Wed. 10/2 Fri. 10/4	**3.2** Images **T4** Relaxation Method **3.4.1-.4.2** Multipole Expansion	
Mon. 10/7 Wed. 10/9 Thurs 10/10	**3.4.3-.4.4** Multipole Expansion *(C 17)* **12.1.1-.1.2, 12.3.1** E to B; **5.1.1-.1.2** Lorentz Force Law: fields and forces	HW4

Materials

Corner mirror (demonstrate Pr. 19)

Announcements

Last Times

Looked more carefully at Laplace's Equation: $\nabla^2 V = 0$

If V satisfies Laplace's equation then:

- It has no local maxima or minima
- Its value at a point equals the average of its value at all equidistant points
- Given specific values on the boundary, there's a unique solution throughout the enclosed space / given specific charges on a conductor, then there's a unique solution throughout any cavity in that conductor.

We just began talking about the Image Charge technique of solving for V (which taps into the uniqueness theorem – if you can find a way to satisfy the boundary conditions, you've got yourself the solution.)

This Time

A little more about using the Image Charge technique

Another approach all together - Relaxation

<u>**Summary**</u>

Methods of Finding Solutions

We are going to discuss three methods of solving Laplace's equation:

(1) *Method of Images* – replace a problem with a simpler equivalent one (based on corollary of the first uniqueness theorem)

(2) *Relaxation Method* – a computational method based on the potential at a point being the average of the values at the same distance (more about Next Time).

(3) *Multipole Expansion* – a method for getting approximate answers for *V* far from a charge distribution

Method of Images.

Images and Mirrors. A conductor acts like a 'mirror' to a charge. In a fundamental way, this isn't just *simile*, this *is* why a mirror (typically made of a thin layer of metal protected by glass) works. That is, if there's a real source on one side, when you look at the conductor, its surface charge arranges itself to produce a field as if there were another charge somewhere on the other side. In the case of you standing in front of the mirror and looking at your own image – you are the source of the light (electric and magnetic fields) that impact the mirror (sure a light bulb shown on you, but in a fundamental way, the light that makes it to the mirror is largely what you generated in response to the bulb's light hitting you.)

That said, the conditions that define an *optical* image aren't the exact same as define a *charge* image. Now, when we figure out where an charge image is, we're asking where do the new fields point back to , when we're asking where an 'optical' image is, we're asking where do the light waves trace back to (where, as we'll learn much later, the electric field of a light wave is *perpendicular* to the direction of its propagation). So these two types of images aren't necessarily in the same place, but some of the intuition you have about one kind of image applies to the other.

For example, only rotationally symmetric surfaces create *true* images. That is to say a plane creates an image, a sphere creates an image (more or less), but a cylinder *doesn't* create an image (of a point).

The exact condition that we're using for placing a charge image is that, since the conductor's surface defines an equipotential,

$$\frac{q_o}{\mathscr{r}_o}+\frac{q_i}{\mathscr{r}_i}=const \text{ for all points on the surface.}$$

(note: if you have a collection of 'object' sources and 'image' sources, then the condition is that the sum over terms like these for all of them comes to 0.)

Plane "mirror"

Given a source a distance z above it, where would you expect the image to be?

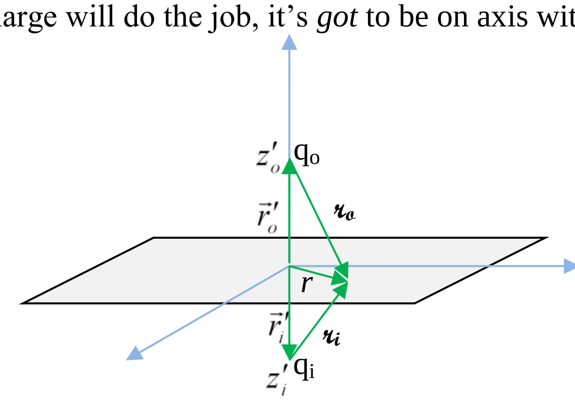
A distance z below it. That comes from our intuition about optical images and plane mirrors.

Voltage: Play that out.

