Mon. 10/7 Wed. 10/9 Thurs 10/10 Fri., 10/11	**3.4.3-.4.4** Multipole Expansion *(C 17)* **12.1.1-.1.2, 12.3.1** E to B; **5.1.1-.1.2** Lorentz Force Law: fields and forces *(C 17)* **5.1.3** Lorentz Force Law: currents	 HW4

Materials

Announcements

A word of explanation regarding Wednesday's broad reading: Often the Magnetic and electric interactions are introduced quite separately from each other and it's only in later chapters (which many folk never get to) that you begin to see how they're really aspects of a single electromagnetic interaction, aspects that we've broken apart more out of historic ignorance and mathematical convenience than a fundamental conceptual distinction. So the Chapter 12 reading helps us to tie electric and magnetic together and so make a smooth transition and build the proper associations in mental framework.

Last Times

We met the notion and tools of a multi-pole expansion and began using finding the first few terms for discrete charge distributions. The multi-pole expansion is much like a Taylor series or Binomial series expansion – a way of making 'good enough' approximations. In this case, we're essentially expanding in terms of the ratio of origin-to-observer-location/origin-charge-source-location. So, if we put our origin near / in the charge source and the source is small compared to the distance from it to the observer, then the first few terms in the expansion series are quite likely good-enough.

This Time

Using Multi-Pole Expansion some more; especially for *continuous* charge distributions.

<u>Summary</u>

Multipole Expansion

Again,

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

Where P_n is the n[th] Legendre polynomial.

$$P_0 = 1$$

$$P_1(u) = u$$

$$P_2(u) = (3u^2 - 1)/2$$

…

So

$$V(r) = \tfrac{1}{4\pi\varepsilon_o}\sum_i \frac{q_i}{r_i} = \tfrac{1}{4\pi\varepsilon_o}\sum_i \frac{q_i}{r}\sum_{n=0}^{\infty}\left(\frac{r_i'}{r}\right)^n P_n(\cos\theta_i') = \tfrac{1}{4\pi\varepsilon_o}\sum_{n=0}^{\infty} r^{-(n+1)}\sum_i r_i'^n P_n(\cos\theta_i')q_i$$

Or for a continuous distribution, we're looking at the integral

$$V(r) = \tfrac{1}{4\pi\varepsilon_o}\sum_{n=0}^{\infty} r^{-(n+1)}\left(\int r_i'^n P_n(\cos\theta_i')\rho(\vec{r}\,')d\tau'\right)$$

Warning – relative to the origin.

If you're quite far from the sources, you say, it's like a point charge way out there; if you get a little closer, you can see that there's some slight polarization – a little more charge on this end than the other – so it's like a point charge + a dipole; you get a little closer and you an resolve 'it's like a point charge + a dipole + a quadrupole,… Just like the first few terms of a Taylor Series Expansion are graphically simple building blocks, the first few terms of a Multipole Series Expansion are like (differential forms of) reasonably simple charge distributions: The monopole, the Dipole, the Quadrupole, the Octopole.

Monopole: far *enough* from a (non-neutral) charge distribution, the voltage looks like that of a point charge

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{r},$$

where the total charge is

$$Q = \sum_i q_i \to \int \rho(\vec{r}\,')d\tau'.$$

We saw many examples of the electric field going to the limit of what it would be for all of the charge treated like a point charge.

What if $Q = 0$? A simple example is the physical dipole – equal and opposite charges ($\pm q$) separated by a distance d. In that case,

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}.$$

Different arrangements of charges have potentials that fall off more quickly as the distance gets large.

$$V(\vec{r}) = V_{\text{mon}}(\vec{r}) + V_{\text{dip}}(\vec{r}) + V_{\text{quad}}(\vec{r}) + \ldots$$

The first two terms are:

$$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r}\sum_i q_i$$

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\sum_i q_i r_i'\cos\theta' = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\sum_i q_i\vec{r}_i\,'\cdot\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\left(\sum_i q_i\vec{r}_i\,'\right)\cdot\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2}$$

where the dipole moment is defined using the property $\vec{r}\,'\cdot\hat{r} = r'\cos\theta'$:

$$\vec{p} \equiv \sum_i q_i \vec{r}_i' .$$

Warning: These position vectors are relative to an origin. Just like a "moment of inertia" exactly what you get depends on the point your measuring against (in that case, the axis of rotation.) For the series to converge the fastest, you want the origin to be in the center of charge, so all *r's* are as small as they can be.

Examples/Exercises:

Problem 3.32(c) (EXERCISE/EXAMPLE): Find the first two terms in the mulitpole expansion for the figure shown below.

