Q_{free} = \int \rho_{free} d\tau = \int \vec{D}\cdot d\vec{a} \qquad \varepsilon_o \vec{E} + \vec{P} \equiv \vec{D}$$

Consider a huge slab of dielectric material initially with uniform field, $\vec{E}_o$ and corresponding uniform polarization and electric displacement $\vec{D}_o = \varepsilon_o \vec{E}_o + \vec{P}_o$



You cut a small spherical hole out of it. What is the field in its center in terms of $\vec{E}_o$ and $\vec{P}_o$?

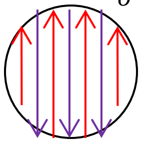
For illustrative purposes only, take the polarization to be anti-parallel to the field, and imagine both to be in the z direction.

By Superposition Principle, cutting out a sphere is the same as inserting a sphere of opposite polarization.

Quoting Example 4.2 (which in turn builds on 3.9), the field *inside* a uniformly polarized sphere is

$$\vec{E}_{sphere} = -\frac{1}{3\varepsilon_o}\vec{P}_{sphere}$$

So, we 'add in ' a sphere of polarization $-\vec{P}_o$



Adding field $\vec{E}_{added} = \frac{1}{3\varepsilon_o}\vec{P}_o$

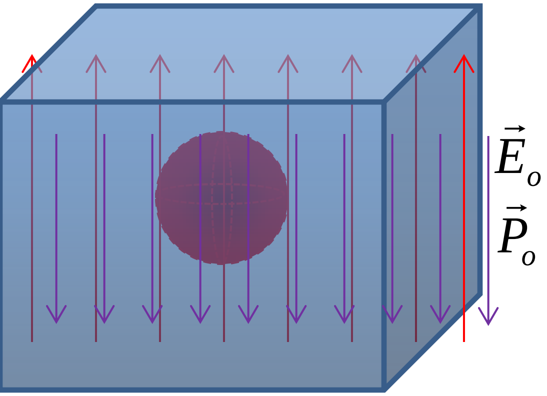
$$\vec{E}_{in.sphere} = \vec{E}_o + \tfrac{1}{3\varepsilon_o}\vec{P}_0$$

Polarization & Electric Displacement

Example

$$Q_{free} = \int \rho_{free} d\tau = \int \vec{D} \cdot d\vec{a} \qquad \varepsilon_o \vec{E} + \vec{P} \equiv \vec{D}$$

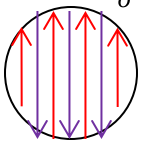
Consider a huge slab of dielectric material initially with uniform field, $\vec{E}_o$ and corresponding uniform polarization and electric displacement $\vec{D}_o = \varepsilon_o \vec{E}_o + \vec{P}_o$



You cut a small spherical hole out of it. What is the field in its center in terms of $\vec{E}_o$ and $\vec{P}_o$?

$$\vec{E}_{sphere} = -\frac{1}{3\varepsilon_o} \vec{P}_{sphere}$$

$$\vec{E}_{in.sphere} = \vec{E}_o + \tfrac{1}{3\varepsilon_o} \vec{P}_0$$



What is the electric displacement in its center in terms of $\vec{D}_o$ and $\vec{P}_o$?

$$\vec{D}_{sphere} = \varepsilon_o \vec{E}_{shere} + \cancel{\vec{P}_{sphere}}$$

There is no material in the sphere, so

$$\vec{D}_{sphere} = \varepsilon_o \left(\vec{E}_o + \tfrac{1}{3\varepsilon_o} \vec{P}_0 \right) \quad \text{where} \quad \vec{E}_o = \tfrac{1}{\varepsilon_o} \left(\vec{D}_o - \vec{P}_o \right)$$

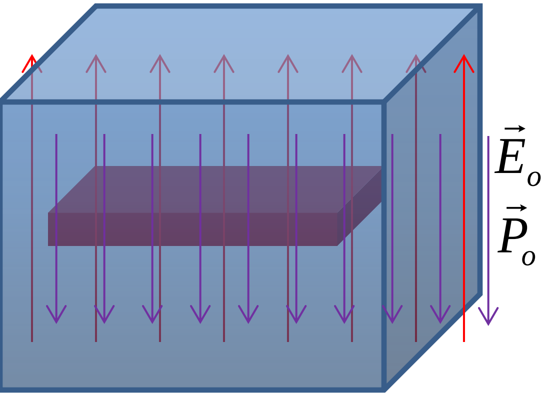
so

$$\vec{D}_{sphere} = \varepsilon_o \left(\tfrac{1}{\varepsilon_o} \left(\vec{D}_o - \vec{P}_o \right) + \tfrac{1}{3\varepsilon_o} \vec{P}_0 \right) \; = \left(\vec{D}_o - \tfrac{2}{3} \vec{P}_0 \right)$$

Polarization & Electric Displacement

$$\vec{P} \equiv \frac{d\vec{p}}{d\tau} \qquad \sigma_b = \vec{P}\cdot\hat{a} \quad \text{and} \quad \rho_b = -\vec{\nabla}\cdot\vec{P} \qquad \varepsilon_o\vec{E} + \vec{P} \equiv \vec{D} \qquad Q_{free} = \int \rho_{free}\, d\tau = \int \vec{D}\cdot d\vec{a}$$

Exercise: Consider a huge slab of dielectric material initially with uniform field, $\vec{E}_o$ and corresponding uniform polarization and electric displacement $\vec{D}_o = \varepsilon_o\vec{E}_o + \vec{P}_o$