Let's see how that jives with our condition (this may feel like too much work for something that, in retrospect is obvious, but it'll give you a sense of how to proceed when you don't have the advantage of 20/20 hindsight):

Symmetry tells you that, if an image charge will do the job, it's *got* to be on axis with the "object" charge.

$$\frac{q_o}{\mathscr{r}_o}+\frac{q_i}{\mathscr{r}_i}=4\pi\varepsilon_o V(z=0)=const$$



$$\frac{q_o}{\sqrt{(\vec{r}-\vec{r}_o')^2}}+\frac{q_i}{\sqrt{(\vec{r}-\vec{r}_i')^2}}=4\pi\varepsilon_o V(z=0)=const$$

$$\frac{q_o}{\sqrt{r^2+r_o'^2-2rr_o'\cos\theta_o}}+\frac{q_i}{\sqrt{r^2+r_i'^2-2rr_i'\cos\theta_i}}=4\pi\varepsilon_o V(z=0)=const$$

But, with the ‘observation point’ in the x-y plane and the charges on the z-axis, the angle between the two vectors is 90°, so cos = 0

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{q_i}{\sqrt{r^2+z_i'^2}}=4\pi\varepsilon_o V(z=0)=const$$

If this is to be a constant, independent of the observation location, it must hold for *all* locations, so, in case we can’t see where this is leading, we can choose a couple of key places; for example, at

$r=\infty$

, the voltage should be 0, which this relation certainly gives. Since this is a constant value, we need the constant to *be 0 everywhere.*

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{q_i}{\sqrt{r^2+z_i'^2}}=0$$

At

$r=0$, $\frac{q_o}{|z_o'|}+\frac{q_i}{|z_i'|}=0 \Rightarrow \frac{q_o}{|z_o'|}=-\frac{q_i}{|z_i'|}$

So,

$$\frac{\frac{q_o}{|z_o'|}}{\sqrt{\left(\frac{r}{z_o'}\right)^2+1}}=-\frac{\frac{q_o}{|z_i'|}}{\sqrt{\left(\frac{r}{z_i'}\right)^2+1}}=\frac{\frac{q_o}{|z_0'|}}{\sqrt{\left(\frac{r}{z_i'}\right)^2+1}}$$

$$\frac{1}{\sqrt{\left(\frac{r}{z_o'}\right)^2+1}}=\frac{1}{\sqrt{\left(\frac{r}{z_i'}\right)^2+1}}$$

Solving for z_i’ gives that

$$z_i' = \pm z_o'$$

Of course, we already assumed it was below the plane, so that leaves

$$z_i' = -z_o'$$

Force

The external field experienced by the real charge is identical to that of a point charge at the image location, so the force is the product of that field and charge.

$$\vec{F}_{cond\to q_o} = q_o\vec{E}_{cond}(\vec{r}_o) = q_o\vec{E}_{q_i}(\vec{r}_o) = \tfrac{1}{4\pi\varepsilon_o}\frac{q_o q_i}{r_{i\to o}^2}\hat{r}_{i\to o} = -\tfrac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z_o)^2}\hat{z}$$

Work / Energy

You have to be careful not to get carried away by the image approach. *Above* the conducting surface, it's as if there were a charge beneath it, but *beneath* the conducting surface, it's not like that at all (in fact, it's as if there were no charge on the other side – no field). So when you calculate the work done in setting up the charge configuration, you have to be careful to only think about bringing in the real charge / to only consider the field where there is one – *outside* the conductor.

$$W = \frac{\varepsilon_o}{2}\int_{all.space} E_{total}^2 d\tau = \frac{\varepsilon_o}{2}\left(\int_{outside} E_{total}^2 d\tau + \int_{inside} E_{total}^2 d\tau\right) = \frac{\varepsilon_o}{2}\left(\int_{outside} E_{total}^2 d\tau + 0\right)$$

Or

$$W = \int_{\infty}^{z_o} \vec{F}_{\to o} d\vec{l}$$

So, putting all this back together, we can now give the voltage at the z=0 plane in terms of the object charge and the image charge. Our uniqueness theorem says that if we've got an expression that solves the boundary condition and obeys Poisson's equation, then we've got *the* expression.