The total charge is $Q = 2q$, so

$$V_{\text{mon}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{2q}{r}.$$

The dipole moment is

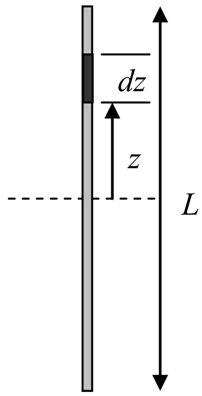
$$\vec{p} = \sum_i q_i \vec{r}_i' = (-q)(0) + (3q)(a\hat{y}) = 3qa\,\hat{y}.$$

The challenge is to find $\hat{y}\cdot\hat{r}$, which is the projection of $\hat{r}$ (see the diagram on the right) in the $\hat{y}$ direction. The projection in the *xy* plane gives $\sin\theta$ and the projection onto the *y* axis gives $\sin\phi$, so $\hat{y}\cdot\hat{r} = \sin\theta\sin\phi$. The dipole term for the potential is

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{3qa}{r^2}\hat{y}\cdot\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{3qa}{r^2}\sin\theta\sin\phi.$$

Example: Suppose a thin rod of length 2*L* lies on the *z* axis and is centered on *z* = 0. If the charge per length of the rod is $\lambda(z) = \lambda_0 (z/L)^3$, what are the first two terms in the mulitpole expansion?

Divided the rod into small segments of length *dz* like the one shown below.



The charge of a segment of the rod between *z* and (*z* + *dz*) is $\lambda\, dz = \lambda_0 (2z/L)^3 dz$. The total charge of the rod is

$$Q = \frac{\lambda_0}{L^3}\int_{-L/2}^{L/2} 8z^3 dz = \frac{8\lambda_0}{L^3}\left[\frac{x^4}{4}\right]_{-L}^{L} = 0.$$

We could also get this result by noticing that the charge distribution is antisymmetric about *z* = 0. The monopole term is $V_{\text{mon}} = 0$.

The dipole moment must be in the *z* direction, so

$$\vec{p} = p_z\hat{z} = \left(\sum_i q_i z_i'\right)\hat{z}.$$

The dipole moment of one segment is

$$dp_z = (dq)z = \left[\lambda_0 (2z/L)^3 dz\right]z.$$

Integrate over the length of the rod to find the total dipole moment:

$$p_z = \frac{8\lambda_0}{L^3}\int_{-L/2}^{L/2} z^4 dz = \frac{8\lambda_0}{L^3}\left[\frac{z^5}{5}\right]_{-L/2}^{L/2} = \frac{\lambda_0 L^2}{10}$$

The dipole term is

$$V_{\text{dip}}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 L^2}{10r^2}\hat{z}\cdot\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda_0 L^2}{10r^2}\cos\theta$$

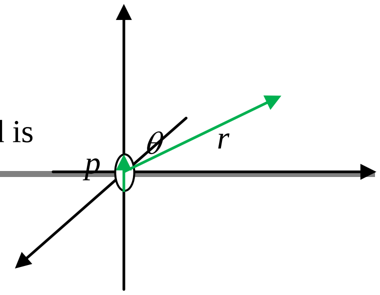
Of course, with an approximate expression for the potential in hand, we can find the corresponding approximate expression for the field.

(see ppt.)

Electric Field of a Dipole

If the dipole moment points in the *z* direction ($\vec{p} = p\hat{z}$), then the potential is

$$V_{\text{dip}}(r,\theta) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}.$$



If there are no other nonzero multipole moments, the electric field is

$$\vec{E} = -\vec{\nabla} V_{\text{dip}} = -\left(\frac{\partial V}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial V}{\partial \theta} \hat{\theta} + \frac{1}{r \sin\theta} \frac{\partial V}{\partial \phi} \hat{\phi} \right)$$

$$= \frac{p}{4\pi\varepsilon_0 r^3} \left(2\cos\theta\, \hat{r} + \sin\theta\, \hat{\theta} \right)$$

Exercise. Problem 3.38: Suppose the charge density on the surface of a sphere of radius *R* is $\sigma(\theta) = k\cos\theta$ in spherical coordinates (θ is the angle from the *z* axis). What is the approximate electric potential far away?

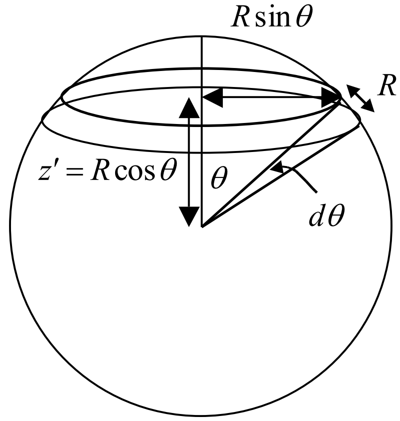
There are opposite charges on the top and bottom halves, so *Q* = 0.

