Fri. 9/13	*(C 16)* **1.6, 2.3.1 -.3.3** Electric Potential	
Mon. 9/16 Wed. 9/18 Thurs 9/19	*(C 16)* **2.3.4-.3.5** Electric Potential **2.4.1-.4.2** Work & Energy in Electrostatics **T3** Contour Plots	HW2

Materials

Last Time

We used Gauss's Law some more

We looked at the curl of E, saw that it was 0.

$$\vec{\nabla}\times\vec{E}=0$$

This Time

Summary

Electric Potential

- **Two Approaches.** There are two approaches to defining the Electric Potential. One is more physical (here's a conceptually reasonable tool) and the other one is more mathematical (such a tool must exist.) We'll take a quick look at both.
  - **Physical Approach.** We're going to take a step back, so we can contextualize this integral, and get a feel for what it means. Cast your memory back to good old classical mechanics. That's all about relating interactions to changes in motion. We do that with forces and momentum, and we do that with work and energy. (In Advanced Classical Mechanics we even do that with Legragians, Hamiltonians, and action.) In the framework of Forces, a natural tool to define is the electric field – force divided by the charge at the receiving end. So the first half of the chapter was about that. In the framework of Energy, a natural tool to define is the electric potential – potential energy divided by the charge at the receiving end. The rest of the chapter is about that.
    - **Review of Work & Energy.**
      - If an external agent, 1, acts on your system, 2, then we'd say that 1 does work on 2 to the tune of
        - $W_{1\to 2}\equiv\int_a^b\vec{F}_{1\to 2}\cdot d\vec{\ell}$. Where *l* we integrate over the path that the point of application of that force to object 2 takes.
      - Alternatively, if we defined our system so that both 1 & 2 were members of it, then, in our great bookkeeping of energy, we'd say that the composite system's *potential* energy changes to the tune of
        - $\Delta P.E._{1\to 2}\equiv-\int_a^b\vec{F}_{1\to 2}\cdot d\vec{\ell}$
          - Subtle difference: now the integral is not just over the motion of object 2 relative to some fixed origin, but its motion relative to object 1 – so the integral is over the *change in separation* of the two. Though quite often, we apply this in situations for which

object 1 (say, the Earth) is approximately stationary during the interaction with object 2 (say, a ball).

- Now, let's make this a little more specific to this class. In the case of an *electric* interaction, the force can be expressed as the product
    - $\vec{F}_{1\to 2} = q_2 \vec{E}_1(\vec{r}_2)$
- So, then
    - $\Delta P.E._{1\to 2} = -\int_a^b q_2 \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell} = -q_2 \int_a^b \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell}$
- That leads to a very natural definition, the **Electric Potential Difference.**
    - $\Delta V_1 \equiv \frac{\Delta P.E._{1\to 2}}{q_2} = -\int_a^b \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell}$

- **Conceptual / Visualization.**
    - **Electric Field and Flow.** If the Electric Field is analogous to an ever-present flow that, should you drop a charge into it, is ready and willing to sweep the charge away, then…
    - **Potential and Elevation.** The Electric Potential Difference is analogous to elevation differences – given the opportunity, a charge dropped at a high elevation will happily 'roll down hill' to a lower elevation.
        - **Topal maps and Equipotential lines.** Just like we map the physical topography of a terrain with a topal map, equi-elevation lines helping us to visualize where the high, low, steep, level, bits are, we map the electrical topography with equipotential lines.

- **Mathematical Approach.** Off we go to Math Land: Section 1.6 presents Helmholtz Theorem(s).
    - If the curl of a vector field ($\vec{F}$) vanishes everywhere, then $\vec{F}$ can be written as the gradient of a scalar potential (*f*):
        - $\vec{F} = \vec{\nabla} f$
    - Or perhaps, say it the other way around, assume you *have* a scalar field (like elevation or temperature), then let's take a look at the curl of its gradient.
        - $\vec{\nabla} \times (\vec{\nabla} f) = ?$
            - Well, $\vec{\nabla} f = \left(\frac{\partial}{\partial x} f\right)\hat{x} + \left(\frac{\partial}{\partial y} f\right)\hat{y} + \left(\frac{\partial}{\partial z} f\right)\hat{z}$
            - Then it's curl is