The voltage in the space above the plane is the same as if we had our object charge and our 'image charge' at their respective locations:

$$V\big|_{z\ge 0} = \tfrac{1}{4\pi\varepsilon_o}\left(\frac{q_o}{\mathscr{r}_o} + \frac{q_i}{\mathscr{r}_i}\right) = \tfrac{1}{4\pi\varepsilon_o}\left(\frac{q_o}{\left(x^2+y^2+(z-z_o')^2\right)^{1/2}} + \frac{-q_o}{\left(x^2+y^2+(z+z_o')^2\right)^{1/2}}\right)$$

- **Induced Surface Charge density**
    - $\sigma = -\varepsilon_o \dfrac{\partial V}{\partial n}$ (where *n* is the direction normal to the surface)
        - **For the infinite sheet in the x-y plane**
            - $\sigma(z=0) = -\varepsilon_o \left.\dfrac{\partial V}{\partial z}\right|_{z=0}$

$$V\Big|_{z\geq 0} = \tfrac{1}{4\pi\varepsilon_o}\left(\frac{q_o}{\left(x^2+y^2+(z-z_o')^2\right)^{1/2}} + \frac{-q_o}{\left(x^2+y^2+(z+z_o')^2\right)^{1/2}}\right)$$

- $$\sigma(z=0) = -\tfrac{q_o}{4\pi}\left(\frac{-(z-z_o')}{\left(x^2+y^2+(z-z_o')^2\right)^{3/2}} - \frac{-(z+z_o')}{\left(x^2+y^2+(z+z_o')^2\right)^{3/2}}\right)\Bigg|_{z=0}$$

$$\sigma(z=0) = -\tfrac{q_o}{4\pi}\left(\frac{2(z_o')}{\left(x^2+y^2+(z_o')^2\right)^{3/2}}\right)$$

Induced Surface Charge

$$q_{surf} = \int \sigma da$$

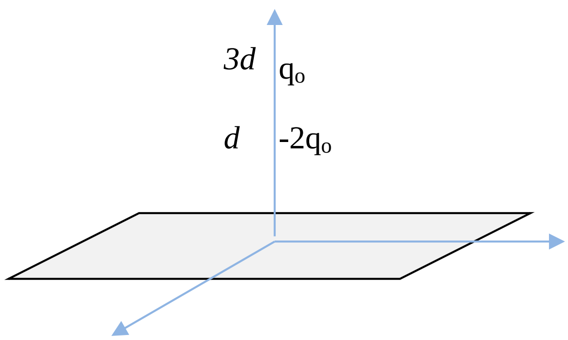
$$q_{surf} = \int -\tfrac{q_o}{4\pi}\left(\frac{2(z_o')}{\left(x^2+y^2+(z_o')^2\right)^{3/2}}\right)dxdy = \int -\tfrac{q_o}{4\pi}\left(\frac{2(z_o')}{\left(s^2+(z_o')^2\right)^{3/2}}\right)sd\phi ds = -q_o\int\left(\frac{(z_o')}{\left(s^2+(z_o')^2\right)^{3/2}}\right)\tfrac{1}{2}ds^2$$

$$q_{surf} = -q_o\left(-\frac{(z_o')}{\left(s^2+(z_o')^2\right)^{1/2}}\Bigg|_{r=0}^{s=\infty}\right) = q_o\left(0-\frac{(z_o')}{\left((z_o')^2\right)^{1/2}}\right) = -q_o$$

(no big surprise!)

This can be a serious phenomenon – you bring a charged object near a conducting surface and, even if (especially if) the surface is grounded, an image charge is induced and *it pulls the charged object toward itself.* Kind of a Narcissus effect (drawn into its own reflection).

Exercise. Where and what are the induced surface charges?



General Procedure

Now, that was an awful lot of work for something that seemed obvious, but now we know how to proceed when the answer *isn't* obvious.

1) Draw picture
2) Appeal to symmetry (and intuition about mirrors)
3) Apply the condition $\frac{q_o}{\mathscr{r}_o} + \frac{q_i}{\mathscr{r}_i} = const$
4) See what you've got to do to remove dependence on the observation location.

