Wed. 9/18 Thurs 9/19 Fri. 9/20	**2.4.1-.4.2** Work & Energy in Electrostatics **T3** Contour Plots **2.4.3-4.4** Work & Energy in Electrostatics	HW2
Mon. 9/23 Wed. 9/25 Thurs 9/26	**2.5** Conductors **3.1-.2** Laplace & Images ***Summer Science Research Poster Session****: Hedco7pm~9pm*	HW3

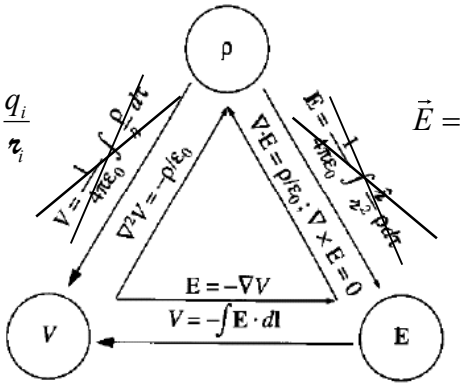
Materials

Computational Tutorial 3

Summary of Electrostatics

Figure 2.35 is a good summary, but the summation equations for V and $\vec{E}$ are much more useful than the integrals ones. We've used all of the relations, except for Poisson's Equation, $\nabla^2 V = -\rho/\varepsilon_0$.

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i} \qquad \vec{E} = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i^2}\hat{r}_i$$



Only 1 Relation's worth of info. Griffith's points out that , while there are 7 different relations here, all the essential information is contained in just one: Coulomb's Law, the others follow from it.

Boundary Conditions

Intro: future use in solving Diff. Eq's. Half of the relations in the above triangle of logic are *differential equations.* We will eventually get in the business of *solving differential equations.* You may remember from past experience solving differential equations that a key step is applying *boundary conditions* – those take you from general to specific solutions to the differential equation. Though we're not now going to solve these differential equations, we do have the tools now to say a few things about how E and V behave across boundaries.

Suppose there is a surface with a charge density σ (may depend on position). How do the electric field and the potential differ across the boundary? Be careful to be consistent about what direction is called positive.

Qualitatively, say you've got *no* charge on the sheet, then the fields above and below should be the same. Now, say you've got some charge density on the sheet, for argument's sake, say it's positive. Then roughly, what does the electric field due to that look like?

(they draw radiating down below and up above)

Now say that the charge density has a gradient, so it's denser on the left edge and less dense on the right, what would the field look like?

(they draw radiating up and to the right above, and down and to the right below.)

So, qualitatively, we expect the perpendicular component of the field to change, pretty dramatically across the sheet, and the parallel component to be unchanged. Let's now get a little mathematical.

Perpendicular component

Apply Gauss's Law for a thin pillbox that extends across the boundary. Let the thickness ε go to zero to get

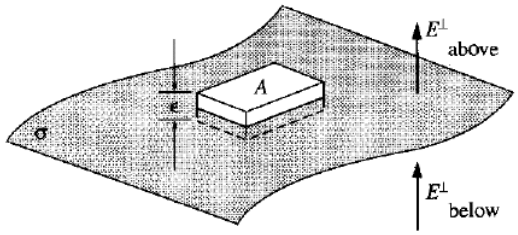
$$\oint \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_o}$$

$$2((E^{||}_{above} + E^{||}_{below})A_{side/\ front-back} + (E^{||}_{above} + E^{||}_{below})A_{side/\ left-right}) + E^{\perp}_{\text{above}}A_{top} - E^{\perp}_{\text{below}}A_{top} = \frac{Q}{\varepsilon_o}.$$

$$0 + + E^{\perp}_{\text{above}}A_{top} - E^{\perp}_{\text{below}}A_{top} = \frac{Q}{\varepsilon_o}$$

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\varepsilon_0}\sigma$$

(the 0 because we'll snug this box right down to the surface, to see the change that happens *right* across the surface)



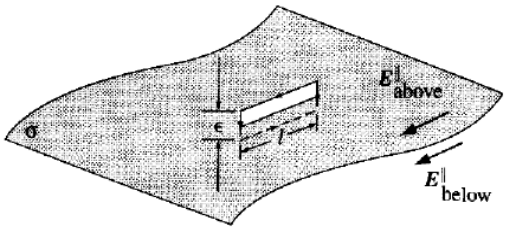
Apply $\oint \vec{E} \cdot d\vec{\ell} = 0$ to a thin rectangular loop that extends across the boundary.