$$\vec{\nabla} \times (\vec{\nabla} f) = \left(\frac{\partial}{\partial y}\left(\frac{\partial}{\partial z} f\right) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial y} f\right)\right)\hat{x} + (..)\hat{y} + (..)\hat{z}$$

$$\vec{\nabla} \times (\vec{\nabla} f) = \left(\frac{\partial^2}{\partial y \partial z} f - \frac{\partial^2}{\partial z \partial y} f\right)\hat{x} + (..)\hat{y} + (..)\hat{z}$$

$$\vec{\nabla} \times (\vec{\nabla} f) = 0$$

regardless of how *f* depends on x,y,or z.

- Naturally, we're free to *name* the gradient of *f:*
    - $\vec{F} \equiv \vec{\nabla} f$
- So, what we've got is
    - When $\vec{\nabla} \times (\vec{F}) = 0$, we can define an *f* such that $\vec{F} \equiv \vec{\nabla} f$.

▪ *Back from Math Land.* The vector field of our interest is $\vec{E}$, so we'll define a scalar "potential" V such that

$$\vec{F} \equiv \vec{\nabla} f$$

$$\downarrow \quad \downarrow$$

- $\vec{E} \quad (V)$

$$so$$

$$\vec{E} = -\vec{\nabla} V$$

- (the 'arbitrary' choice to define as a negative will pay off)

- Integrating both sides gives
    - $\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \vec{\nabla} V \cdot d\vec{l}$
    - Applying the fundamental theorem of calculus:

$$\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \vec{\nabla} V \cdot d\vec{l}$$

- $\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b \left(\frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \right)$

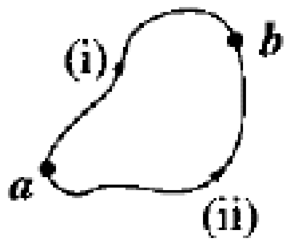
$$\int_a^b \vec{E} \cdot d\vec{l} = -\int_a^b dV = -(V_b - V_a)$$

- Which is exactly what we decided to define based on physical reasoning.

Path Independence: Notice that this integral yields the difference of the voltage evaluated at two locations. Obviously, if those two locations are the *same* one location, i.e., we integrated along a closed path, we'd get

$$\oint \vec{E} \cdot d\vec{l} = -(V_a - V_a) = 0$$

Clearly, $\oint \vec{E} \cdot d\vec{\ell} = 0$ doesn't depend on the path of integration. In terms of my analogy to elevation, this integral is like the change in gravitational potential energy when you execute a round trip – nothing. More generally, even over a short segment, the integral, $\int_a^b \vec{E} \cdot d\vec{\ell}$ is independent of the path (see Fig. 2.30 below).



The proof goes something like this:

$$\oint \vec{E}\cdot d\vec{l} = 0$$

$$\int_{a\ (path.i)}^{b} \vec{E}\cdot d\vec{l} + \int_{b\ (path.ii)}^{a} \vec{E}\cdot d\vec{l} = 0$$

$$\int_{a\ (path.i)}^{b} \vec{E}\cdot d\vec{l} = -\int_{b\ (path.ii)}^{a} \vec{E}\cdot d\vec{l} = \int_{a\ (path.ii)}^{b} \vec{E}\cdot d\vec{l}$$

For any two paths *i* and *ii*.

***Potential* difference.** This means that it makes physical sense to think of V as a "potential" – if you consider moving from point a to b, no matter the path, you have the potential to change by the corresponding voltage. That's like saying that mowing the lawn has the potential to earn you $5 – you don't get paid more or less depending on how you zigg-zagg across the grass.

Potential at a *point*. Often it's convenient to define the electric potential at a point *P* as the line integral from a *reference point* $\mathcal{O}$ (where we'll define $V = 0$):

$$V(P) \equiv -\int_{\mathcal{O}}^{P} \vec{E}\cdot d\vec{\ell} \qquad [2.21]$$

It does not matter what path you choose to calculate *V*, but <u>you must pick a path</u> to perform a line integral!

Relation to Curl. In some sense, all this hinges on

$$\vec{\nabla}\times\vec{E} = 0.$$

Because E has no curl, it can be expressed as the gradient of a scalar potential.