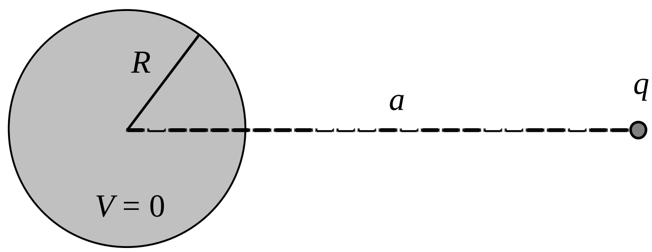
a. Mathematically, you've got 3 free parameters: the constant, the image charge's value, and the image charge's location.
b. Since the relation should be true for *all* observation locations on the conducting surface, choose easy ones to help you determine the three parameter's values.

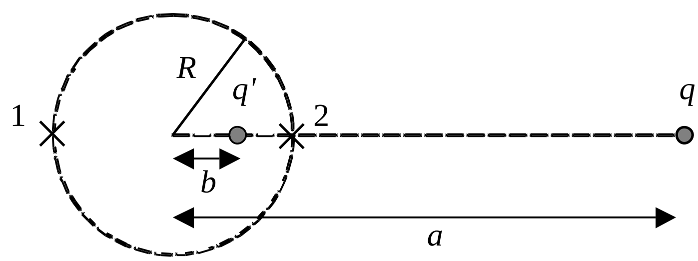
5) If you've got a solution that works for the boundary and satisfies Poisson's equation, then you've got *the* solution.

Sphere.

Now we'll apply this process to a sphere. Okay, actually, we'll just do bits and pieces of the process to show where the values of q_i and r'_i come from, we won't *prove* that they work in the general case (that is, that all r dependence disappears).

Grounded metal sphere of radius *R* and a charge *q* at a distance *a* from the center of the sphere.





Find *b* and q' from the two conditions that the potential is zero at points 1 and 2:

$$V(1) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{R+b} + \frac{q}{R+a}\right) = 0,$$

$$V(2) = \frac{1}{4\pi\varepsilon_0}\left(\frac{q'}{R-b} + \frac{q}{a-R}\right) = 0.$$

Solve each condition for q' and equate them to get

$$q' = -q\frac{(R+b)}{a+R} = -q\frac{(R-b)}{a-R},$$

$$(a-R)(R+b) = (a+R)(R-b),$$

$$2ab = 2R^2,$$

$$\boxed{b = R^2/a}.$$

Substitute *b* back in to get

$$q' = -q\frac{\left(R + R^2/a\right)}{a+R} = -\frac{q}{a}\frac{\left(Ra + R^2\right)}{a+R},$$

$$\boxed{q' = -\frac{R}{a}q}.$$

We should also write an expression for *V* of the image configuration that demonstrates that *V* = 0 on the surface of the sphere (Problem 2.7), but that is much harder.

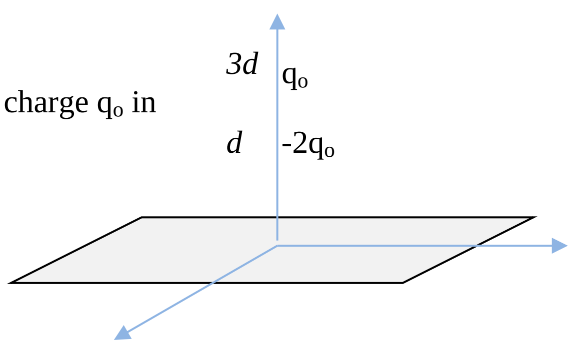
So, we can write the voltage anywhere *outside* the sphere as that due to the real charge and the appropriately charged & placed image charge.

Limitations and Extensions of method. This process is quite nice, but of limited utility – it works exactly for only very few surfaces (infinite flat plate, sphere, infinite cylinder, and maybe the ellipse?) But it can be extended in two ways

1) An infinite series of image charges can be used to model some more complicated surfaces.
2) A conductor made of such building blocks can be handled by introducing image charges for each block.
    a. Warning: Like regular images and mirrors – the conductors reflect not only the object charge but also each other's image charges.