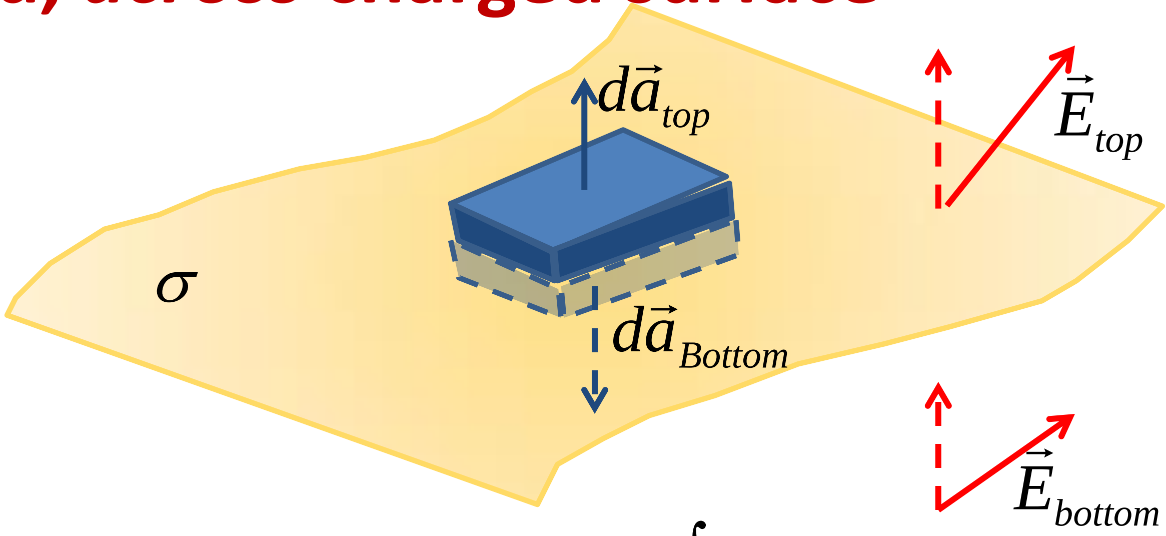
You cut out a wafer-shaped cavity perpendicular to $\vec{P}_o$.

What is the field in its center in terms of $\vec{E}_o$ and $\vec{P}_o$?

Hint: Think of *inserting* the appropriate waver-sized capacitor.

What is the electric displacement in its center in terms of $\vec{D}_o$ and $\vec{P}_o$?

Boundary Conditions
Electric field, *across* charged surface



$$\vec{\nabla}\cdot\vec{E} = \frac{\rho_{encl}}{\varepsilon_o}$$

$$\oint \vec{E}\cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_o}$$

$$\int \vec{E}_{top}\cdot d\vec{a}_{top} + \int \vec{E}_{bottom}\cdot d\vec{a}_{bottom} + \int \vec{E}_{sides}\cdot d\vec{a}_{sides} = \frac{Q_{encl}}{\varepsilon_o} \qquad = \frac{\int \sigma da_{surface}}{\varepsilon_o}$$

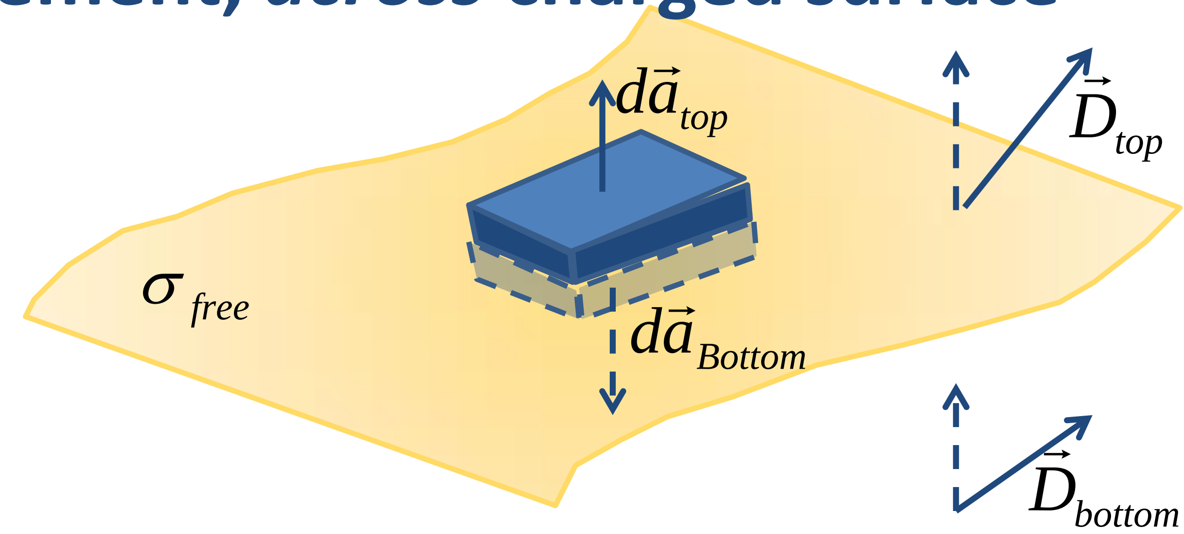
Send side height / area to 0

$$\int \vec{E}_{top}\cdot d\vec{a}_{top} + \int \vec{E}_{bottom}\cdot d\vec{a}_{bottom} = \frac{\int \sigma da_{surface}}{\varepsilon_o}$$

$$E_{\perp top}A + E_{\perp bottom}A(-1) = \frac{\sigma A}{\varepsilon_o}$$

$$E_{\perp top} - E_{\perp bottom} = \frac{\sigma}{\varepsilon_o}$$

Boundary Conditions
Electric Displacement, *across* charged surface



$$\vec{\nabla} \cdot \vec{D} = \rho_{free.encl}$$

$$\oint \vec{D} \cdot d\vec{a} = Q_{free.encl}$$

$$\int \vec{D}_{top} \cdot d\vec{a}_{top} + \int \vec{D}_{bottom} \cdot d\vec{a}_{bottom} + \int \vec{D}_{sides} \cdot d\vec{a}_{sides} = Q_{free.encl} \quad = \int \sigma_{free} da_{surface}$$