Similarly, the perpendicular components' contributions vanish if we make our loop snug right to the surface, so

$$\oint \vec{E} \cdot d\vec{\ell} = 0$$

$$E^{||}_{above}l - E^{||}_{below}l = 0$$

$$\vec{E}^{||}_{\text{above}} = \vec{E}^{||}_{\text{below}}.$$



The results for the two components can be summarized as

$$\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{n},$$

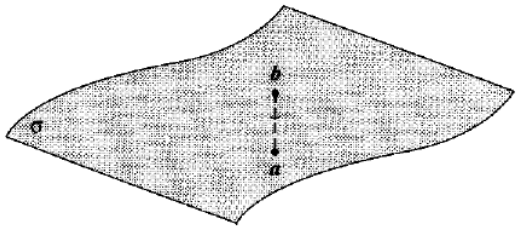
where $\hat{n}$ is a unit vector pointing from "below" to "above."

Find the potential difference for a short path across the boundary.

$$\Delta V = -\int_a^b \vec{E} \cdot d\vec{\ell}$$

Shrink the length of the path to get

$$V_{\text{above}} = V_{\text{below}}$$



Of course, since

$\vec{E} = -\vec{\nabla}V$ and $\vec{E}_{\text{above}} - \vec{E}_{\text{below}} = \frac{\sigma}{\varepsilon_0}\hat{n}$ we have

$$\vec{\nabla}V_a - \vec{\nabla}V_b = -\frac{\sigma}{\varepsilon_0}\hat{n}$$

$$\frac{\partial}{\partial n}V_a - \frac{\partial}{\partial n}V_b = -\frac{\sigma}{\varepsilon_0}$$

Don't worry about "normal derivatives."

Ex. Boundary Conditions at a Charged Spherical Shell

Confirm that the electric field for a spherical shell of radius *R* with a charge *Q* satisfies the boundary conditions stated above.

$$\vec{E}(\vec{r}) = \begin{cases} 0 & r < R \\ \dfrac{1}{4\pi\varepsilon_0}\dfrac{Q}{r^2}\hat{r} & r > R \end{cases}$$

Let's call the outside of the sphere "above" and the inside "below." There is no component of the electric field parallel to the boundary, so $\vec{E}^{\parallel}_{\text{above}} = \vec{E}^{\parallel}_{\text{below}}$ is trivial.

The normal is $\hat{n} = \hat{r}$. The charge per area is

$$\sigma = \frac{Q}{4\pi R^2}.$$

The two sides of the second condition are

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2} - 0 = \frac{1}{4\pi\varepsilon_0}\frac{Q}{R^2},$$

$$\frac{\sigma}{\varepsilon_0} = \frac{\left(Q/4\pi R^2\right)}{\varepsilon_0},$$

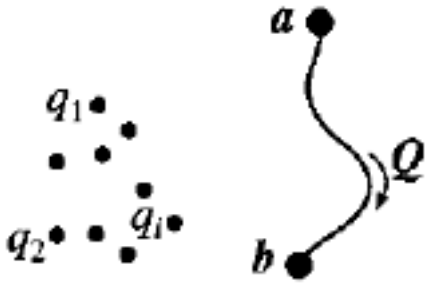
which are equal as they should be.

Work to Move a Charge

We *motivated* the introduction of the Electric Potential Difference by noting that it was essentially the Change in Electric Potential Energy for moving a charge through a field, divided by that charge. Of course, the change in electric potential energy is just the opposite of the work that you have to do to cause that change; the book takes the Work perspective.

Now we'll look more closely at that. Today, we'll mostly look at discrete / countable charge distributions. Next time we'll think carefully about how to generalize the math (without tempting over-counting) to handle continuous charge distributions.

Suppose there are a bunch of stationary "source" charges. How do you calculate the amount of work required to move a "test" charge *Q* from point *a* to point *b*?