By symmetry, the dipole moment must be in the *z* direction, so

$$\vec{p} = p_z \hat{z} = \left(\sum_i q_i z_i' \right) \hat{z}.$$

(subtly, identifying this is key, because, z-hat is a Cartesian coordinate, so it's a *constant* as we go about integrating over all the point charges.)

The charge on a thin ring between the angles θ and $\theta + d\theta$ is at the same value of z'.



The area of the ring shown above is $(2\pi r)(R\,d\theta) = 2\pi R^2 \sin\theta\, d\theta$, so its dipole moment is

$$dp_z = q_i z_i' = \left[\left(2\pi R^2 \sin\theta\, d\theta \right) \left(k\cos\theta \right) \right] \left(R\cos\theta \right).$$

Add (integrate) up the contributions from rings at all angles (θ) to get

$$p_z = 2\pi k R^3 \int_0^{\pi} \cos^2\theta \sin\theta\, d\theta.$$

Make the change of variables $u = \cos\theta$, so $du = -\sin\theta\, d\theta$ and

$$p_z = -2\pi k R^3 \int_1^{-1} u^2 du = -2\pi k R^3 \left[\frac{u^3}{3} \right]_1^{-1} = \frac{4\pi R^3 k}{3}.$$

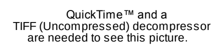
The total charge of the sphere is zero, so far away the approximate potential is

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{4\pi R^3 k}{3r^2}\hat{z}\cdot\hat{r} = \frac{kR^3}{3\varepsilon_0}\frac{\cos\theta}{r^2}.$$

This is the same as the exact answer (Eq. 3.87) for $r > R$, so all of the higher multipoles are zero.

Origin of the Coordinates for Multipole Expansion

Suppose the origin of the coordinate system is shifted by a vector $\vec{a}$ (as shown below).

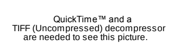


The monopole moment does not change, since the total charge Q is unchanged.

The dipole moment in the new coordinate system is

$$\begin{aligned}\bar{\vec{p}} &= \int \bar{\vec{r}}'\rho(\vec{r}')\,d\tau' = \int (\vec{r}' - \vec{a})\rho(\vec{r}')\,d\tau' \\ &= \int \vec{r}'\rho(\vec{r}')\,d\tau' - \vec{a}\int \rho(\vec{r}')\,d\tau' = \vec{p} - Q\vec{a}\end{aligned}$$

If $Q = 0$, then $\bar{\vec{p}} = \vec{p}$. However, if $Q \neq 0$, then the dipole moment does depend on the choice of the origin.

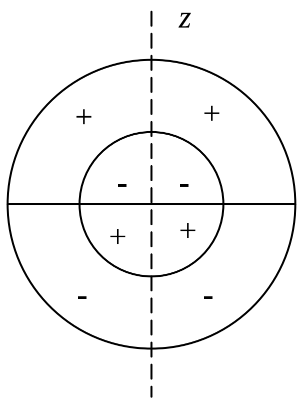


Exercise: A solid sphere of radius R has a charge density of $\rho = \rho_0\left(2 - \frac{R}{r}\right)\sin(2\theta)$.

a. Note: in spherical coordinates, θ runs only from 0 to π; the other half of space is covered by running ϕ from 0 to 2π. Make a sketch of the charge distribution. What is the sign of the charge in different regions?

$$\left(2 - \frac{R}{r}\right)\begin{cases} < 0 & r < R/2 \\ = 0 & r = R/2 \\ > 0 & r > R/2 \end{cases}$$

$$\sin(2\theta)\begin{cases} \geq 0 & 0 < \theta < \pi/2 \\ \leq 0 & \pi/2 < \theta < \pi \end{cases}$$



b. Find the first two terms in the mulitpole expansion of the electric potential.

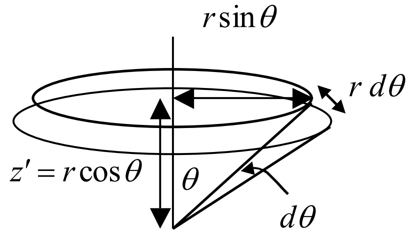
Because the distribution about the *xy* axis is antisymmetric, $Q = 0$ and $V_{\text{mon}} = 0$.