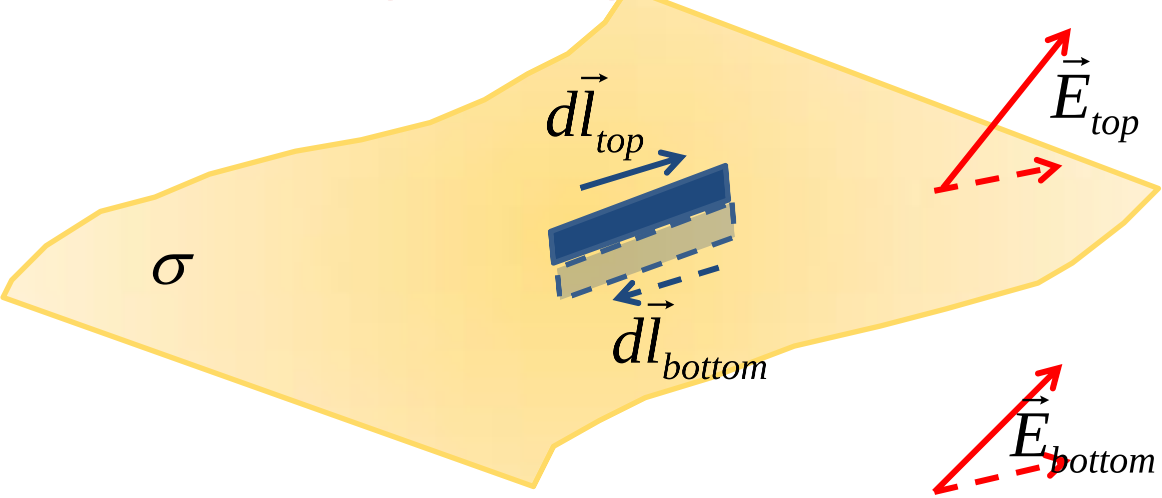
Send side height / area to 0

$$\int \vec{D}_{top} \cdot d\vec{a}_{top} + \int \vec{D}_{bottom} \cdot d\vec{a}_{bottom} = \int \sigma_{free} da_{surface}$$

$$D_{\perp top} A + D_{\perp bottom} A(-1) = \sigma_{free} A$$

$$D_{\perp top} - D_{\perp bottom} = \sigma_{free}$$

Boundary Conditions
(static) Electric field, *along* charged surface



$$\vec{\nabla}\times\vec{E}=0$$

$$\oint\vec{E}\cdot d\vec{l}=0$$

$$\int\vec{E}_{top}\cdot d\vec{l}_{top}+\int\vec{E}_{bottom}\cdot d\vec{l}_{bottom}+\int\vec{E}_{sides}\cdot d\vec{l}_{sides}=0$$

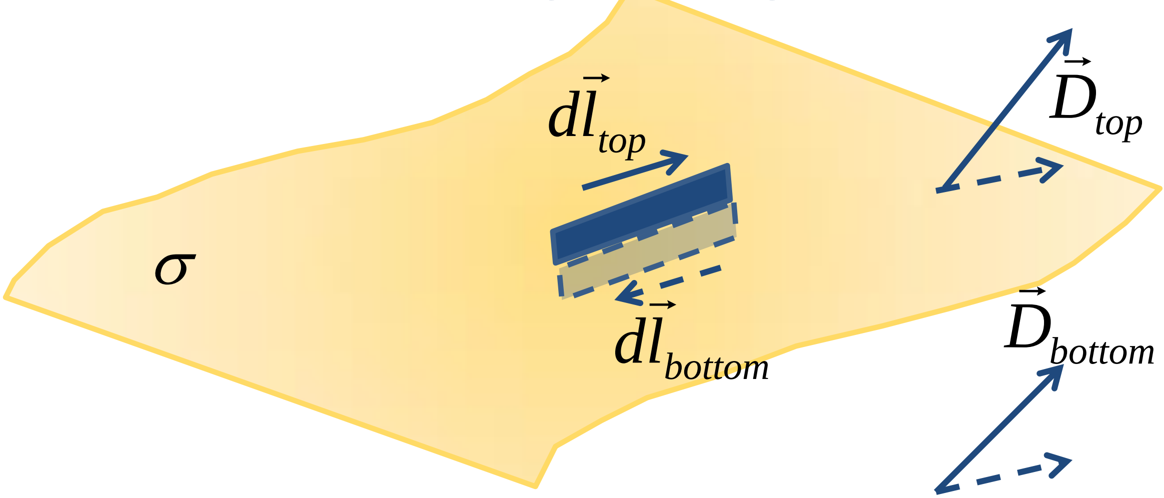
Send side height to 0

$$\int\vec{E}_{top}\cdot d\vec{l}_{top}+\int\vec{E}_{bottom}\cdot d\vec{l}_{bottom}=0$$

$$E_{||top}L+E_{||bottom}L(-1)=0$$

$$E_{||top}-E_{||bottom}=0$$

Boundary Conditions
(static) Electric displacement, *along* charged surface



$$\vec{\nabla}\times\vec{D}=\vec{\nabla}\times\vec{P}$$

$$\oint\vec{D}\cdot d\vec{l}=\oint\vec{P}\cdot d\vec{l}$$

$$\int\vec{D}_{top}\cdot d\vec{l}_{top}+\int\vec{D}_{bottom}\cdot d\vec{l}_{bottom}+\int\vec{D}_{sides}\cdot d\vec{l}_{sides}=\int\vec{P}_{top}\cdot d\vec{l}_{top}+\int\vec{P}_{bottom}\cdot d\vec{l}_{bottom}+\int\vec{P}_{sides}\cdot d\vec{l}_{sides}$$