The minimum force that must be exerted is opposite of the electrical force due to the source charges, so

$$W_{a\to b} = \int_a^b \vec{F}_{exerted}\cdot d\vec{\ell} = -\int_a^b \vec{F}_{electric}\cdot d\vec{\ell} = -\int_a^b \left(Q\vec{E}\right)\cdot d\vec{\ell} = -Q\int_a^b \vec{E}\cdot d\vec{\ell} = Q\left[V(b)-V(a)\right].$$

In other words, the *potential difference* between two points is the work per charge required to move a charge between the points:

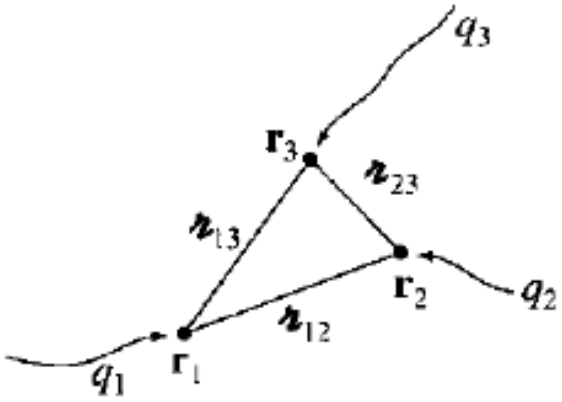
$$V(b)-V(a) = \frac{W_{a\to b}}{Q}.$$

If $V(\infty)=0$ is chosen for a localized charge distribution, the electric potential at a location $\vec{r}$ is related to the amount of work that would be required to move an additional (test) charge to that location from infinity (or approximately from very far away):

$$W_{\infty\to\vec{r}} = Q\left[V(\vec{r})-V(\infty)\right] = QV(\vec{r}).$$

Work to assemble / Energy in a Point Charge Distribution

How much energy is required to assemble a collection of point charges?



Imagine the charges initially very far from each other. Calculate the work required to bring the charges in one at a time. No work ($W_1=0$) required to put the first charge in place because there is nothing else around. The work required to put the second charge in place is

$$W_2 = q_2 V(\text{due to } q_1) = q_2\left(\frac{1}{4\pi\varepsilon_0}\frac{q_1}{\boldsymbol{r}_{12}}\right).$$

The work required to put the third charge in place is

$$W_3 = q_3 V(\text{due to } q_1, q_2) = q_3\left[\frac{1}{4\pi\varepsilon_0}\left(\frac{q_1}{\boldsymbol{r}_{13}}+\frac{q_2}{\boldsymbol{r}_{23}}\right)\right],$$

and so on. The total work can be written as the sum of products all unique pairs of charges over their separations:

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{\boldsymbol{r}_{ij}}.$$

If the energy required to assemble a configuration is positive, that amount of energy is stored (and can be released). For example, two positive charges brought together will fly apart when released.

Example: **Nuclear Fission of Uranium-238 ($^{238}_{92}\mathrm{U}$)**

Let's estimate how much energy is released when a $^{238}_{92}\mathrm{U}$ nucleus (92 protons and 238 total nucleons) fissions. Uranium can break into a variety of products, but we'll assume that it

goes into two identical nuclei with 46 protons and 119 total nucleons each ($^{119}_{46}$Pd, a Palladium isotope). The radius for a uranium nucleus is about $10\ fm = 10\times10^{-15}\text{m} = 10^{-14}\text{m}$, so let's assume that the two "daughter" nuclei start a distance $d = 2\times10^{-14}\text{m}$ apart. For simplicity, we'll treat the nuclei as point charges. The energy released when the two nuclei fly apart (work to bring them together) is

$$W = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{d} = \frac{1}{4\pi\varepsilon_0}\frac{(46e)^2}{d} = \frac{1}{4\pi\left(8.85\times10^{-12}\text{C}^2/\text{Nm}^2\right)}\frac{\left[(46)(1.6\times10^{-19}\text{C})\right]^2}{(2\times10^{-14}\text{m})}$$

$$= 2.4\times10^{-11}\text{J} = 2.4\times10^{-4}\text{ergs}$$

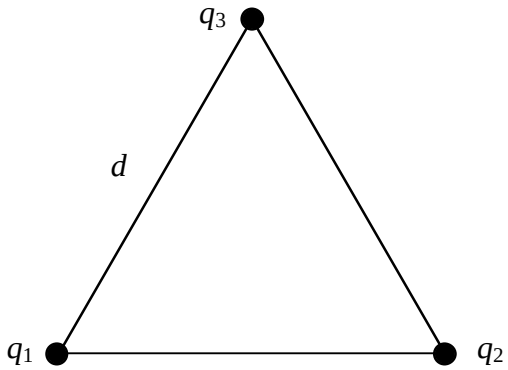
That may seem small, but it is very large for a single reaction! The units of "ergs" (1 J = 10^7 ergs) are sometimes used for chemical reactions. TNT will release about 4×10^{10}ergs/gram. How does that compare to the energy that could be released by pure $^{238}_{92}$U? There are 238 grams/mole of $^{238}_{92}$U, so there are about $6\times10^{23}\text{nuclei}/238$ grams and