<u>**Examples/Exercises**</u>:

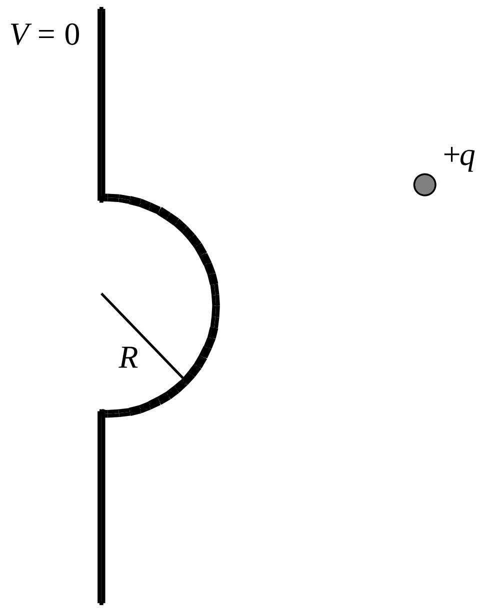
Problem 3.6. Find the force on the charge q_o in

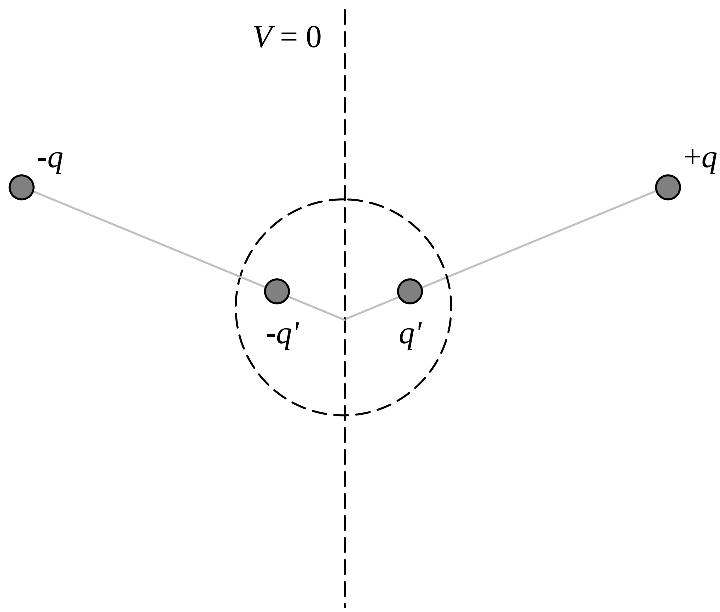


Place image charges $+2q$ at $z = -d$ and $-q$ at $z = -3d$. Total force on $+q$ is

$$\mathbf{F} = \frac{q}{4\pi\epsilon_0}\left[\frac{-2q}{(2d)^2} + \frac{2q}{(4d)^2} + \frac{-q}{(6d)^2}\right]\hat{\mathbf{z}} = \frac{q^2}{4\pi\epsilon_0 d^2}\left(-\frac{1}{2} + \frac{1}{8} - \frac{1}{36}\right)\hat{\mathbf{z}} = \boxed{-\frac{1}{4\pi\epsilon_0}\left(\frac{29q^2}{72d^2}\right)\hat{\mathbf{z}}.}$$

What is the equivalent image charge configuration? (a hemispherical bump on a plane)





Problem 3.7ab

[]