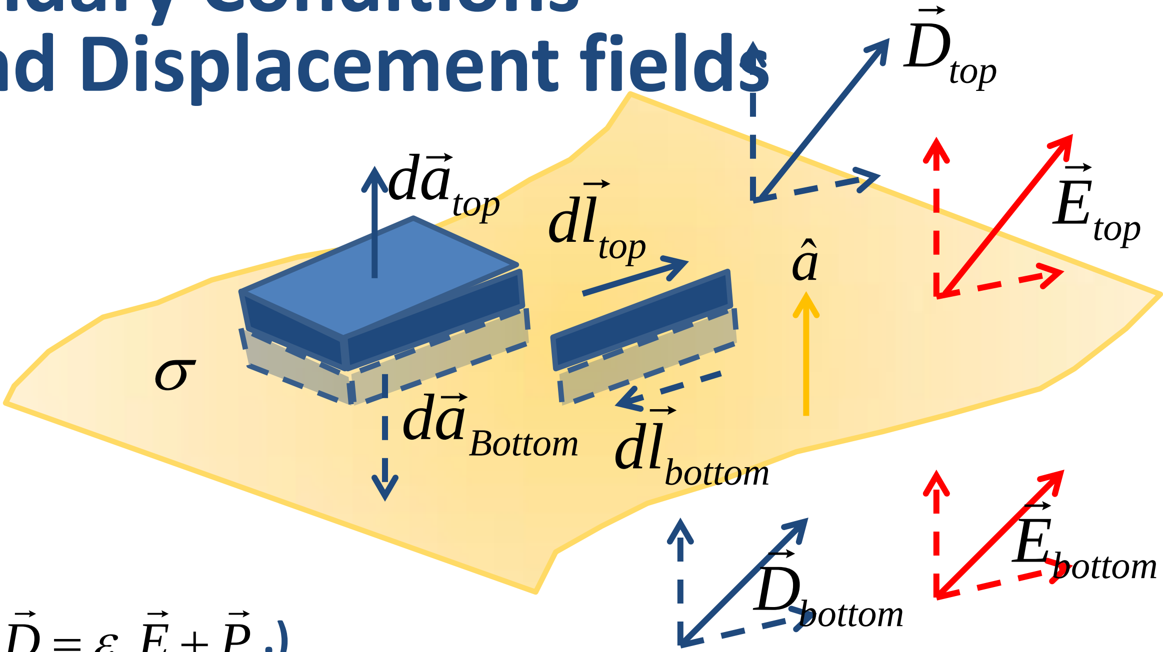
Send side height to 0

$$\int\vec{D}_{top}\cdot d\vec{l}_{top}+\int\vec{D}_{bottom}\cdot d\vec{l}_{bottom}=\int\vec{P}_{top}\cdot d\vec{l}_{top}+\int\vec{P}_{bottom}\cdot d\vec{l}_{bottom}$$

$$D_{\|top}L+D_{\|bottom}L(-1)=P_{\|top}L+P_{\|bottom}L(-1)$$

$$D_{\|top}-D_{\|bottom}=P_{\|top}-P_{\|bottom}$$

Boundary Conditions
Electric and Displacement fields



Along

$$E_{\|top} - E_{\|bottom} = 0$$

$$D_{\|top} - D_{\|bottom} = P_{\|top} - P_{\|bottom}$$

(could have guessed as much from $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$.)

Across

$$E_{\perp top} - E_{\perp bottom} = \frac{\sigma}{\varepsilon_o} \quad \text{or} \quad \vec{E}_{top} \cdot \hat{a} - \vec{E}_{bottom} \cdot \hat{a} = \frac{\sigma}{\varepsilon_o}$$

$$D_{\perp top} - D_{\perp bottom} = \sigma_{free} \qquad \vec{D}_{\perp top} \cdot \hat{a} - \vec{D}_{\perp bottom} \cdot \hat{a} = \sigma_{free}$$

(could have guessed as much from $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$ and $\sigma - \sigma_{free} = \sigma_b = \vec{P} \cdot \hat{a}$.)

Boundary Conditions
Electric and Displacement fields

$$\vec{D} = \varepsilon_o \vec{E} + \vec{P}$$

$$\sigma - \sigma_{free} = \sigma_b = \vec{P} \cdot \hat{a}$$

Along

$$E_{\|top} - E_{\|bottom} = 0$$

$$D_{\|top} - D_{\|bottom} = P_{\|top} - P_{\|bottom}$$

Across

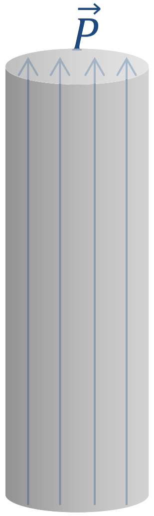
$$E_{\perp top} - E_{\perp bottom} = \frac{\sigma}{\varepsilon_o}$$