$$\left(\frac{2.4\times10^{-4}\text{ergs}}{\text{nucleus}}\right)\left(\frac{6\times10^{23}\text{nuclei}}{238\text{ grams}}\right) \approx 6\times10^{17}\text{ergs/gram}$$

which is over 10 million times larger! (Note that relativity is not required!)

Exercises:

Suppose 3 charges sit on the corners of an equilateral triangle with sides of length *d*.



a. Find the work required to move q_3 to the point halfway between q_1 and q_2.

We need to know the potential due to q_1 and q_2 (the source charges) at (a) the initial location and (b) the final location.

$$V(a) = \frac{q_1}{4\pi\varepsilon_0 d} + \frac{q_2}{4\pi\varepsilon_0 d} = \frac{(q_1+q_2)}{4\pi\varepsilon_0 d}$$

$$V(b) = \frac{q_1}{4\pi\varepsilon_0 (d/2)} + \frac{q_2}{4\pi\varepsilon_0 (d/2)} = \frac{2(q_1+q_2)}{4\pi\varepsilon_0 d}$$

Multiply the potential difference times q_3 (the test charge) to find the work:

$$W_{a\to b} = q_3\left[V(b) - V(a)\right] = q_3\left[\frac{2(q_1+q_2)}{4\pi\varepsilon_0 d} - \frac{(q_1+q_2)}{4\pi\varepsilon_0 d}\right] = \frac{(q_1+q_2)q_3}{4\pi\varepsilon_0 d}$$

b. How much work does it take to assemble the original configuration?

The work required to put the first charge in place is

$$W_1 = 0.$$

because there is nothing else around. The work required to put the second charge in place is

$$W_2 = \frac{q_2}{4\pi\varepsilon_0}\left(\frac{q_1}{d}\right) = \frac{q_1 q_2}{4\pi\varepsilon_0 d}.$$

The work required to put the third charge in place is

$$W_3 = \frac{q_3}{4\pi\varepsilon_0}\left(\frac{q_1}{d} + \frac{q_2}{d}\right) = \frac{(q_1+q_2)q_3}{4\pi\varepsilon_0 d}.$$

The total work to assemble the configuration of charges is

$$W_{\text{total}} = W_1 + W_2 + W_3 = \frac{(q_1q_2 + q_1q_3 + q_2q_3)}{4\pi\varepsilon_0 d}.$$

This includes all unique pairs (e.g. - including q_2q_1 would be double counting). This example is simple because all of the distances between charges are the same.

Computational Tutorial

Making contour plots with Python

- download the file and try each example by removing comments
- do the exercise at the end

Preview

More about energy related to *continuous* charge distributions.

We'll also have some time for HW questions or review. Come with questions!

Summary

Energy of a Continuous Charge Distribution

The work to assemble a discrete charge distribution is

$$W = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{j<i}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\left[\frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{n}\sum_{i\neq j}\frac{q_i q_j}{r_{ij}}\right] = \frac{1}{2}\sum_{i=1}^{n} q_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{i\neq j}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{n} q_i V(P_i).$$

The ½ in the second expression is because of double counting. The second sum can be rewritten to give the potential at the location of charge q_i due to all of the other charges – called $V(P_i)$.

The final form can be turned into a volume integral:

$$W = \frac{1}{2}\int \rho V \, d\tau, \qquad [2.43]$$

because the charge in a small volume $d\tau$ is $\rho\, d\tau$. Using a product rule and the divergence theorem, this can be rewritten as

$$W = \frac{\varepsilon_0}{2}\int_{\text{all space}} E^2 \, d\tau. \qquad [2.45]$$

First use the differential form of Gauss's Law, $\vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0$, to write

$$W = \frac{\varepsilon_0}{2}\int (\vec{\nabla}\cdot\vec{E}) V \, d\tau.$$

The product rule:

$$\vec{\nabla}\cdot(\vec{E}V) = (\vec{\nabla}\cdot\vec{E})V + \vec{E}\cdot(\vec{\nabla}V)$$

and the relation $\vec{\nabla}V = -\vec{E}$ to rewrite the integrand as

$$W = \frac{\varepsilon_0}{2}\left[\int \vec{\nabla}\cdot(\vec{E}V)\, d\tau + \int E^2 d\tau\right].$$