The distribution is symmetric about the *z* axis, so the dipole moment only has a *z* component:

$$\vec{p} = p_z\hat{z} = \left(\sum_i q_i z_i'\right)\hat{z}$$

The sphere must be divided into segments that have the same *z* component. We also want to be able to calculate the charge of each segment easily. Since the charge density depends on r and θ, use thin rings between the radii r and $r + dr$ and between the angles θ and $\theta + d\theta$. The volume of such a ring is

$$[2\pi(r\sin\theta)](r\,d\theta)dr = 2\pi r^2 \sin\theta\, d\theta\, dr$$



The z component of the ring's position is $z' = r\cos\theta$, so the z component of its dipole moment is

$$dp_z = dq\, z' = \left[\rho_0\left(2-\frac{R}{r}\right)\sin(2\theta)\cdot\left(2\pi r^2\sin\theta\, d\theta\, dr\right)\right](r\cos\theta)$$
$$= 2\pi\rho_0(2r-R)r^2 dr\sin(2\theta)\sin\theta\cos\theta\, d\theta = 2\pi\rho_0(2r-R)r^2 dr\sin(2\theta)\tfrac{1}{2}\sin(2\theta)d\theta$$
$$= \pi\rho_0(2r-R)r^2 dr\sin^2(2\theta)d\theta = \tfrac{1}{2}\pi\rho_0(2r-R)r^2 dr\sin^2(2\theta)d(2\theta) = \tfrac{1}{2}\pi\rho_0(2r-R)r^2 dr\sin^2(\alpha)d\alpha$$

Integrate over r from 0 to R and θ from 0 to π to get the dipole moment for the whole sphere. The two integrals can be done separately.

$$\int_0^R (2r-R)r^2 dr = \left[\frac{2r^4}{4} - \frac{Rr^3}{3}\right]_0^R = R^4\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{R^4}{6}$$

Using the relation $\sin(2\theta) = 2\sin\theta\cos\theta$,

$$\int_0^\pi \sin(2\theta)\sin\theta\cos\theta\, d\theta = 2\int_0^\pi \sin^2\theta\cos^2\theta\, d\theta = 2\left[\frac{1}{32}\left(4\theta - \sin(4\theta)\right)\right]_0^\pi = \frac{\pi}{4}$$

These give (in the positive z direction):

$$p_z = \int_{\text{sphere}} dp_z = 2\pi\rho_0\left(\frac{R^4}{6}\right)\left(\frac{\pi}{4}\right) = \frac{\rho_0\pi^2 R^4}{12}$$

- and

$$V_{dip}(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} = \frac{1}{4\pi\varepsilon_0}\frac{\rho_0\pi^2 R^4}{12 r^2}\hat{z}\cdot\hat{r} = \frac{\rho_0\pi R^4}{48\varepsilon_0}\frac{\cos\theta}{r^2}$$

Preview

For Wednesday, you'll read about magnetic forces. We'll talk about the production of magnetic forces after that.

"Can we do an example problem finding the electric field of a dipole in different coordinates and configurations?"
Jessica Hide responses Post a response
Admin
Flag as inappropriate

Maybe an easy one in polar coordinates to start out with?

Casey P, AHoN swag 4 liphe

And then maybe a more challenging one.

Spencer

and then a REALLY challenging one.

Rachael Hach

"It seems that the idea of a dipole moment is rather important. How are we going to use this and apply it in the future? And are quadrupoles/higher order -pole moments just not quite as common as mono- and dipole moments?"
Casey McGrath Post a response
Admin
Flag as inappropriate

"Those field contours on figure 3.37a for a pure dipole are weirding me out a little bit. Do you think we could go into a bit more conceptual detail about what exactly is going on when you confine the dipole to the origin?"
Rachael Hach Post a response
Admin
Flag as inappropriate

"I would also like to see various examples in different coordinate systems, etc."
Sam Post a response
Admin
Flag as inappropriate

"Can we do a problem using multipole expansion but finding the potential over a volume rather than point charges?"
Jessica Post a response
Admin

Drop off in potential, field: Can we get a better explanation of the potentials fall offs, and how it relates to the pure and physical dipoles?

How many terms to keep? Yeah...how are we supposed to know how many pole terms we should use in the approximation if monopole and dipole (altered) are not enough?

another discrete charge distribution example: Can we do problem 3.32 for practice?

Move the origin, change the dipole: can we go over how the origin shift changes the dipole moment and do some examples involving that?

coordinate-free representation of dipole field: Please do problem 3.36.

Electric field of dipole in other coordinates: Is it possible to show how the electric field of a dipole would look like in Cartesian or cylindrical coordinates?

Breaking up one into several? (summing over distribution): This is something that I wondered as well. A single large dipole is one thing, but splitting a distribution up into several is where I lose my understanding.

Continuous charge distribution: Could we see an example of a multipole expansion with a continuous charge distribution?

Aye. My question at the end of the section as well. Would you just find the "center of mass", or more the actual center of charge?