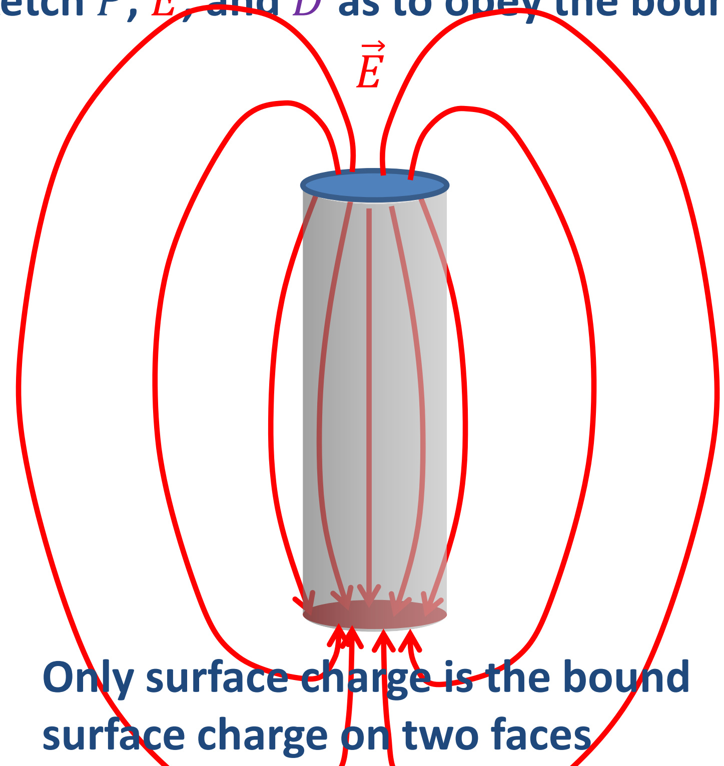
$$D_{\perp top} - D_{\perp bottom} = \sigma_{free}$$

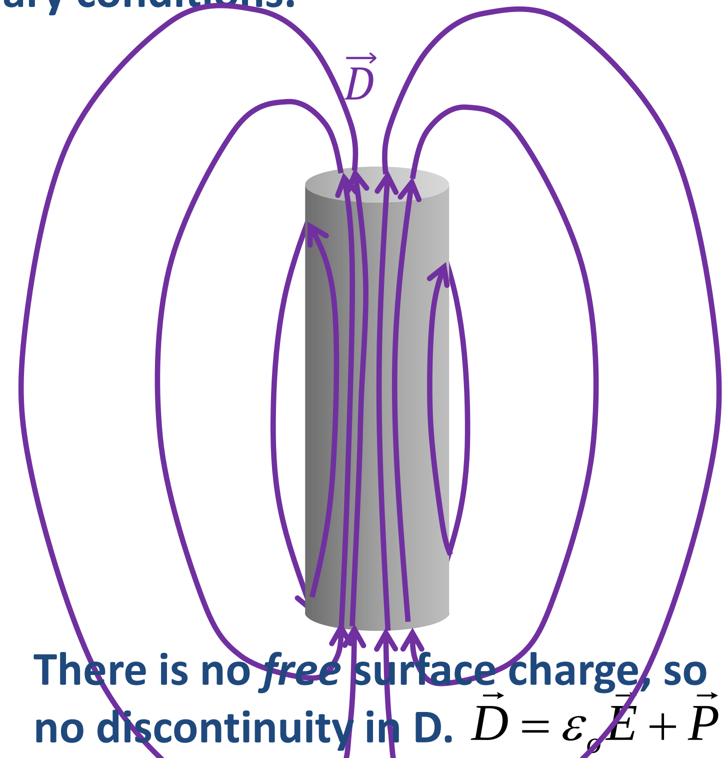
Exercise

Bar Electret (like an electric bar magnet): uniform P along axis

Sketch $\vec{P}$, $\vec{E}$, and $\vec{D}$ as to obey the boundary conditions.



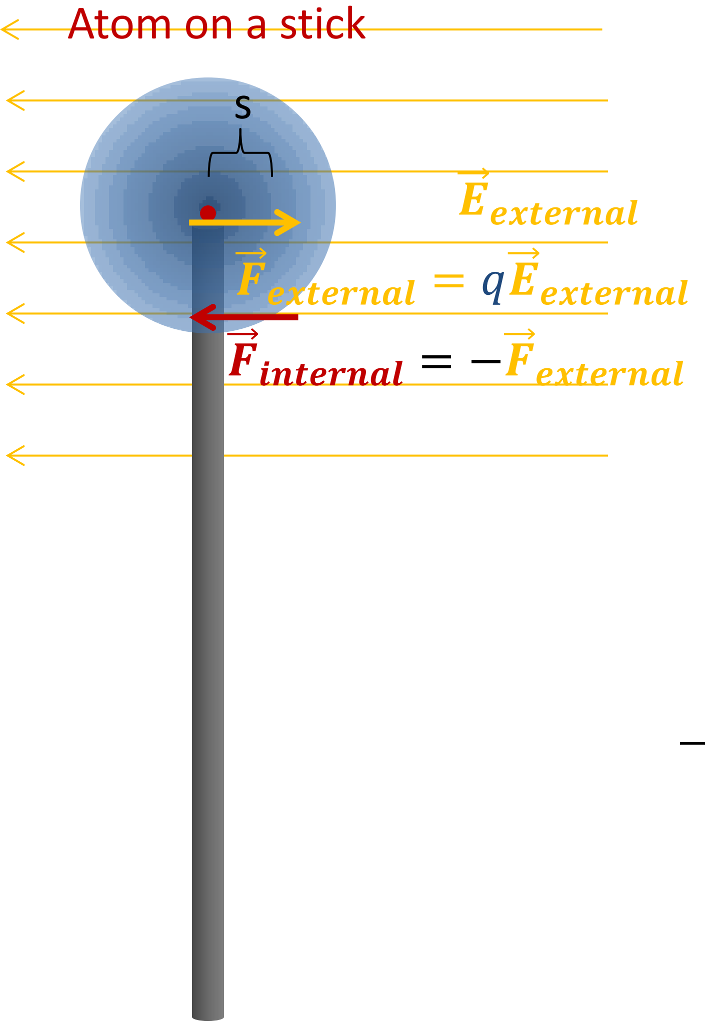




Only surface charge is the bound surface charge on two faces

There is no *free* surface charge, so no discontinuity in D. $\vec{D} = \varepsilon_o \vec{E} + \vec{P}$