Apply the divergence theorem, $\int_V (\vec{\nabla}\cdot\vec{A})\, d\tau = \oint_S \vec{A}\cdot d\vec{a}$, to the first integral to get

$$W = \frac{\varepsilon_0}{2}\left[\int_{Surface} V\vec{E}\cdot d\vec{a} + \int_{Volume} E^2 d\tau\right].$$

Originally, the integral was just over the region were the charge resided, but we can expand it to regions where $\rho = 0$ without changing the answer. If we let the volume increase to all of space, the first integral goes to zero because *V* and *E* get smaller far away from the charge. This gives us the result in Eq. 2.45.

Eq. 2.45 implies that $\varepsilon_0 E^2/2$ is the energy per volume stored, so we can say that energy is stored *in the electric field*.

Note that the superposition principle does not hold for electrostatic energy because it depends on the square of the field strength at each location. If the charge everywhere is doubled, the total energy quadruples.

Example/Exercise:

Problem 2.32 – Energy Stored in a Uniformly Charged Solid Sphere

Suppose the sphere has a radius *R* and a total charge *q* (positive).

To use Eq. 2.45, we need to know the electric field everywhere. The simplest method is to use Gauss's Law,

$$\oint_S \vec{E} \cdot d\vec{a} = \frac{1}{\varepsilon_0} Q_{\text{enc}}.$$

By symmetry, the electric field must point radially outward. For a spherical Gaussian surface of radius *r*, the electric flux is $E\left(4\pi r^2\right)$. The amount of charge enclosed depends on the size of *r*:

$$Q_{enc} = \begin{cases} \frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3} q = \frac{q r^3}{R^3} & r < R \\ q & r > R \end{cases}$$

For $r < R$, the fraction is the ratio of the volume enclosed to the total volume.

Applying Gauss's Law gives

$$\vec{E} = \begin{cases} \frac{1}{4\pi\varepsilon_0} \frac{qr}{R^3} \hat{r} & r < R \\ \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \hat{r} & r > R \end{cases}$$

The square of the electric field will be approximately constant in a thin spherical shell between *r* and *r* + *dr*. The energy for such a shell is

$$dW = \frac{\varepsilon_0}{2} E^2(r)\left[4\pi r^2 dr\right].$$

The integral over all of space must be broken into two parts because of the different regions for *E*:

$$W = \frac{\varepsilon_0}{2}\left\{\int_0^R\left(\frac{1}{4\pi\varepsilon_0}\frac{qr}{R^3}\right)^2\left[4\pi r^2 dr\right]+\int_R^\infty\left(\frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\right)^2\left[4\pi r^2 dr\right]\right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{\frac{1}{R^6}\int_0^R r^4 dr+\int_R^\infty r^{-2}dr\right\} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{\frac{1}{R^6}\left[\frac{r^5}{5}\right]_0^R+\left[\frac{-1}{r}\right]_R^\infty\right\}$$

$$= \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{\frac{1}{5R}+\frac{1}{R}\right\} = \frac{1}{4\pi\varepsilon_0}\frac{q^2}{2}\left\{\frac{6}{5R}\right\} = \frac{1}{4\pi\varepsilon_0}\frac{3q^2}{5R}$$

Preview

For Friday, you'll read about conductors (Section 2.5)

You will need a copy of the textbook when we reach Ch. 3 soon (definitely by Oct. 1)

Exam #1 will take place on Friday, Oct. 5. It will cover HW #1-3 or up to Section 2.4.

Remaining Time

Time for questions about HW #3 (or any review of HW #1 & #2)

You can also work on the computational problem (IE 11) from HW #3 – can email results because they are more complicated and use much more ink