With that in mind…

<u>Exercise</u>:

For the following electric field:

$$\vec{E} = ky\hat{x} + kx\hat{y}$$

a. Check that $\vec{E}$ is a valid electrostatic field (curl-less.)

$$\vec{\nabla} \times \vec{E} = k \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & 0 \end{vmatrix} = k\left[\hat{x}(0-0) + \hat{y}(0-0) + \hat{z}(1-1)\right] = 0$$

Later on, when we're dealing with time-varying electric fields, we'll see that there *is* curl, and this will no longer be the case. Or, rather, we'll need to generalize our definition of a "potential."

Why use the potential instead of the electric field? Essentially, because scalars are easier to work with then vectors.

- **Other things to know about *V*.**
    - **Arbitrary Reference Point.** The location of the reference point is arbitrary. Moving it will just shift the electric potential by a constant.
    - **Superposition Principle.** The potential obeys the principle of superposition. If V_1, V_2, … are the potentials of individual point charges q_1, q_2, … then the total potential of all charges is $V = V_1 + V_2 + \ldots$
    - **Units.** The units of potential are volts = (joules per coulomb).

<u>Example</u>: Potential of a Point Charge at a Distance *r* (similar to Ex. 2.6)

- For this problem (and most finite distributions of charge), infinity is used as the reference point. We might as well define the origin to be at the location of the charge since there is only one. In that case, the electric field at *r* is
- $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$$
- Switching limits introduces a minus sign. Integrate along a radial line (outward after swapping limits), so $d\vec{\ell} = dr\,\hat{r}$. The potential at a distance *r* is
- $$V(r) = -\int_{\infty}^{r} \vec{E}\cdot d\vec{\ell} = +\int_{r}^{\infty} \vec{E}\cdot d\vec{\ell} = \frac{q}{4\pi\varepsilon_0}\int_{r}^{\infty}\left(\frac{1}{r^2}\hat{r}\right)\cdot(dr\,\hat{r}) = \frac{q}{4\pi\varepsilon_0}\int_{r}^{\infty}\frac{dr}{r^2}$$
  $$= \frac{q}{4\pi\varepsilon_0}\left[\frac{-1}{r}\right]_r^{\infty} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$
- The potential of a distribution of charges can be calculated based on this and the superposition principle.

<u>Example</u>: Prob. 2.22 - Potential of a Uniform Line Charge

- Find the potential for an infinite line with uniform charge per length λ. The electric field (found using Gauss's Law in Prob. 2.13 – exercise for students) is
- $$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{s}\hat{s}$$
- In this case, infinity cannot be used as the reference point because the charge distribution is infinite. Choose the reference point at $s = a$. It doesn't matter what

direction the reference point is from the line because of symmetry. In other words, you can move around the wire at a fixed radius without changing the potential because $d\vec{\ell}\perp\vec{E}$. Integrate along a radial path:

$$V(s) = -\int_a^s \vec{E}\cdot d\vec{\ell} = -\frac{2\lambda}{4\pi\varepsilon_0}\int_a^s\left(\frac{\hat{s}}{s}\right)\cdot(ds\,\hat{s}) = -\frac{2\lambda}{4\pi\varepsilon_0}\int_a^s\frac{ds}{s}$$

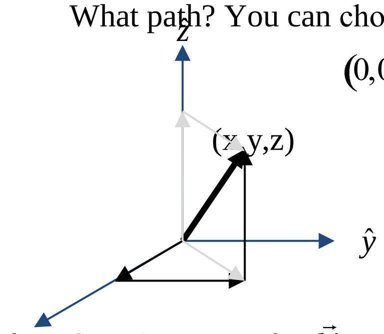
- $$= -\frac{2\lambda}{4\pi\varepsilon_0}[\ln s]_a^s = -\frac{2\lambda}{4\pi\varepsilon_0}(\ln s - \ln a) = -\frac{2\lambda}{4\pi\varepsilon_0}\ln\left(\frac{s}{a}\right)$$
- Check that this is correct:
- $$\vec{\nabla}V = -\frac{2\lambda}{4\pi\varepsilon_0}\frac{\partial}{\partial s}\left[\ln\left(\frac{s}{a}\right)\right]\hat{s} = -\frac{2\lambda}{4\pi\varepsilon_0}\frac{\partial}{\partial s}[\ln s - \ln a]\hat{s} = -\frac{1}{4\pi\varepsilon_0}\frac{2\lambda}{s}\hat{s} = -\vec{E}$$.