Recall: Atom's Response to Electric Field



For *small* stretch, first term in Taylor Series (Hook's Law)

$$F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial s}\Big|_{s=0}\right| s + \ldots$$

$$F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial (qs)}\Big|_{s=0}\right| (qs) + \ldots$$

$$F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial p}\Big|_{s=0}\right| p + \ldots$$

Electric Dipole moment

$$-F_{ext} = F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial p}\Big|_{s=0}\right| p + \ldots$$

$$-qE_{ext} = F_{\text{int}} \approx -\left|\frac{\partial F_{\text{int}}}{\partial p}\Big|_{s=0}\right| p + \ldots$$

So, for small enough stretch, weak enough field

$p \propto -qE_{ext}$ or $\vec{p} \approx \alpha \vec{E}_{ext}$ α≡polarizability

Linear Dielectrics

Point along field

Linearly proportional

Chunk of induced dipoles

for individual induced dipole

$$\vec{p} \approx \alpha \vec{E}$$

Everyone's field but its own

For chunk of induced dipoles

Polarization = Dipole density

$$\vec{P} \equiv \frac{d\vec{p}}{d\tau}$$

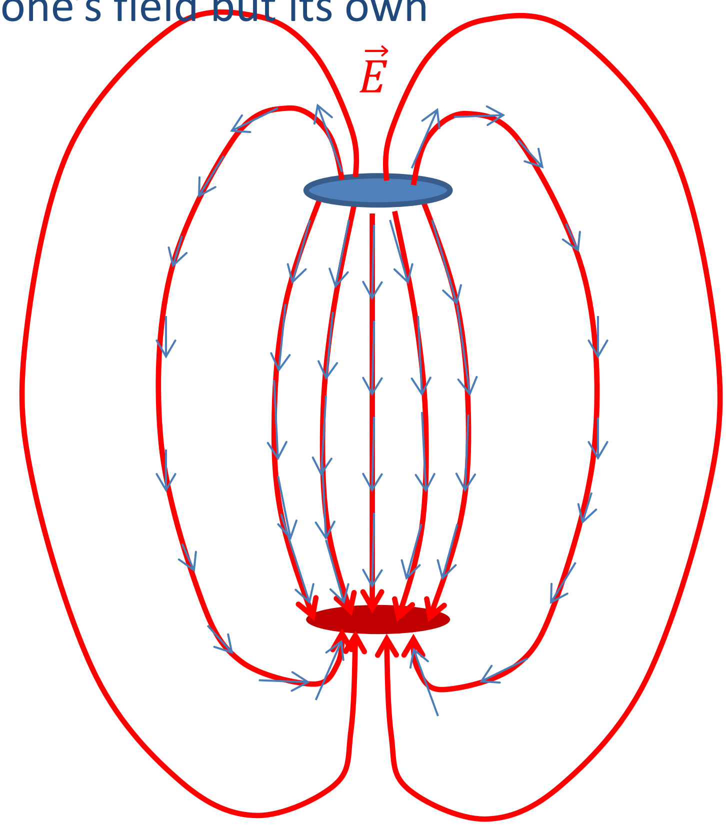
so

$$\vec{P} = \frac{d\left(\alpha \vec{E}\right)}{d\tau} = \left(\frac{d\alpha}{d\tau}\right)\vec{E}$$

Define "electric susceptibility" to be the proportionality constant (and provide convenient factor of ε_o.)

$$\chi_e \equiv \frac{1}{\varepsilon_o}\frac{d\alpha}{d\tau}$$

$$\vec{P} = \varepsilon_0 \chi_e \vec{E}$$



$$\overbrace{\vec{D} = \varepsilon_o \vec{E} + \vec{P}}^{\text{Always}} = \overbrace{\varepsilon_o \vec{E} + \left(\varepsilon_o \chi_e \vec{E}\right)}^{\text{linear dielectric}} = \underbrace{\varepsilon_o \left(1 + \chi_e\right)}\vec{E}$$

$$\vec{D} = \varepsilon \vec{E}$$

$$\vec{D} = \varepsilon_o \varepsilon_r \vec{E}$$

Permittivity (of not-so-free space)

$$\varepsilon \equiv \varepsilon_0 \left(1 + \chi_e\right)$$

Dielectric Constant

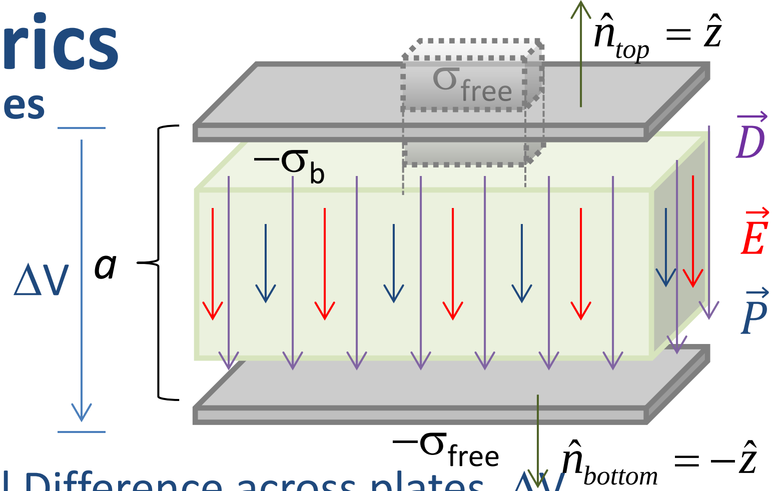
$$\varepsilon_r \equiv \tfrac{\varepsilon}{\varepsilon_o} = \left(1 + \chi_e\right)$$

Or, in terms of polarization

$$\vec{D} = \left(\tfrac{1}{\chi_e} + 1\right)\vec{P} \quad \text{or} \quad \tfrac{\chi_e}{\chi_e + 1}\vec{D} = \vec{P}$$

Linear Dielectrics

Chunk of induced dipoles



Example: consider a simplified version of problem 4.18 . Say we have only one dielectric material, of constant ε_r between two capacitor plates distance *a* apart.

a. Electric Displacement, D.