Tutorials

```
from pylab import *

xlist = linspace(-3.0, 3.0, 40)
ylist = linspace(-3.0, 3.0, 30)
X, Y = meshgrid(xlist, ylist)
print X
print Y
Z = sqrt(X**2 + Y**2)
print Z

figure()

### Example 1 ###
#CP1 = contour(X, Y, Z) # replace this
#clabel(CP1, inline=True, fontsize=10) # replace this
### Example 1 ###

### Example 2 ###
#CP2 = contourf(X, Y, Z)
#colorbar(CP2)
### Example 2 ###

### Example 3 ###
#levels = [ 0, 0.5, 1, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0, 4.5, 5.0]
#CP3 = contour(X, Y, Z, levels, colors = 'k')
#clabel(CP3, colors = 'k', fmt = '%3.1f', fontsize=14)
#CP4 = contourf(X, Y, Z, levels)
#colorbar(CP4)
### Example 3 ###

title('Contour Plot')
xlabel('x (cm)')
ylabel('y (cm)')
show()
```

```
from __future__ import division
from pylab import *

xlist = linspace (-10.0, 10.0, 200)
ylist = linspace (-10.0, 10.0, 200)
X, Y = meshgrid(xlist, ylist)
#print X,
#print Y
Z = (X**2)/9 - (Y**2)/4
#print Z

#the format is something like
#(x1, x2, x3, x4)...
#(x1, x2, x3, x4)...
#(x1, x2, x3, x4)...
#.....................

#(y1, y1, y1, y1)...
#(y2, y2, y2, y2)...
#(y3, y3, y3, y3)...
#.....................

#so then the z values use
#(z(x1,y1), z(x2,y1), z(x3,y1), z(x4,y1)...
#(z(x1,y2), z(x2,y2), z(x3,y2), z(x4,y2)...
#(z(x1,y3), z(x2,y3), z(x3,y3), z(x4,y3)...
#.....................................

figure()
#use the two following lines for a contour plot
#CP1 = contour (X, Y, Z)
#clabel(CP1, inline = True, fontsize = 10)

#or, use the two following lines for a filled contour plot
#CP2 = contourf(X,Y,Z)
#colorbar(CP2)

#or, use these five lines to do both!
#levels = [0.0, 0.5, 1, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0]
levels = linspace (-24,12,10)
CP3 = contour(X, Y, Z, levels, colors='k')
clabel(CP3, colors='k', fmt='%2.1f', fontsize=14)
CP4 = contourf(X,Y,Z, levels)
colorbar(CP4)

title('Contour Plot')
xlabel('x (cm)')
ylabel('y (cm)')
show()
```

"Can we do an example problem using eqn 2.42, where we find how much work it takes to put together a configuration of point charges?"
Jessica Hide responses Post a response
Admin
Jessica took my question.

Spencer
Spencer took my comment.

Casey P, AHoN swag 4 liphe
On a side note, I like the way Griffiths described bringing in all the charges, by taking their quantitative influences into account then nailing them down. My brain was already trying to manage point charges repelling each other in addition to calculating the work on the next point charge entering the system. Good thing it's not that!

Rachael Hach

Flag as inappropriate

"I thought the definition of Work was W= - integral(F*dl) , did they forget a negative sign here in front? And so equation 2.39 is the definition of potential energy (with a reference at infinity)? Can we talk more about this idea of a...."
Casey McGrath Hide responses Post a response
Admin
.... a reference point at infinity and why we keep doing it so often?

Casey McGrath
I think you're thinking of potential energy. I think it's fine how it is.

Casey P, AHoN swag 4 liphe
Doesn't the sign depend on how you define L and which way you integrate it?

Spencer
I could be way off but I got the impression that since they defined W on pg. 91 as a potential DifFeReNce, that a sign wouldn't matter.

Rachael Hach

Flag as inappropriate

"Maybe this gets addressed later but in this section we are only ever dealing with work associated with assembling point charge distributions but could that sum turn into an integral to find work associated with a continuous distribution?"
Ben Kid Post a response
Admin

Flag as inappropriate

"Why does Griffiths seem so keen to differentiate potential from potential energy, even as he states that potential is potential energy per unit charge? Even just this description makes the concept make more sense than it did before."
Freeman, Nappleton Post a response
Admin

Flag as inappropriate

"Could we go over physical effects caused by the laws? It helps me visualize what is going on. On page 101 Griffiths both describes the spreading of charge to the surface by repulsion and the reasons for not being shocked by your car in a thunderstorm"
Anton Post a response
Admin

Flag as inappropriate

"Could we please go over the argument made in section 4.2 and see an example problem where the info from that section would be useful?"
Sam Post a response
Admin

Flag as inappropriate

"I was wondering if we could go over Poisson's Equation and it's relationship to Gauss's law. It is equation 2.24 and it comes up in both chapters 2 and 3."
Anton Post a response
Admin