Exercise: B Find $V(x,y,z)$ {relative to V(0,0,0)} if $\vec{E} = ky\hat{x} + kx\hat{y}$

$$V(X,Y,Z) = -\int_{(0,0,0)}^{(X,Y,Z)}\vec{E}\cdot d\vec{\ell}$$

What path? You can choose any path to follow; here's a simple one:

$$(0,0,0)\rightarrow(X,0,0)\rightarrow(X,Y,0)\rightarrow(X,Y,Z)$$



Step 1: $y = z = 0,\ d\vec{\ell}_1 = dx\,\hat{x},\ x: 0\rightarrow X,\ \vec{E}\cdot d\vec{\ell}_1 = ky\,dx = 0$

$$\int\vec{E}\cdot d\vec{\ell}_1 = 0$$

Step 2: $x = X,\ z = 0,\ d\vec{\ell}_2 = dy\,\hat{y},\ y: 0\rightarrow Y,\ \vec{E}\cdot d\vec{\ell}_2 = kx\,dy = kX\,dy$

$$\int\vec{E}\cdot d\vec{\ell}_2 = kX\int_0^Y dy = kX[y]_0^Y = kXY$$

Step 3: $x = X,\ y = y_0,\ d\vec{\ell}_3 = dz\,\hat{z},\ z: 0\rightarrow Z,\ \vec{E}\cdot d\vec{\ell}_3 = 0$

$$\int\vec{E}\cdot d\vec{\ell}_3 = 0$$

The electric potential is

$$V(X,Y,Z) = -\int_{(0,0,0)}^{(X,Y,Z)}\vec{E}\cdot d\vec{\ell} = -kXY \quad\text{or}\quad V(x,y,z) = -kxy$$

<u>**Exercise**</u>:

a. Check that $\vec{\nabla}V = -\vec{E}$ for your answer to (b).

$$\vec{\nabla}V = \frac{\partial}{\partial x}(-kxy)\hat{x} + \frac{\partial}{\partial y}(-kxy)\hat{y} + \frac{\partial}{\partial z}(-kxy)\hat{z} = -ky\hat{x} - kx\hat{y} = -\vec{E}$$

Poisson's Equation and Laplace's Equation

The differential equation relating the electric field and the potential,

$$\vec{E} = -\vec{\nabla} V,$$

can be combined with the differential statement of Gauss's Law,

$$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0,$$

to get a single differential equation for the electric potential:

$$\vec{\nabla} \cdot \vec{E} = \vec{\nabla} \cdot \left(-\vec{\nabla} V\right) = -\nabla^2 V = \rho / \varepsilon_0 . \qquad [2.24]$$

What we mean by del-squared is

$$\vec{\nabla} \cdot \left(\vec{\nabla} V\right) = \vec{\nabla} \cdot \left(\left(\frac{\partial}{\partial x} V \right) \hat{x} + \left(\frac{\partial}{\partial y} V \right) \hat{y} + \left(\frac{\partial}{\partial z} V \right) \hat{z} \right) = \frac{\partial^2}{\partial x^2} V + \frac{\partial^2}{\partial y^2} V + \frac{\partial^2}{\partial z^2} V$$

The **Laplacian** operator ∇^2 means take the divergence of the gradient (order matters, only one makes sense!). Usually, ***Poisson's Equation*** is written as

$$\boxed{\nabla^2 V = -\rho / \varepsilon_0}$$

In regions where there is <u>no charge</u>, ***Laplace's Equation*** holds

$$\nabla^2 V = 0.$$

Chapter 3 is all about solving this final equation.

<u>Preview</u>

For Wednesday, you'll read about calculating the potential for a localized charge distribution (finding *V* without first knowing $\vec{E}$ is more useful – can use it to find $\vec{E}$).

You'll also learn about *boundary conditions* for the electric field and the electric potential.

Time for questions on HW #2 (or work on the computer problem)