Gaussian box $\oint D \cdot d\vec{a} = Q_{free.encl}$

Expect only perpendicular to surface and only inside capacitor

$$D_{outside} A_{top} + D_{inside} A_{bottom} = Q_{free.encl}$$

$$0 + D_{inside} A_{bottom} = Q_{free.encl}$$

$$D_{inside} = \frac{Q_{free.encl}}{A_{bottom}} = \sigma_{free} \qquad \vec{D} = \begin{cases} -\sigma_{free}\hat{z} \text{ inside} \\ 0 \qquad \text{outside} \end{cases}$$

b. Electric Field, E.

$$\vec{E} = \frac{\vec{D}}{\varepsilon} = \frac{-\sigma_{free}}{\varepsilon}\hat{z} = -\frac{\sigma_{free}}{\varepsilon_r \varepsilon_o}\hat{z}$$

c. Polarization, P.

$$\vec{P} = \vec{D} - \varepsilon_o \vec{E} = \left(-\sigma_{free}\hat{z}\right) - \left(-\frac{\sigma_{free}}{\varepsilon_r}\hat{z}\right)$$

$$= -\left(1 - \tfrac{1}{\varepsilon_r}\right)\sigma_{free}\hat{z}$$

d. Potential Difference across plates, ΔV.

$$\Delta V = -\int_{bottom}^{top} \vec{E} \cdot d\vec{l} = -\int_{bottom}^{top} \left(-\frac{\sigma_{free}}{\varepsilon_r \varepsilon_o}\hat{z}\right) \cdot d\vec{z} = \frac{\sigma_{free}}{\varepsilon_r \varepsilon_o} a$$

e. Bound charge, σ_b and ρ_b.

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = 0$$

$$\sigma_b = \vec{P} \cdot \hat{n} = \begin{pmatrix} \vec{P}\Big|_{top} \cdot \hat{n}_{top} = \left(-\left(1 - \tfrac{1}{\varepsilon_r}\right)\sigma_{free}\hat{z}\right) \cdot \hat{z} = -\left(1 - \tfrac{1}{\varepsilon_r}\right)\sigma_{free} \\ \vec{P}\Big|_{bottom} \cdot \hat{n}_{bottom} = \left(-\left(1 - \tfrac{1}{\varepsilon_r}\right)\sigma_{free}\hat{z}\right) \cdot \left(-\hat{z}\right) = \left(1 - \tfrac{1}{\varepsilon_r}\right)\sigma_{free} \end{pmatrix}$$

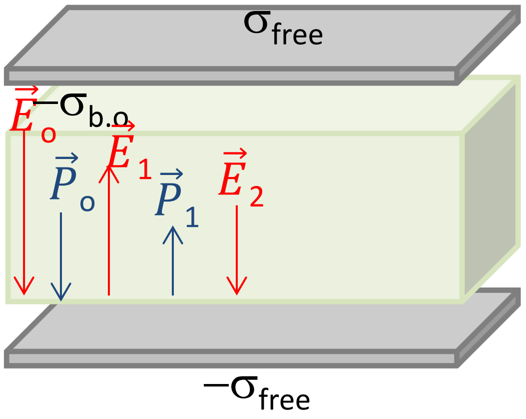
f. E from charge distribution.

$$\oint \vec{E} \cdot d\vec{a} = \frac{Q_{encl}}{\varepsilon_o} \qquad E_{inside} A_{bottom} = \frac{Q_{f.encl} + Q_{b.encl}}{\varepsilon_o}$$

$$E_{inside} = \frac{\sigma_f + \left(-\sigma_b\right)}{\varepsilon_o} = \frac{\sigma_f - \left(1 - \tfrac{1}{\varepsilon_r}\right)\sigma_f}{\varepsilon_o} = \frac{\sigma_f}{\varepsilon_r \varepsilon_o}$$

Linear Dielectrics

Example: Alternate / iterative perspective on field in dielectric. Consider again a simple capacitor with dielectric. We'll find the electric field in terms of what it would have been without the dielectric. We'll do this iteratively and build a series solutions.



0. Say we start with no dielectric. Initially there's the field simply due to the free charge; E_o.

We insert the dielectric and that field induces a polarization,

$$\vec{P}_o = \varepsilon_o \chi_e \vec{E}_o$$

and the associated surface charges contribute a field of their own,

$$\vec{E}_1 = \frac{\sigma_{b.o}}{\varepsilon_o}\hat{z}$$ where $\sigma_{b.o} = \vec{P}_0 \cdot \hat{n}$ so $\vec{E}_1 = -\vec{P}_0 / \varepsilon_0 = -\chi_e \vec{E}_0$

in the opposite direction.