Relaxation Method

Go over the handout – download the example from the course webpage

Have students do the exercises – bring Python program with solutions (email to self)

```
from pylab import *

NX = 6
NY = 5
maxdiff = 1e-6
diff = 1.0 # must start bigger than 'maxdiff'
V = zeros((NY,NX),float) # fill 'V' with zeros

# Enter the boundary conditions for 'V' other than 0
for i in arange(0,NY,1): # j from 0 to NY-1
    V[i,NX-1] = 1.0
print V # print the starting values

while diff > maxdiff: # repeat until 'diff' is small
    diff = 0.0 # reset for each iteration
    # Loop over all of the interior points
    for i in arange(1,NY-1,1): # i from 1 to NY-2
        for j in arange(1,NX-1,1): # j from 1 to NX-2
            newVij = (V[i-1,j]+V[i+1,j]
                     +V[i,j-1]+V[i,j+1])/4.0
            lastdiff = abs(newVij-V[i,j])
            if lastdiff > diff:
                diff = lastdiff
            V[i,j] = newVij
print V # print the final values
```

```
# Tutorial 4 Example 1
from __future__ import division
from numpy import *
from pylab import *

w = 1.5 #mixing constant for 'overrelaxation' (actually, looks more like 'underrelaxation' to me since it retains some of the old value
NX = 6
NY = 5
maxdiff = 1e-6
diff = 1.0 # must start bigger than 'maxdiff'
V = zeros((NY, NX), float) #fill 'V' with zeros but specifying as floats rather than integers

#Enter the boundary conditions for 'V' other than 0
for i in arange (0, NY, 1):  # i from 0 to NY-1
    V[i, NX-1] = 1.0
print V # print the starting values

while diff > maxdiff:  # repeat until 'diff' is small
    diff = 0.0 # reset for each iteration
    # Loop over all of the interior points
    for i in arange (1, NY-1, 1): # i from 1 to NY-2
        for j in arange(1, NX-1, 1): # j from 1 to NX-2
            newVij = w*(V[i-1,j] + V[i+1,j] + V[i,j-1]+V[i,j+1])/4 + (1-w)*V[i,j]
            lastdiff = abs(newVij-V[i,j])
            if lastdiff > diff:
                diff = lastdiff
            V[i,j] = newVij
print V #print the final vlaues

# The following code is based on Tutorial 3  It creates a line contour plot overlayed on a filled contour plot
xlist = linspace(0.0, 5.0, NX)
ylist = linspace(-2.0, 2.0, NY)
X, Y = meshgrid(xlist, ylist)
levels = linspace(0, 1, 10)

figure()
CP1 = contour(X,Y,V)
CP2 = contourf(X,Y,V, levels)
clabel(CP1, inline = True, fontsize = 10)
show()
```

```
# Tutorial 4 Example 1
from __future__ import division
from numpy import *
from pylab import *

w = 1.5 #mixing constant for 'overrelaxation' (actually, looks more like 'underrelaxation' to me since it retai
NX = 50
NY = NX
maxdiff = 1e-6
diff = 1.0 # must start bigger than 'maxdiff'
V = zeros((NY, NX), float) #fill 'V' with zeros but specifying as floats rather than integers

#Enter the boundary conditions for 'V' other than 0
for i in arange (0, NY, 1):  # i from 0 to NY-1
    V[i, NX-1] = 0.7
    V[i, 0] = 0.3
for j in arange (0, NX, 1):  # i from 0 to NX-1
    V[0, j] = 1
print V # print the starting values

while diff > maxdiff:  # repeat until 'diff' is small
    diff = 0.0 # reset for each iteration
    # Loop over all of the interior points
    for i in arange (1, NY-1, 1): # i from 1 to NY-2
        for j in arange(1, NX-1, 1): # j from 1 to NX-2
            newVij = w*(V[i-1,j] + V[i+1,j] + V[i,j-1]+V[i,j+1])/4 + (1-w)*V[i,j]
            lastdiff = abs(newVij-V[i,j])
            if lastdiff > diff:
                diff = lastdiff
            V[i,j] = newVij
print V #print the final vlaues

# The following code is based on Tutorial 3  It creates a line contour plot overlayed on a filled contour plot
xlist = linspace(0.0, 11.0, NX)
ylist = linspace(-3.0, 3.0, NY)
X, Y = meshgrid(xlist, ylist)
levels = linspace(0, 1, 10)

figure()
CP1 = contour(X,Y,V)
CP2 = contourf(X,Y,V, levels)
clabel(CP1, inline = True, fontsize = 10)
show()
```

"can we go over how the uniqueness theorems apply to method of images and also describe typical situations in which it would be useful?"
Sam Post a response
Admin
Flag as inappropriate

"Can we talk about why the energy of a single charge and conducting plane is half that of two charges? I'm having a hard time visualizing this."
Spencer Post a response
Admin
Flag as inappropriate

"What is a "locus"? its mentioned in a footnote as a geometric argument in Example 3.2."
Casey P, AHoN swag 4 liphe Hide response Post a response
Admin
Flag as inappropriate

I've heard that word before. Isn't it just such that you have a certain condition, and every function that you have that satisfies that condition is a member of that locus?

I mean I think. Generally speaking.

Rachael Hach

"Could we talk about the Force and Energy section a bit - that part was a little unclear to me."
Casey McGrath Hide response Post a response
Admin

I agree with casey, I didn't really understand this section.

Jessica