1. This field induces a little counter polarization,

$$\vec{P}_1 = \varepsilon_0 \chi_e \vec{E}_1 = -\varepsilon_0 \chi_e^2 \vec{E}_0$$

Which is means a surface charge and resulting field contribution of its own

$$\vec{E}_2 = -\vec{P}_1 / \varepsilon_0 = \left(-\chi_e\right)^2 \vec{E}_0$$

2. See a pattern?

$$\vec{E} = \vec{E}_o + \left(-\chi_e\right)\vec{E}_o + \left(-\chi_e\right)^2 \vec{E}_o + \ldots$$

$$\vec{E} = \sum_{n=0}^{\infty} \left(-\chi_e\right)^n \vec{E}_o$$

As long as $\chi_e < 1$, this converges to

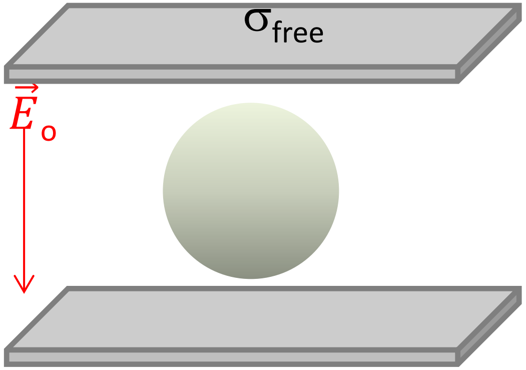
$$\vec{E}_{inside} = \left(\frac{1}{1+\chi_e}\right)\vec{E}_0 = \frac{1}{\varepsilon_r}\vec{E}_0$$

Same result as we got previously

Linear Dielectrics

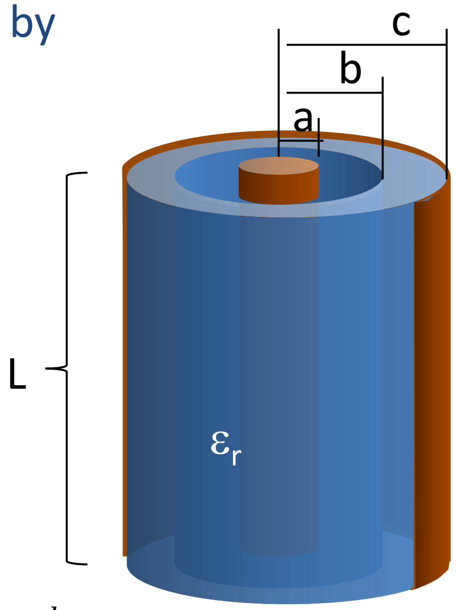
Exercise: Try it for your self. A sphere made of linear dielectric material is placed in an otherwise uniform electric field $\vec{E}_{\text{o}}$. Find the electric field inside the sphere in terms of the material's dielectric constant, ε_r.

You can take it as a given that a sphere of uniform polarization contributes field $\vec{E} = -\vec{P}/3\varepsilon_0$



$$\vec{E} = \vec{E}_o +$$

Example: A coaxial cable consists of a copper wire of radius *a* surrounded by a concentric copper tube of inner radius *c*. The space between is partially filled (from *b* to *c*) with material of dielectric constant ε_r as shown below. Find the capacitance per length of the cable.



For the sake of reasoning this out, say there's charge Q uniformly distributed along the surface of the central wire.

$$C \equiv \left|\frac{Q}{\Delta V}\right|$$

$$\Delta V = -\int_a^c \vec{E}\cdot d\vec{l}$$

$$\vec{E} = \begin{cases} \frac{1}{\varepsilon_o}\vec{D} & a < s < b \\ \frac{1}{\varepsilon_o\varepsilon_r}\vec{D} & b < s < c \end{cases}$$

Gaussian cylinder of some radius a<s<c.

$$\oint \vec{D}\cdot d\vec{a} = Q_{f.encl}$$

$$D 2\pi s L = Q$$

$$D = \frac{Q}{2\pi s L}$$

$$\vec{E} = \begin{cases} \frac{Q}{\varepsilon_o 2\pi s L}\hat{s} & a < s < b \\ \frac{Q}{\varepsilon_o\varepsilon_r 2\pi s L}\hat{s} & b < s < c \end{cases}$$

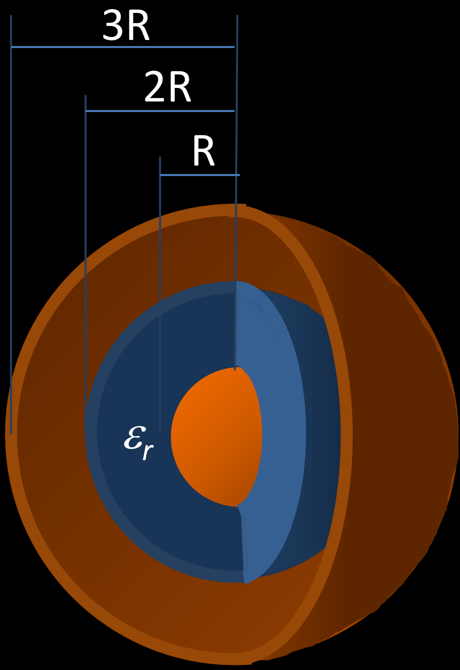
$$\Delta V = -\int_a^c \vec{E}\cdot d\vec{l} = -\int_a^b \vec{E}\cdot d\vec{l} - \int_b^c \vec{E}\cdot d\vec{l}$$

$$\Delta V = -\int_a^b \frac{Q}{\varepsilon_o 2\pi s L} ds - \int_b^c \frac{Q}{\varepsilon_o\varepsilon_r 2\pi s L} ds$$

$$\Delta V = -\frac{Q}{\varepsilon_o 2\pi L}\left(\ln\left(\tfrac{b}{a}\right) + \tfrac{1}{\varepsilon_r}\ln\left(\tfrac{c}{b}\right)\right)$$

$$\frac{C}{L} \equiv \frac{\varepsilon_o 2\pi}{\left(\ln\left(\frac{b}{a}\right) + \frac{1}{\varepsilon_r}\ln\left(\frac{c}{b}\right)\right)}$$

Exercise: There are two metal spherical shells with radii R and 3R. There is material with a dielectric constant $\varepsilon_r = 3/2$ between radii R and 2R. What is the capacitance?



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