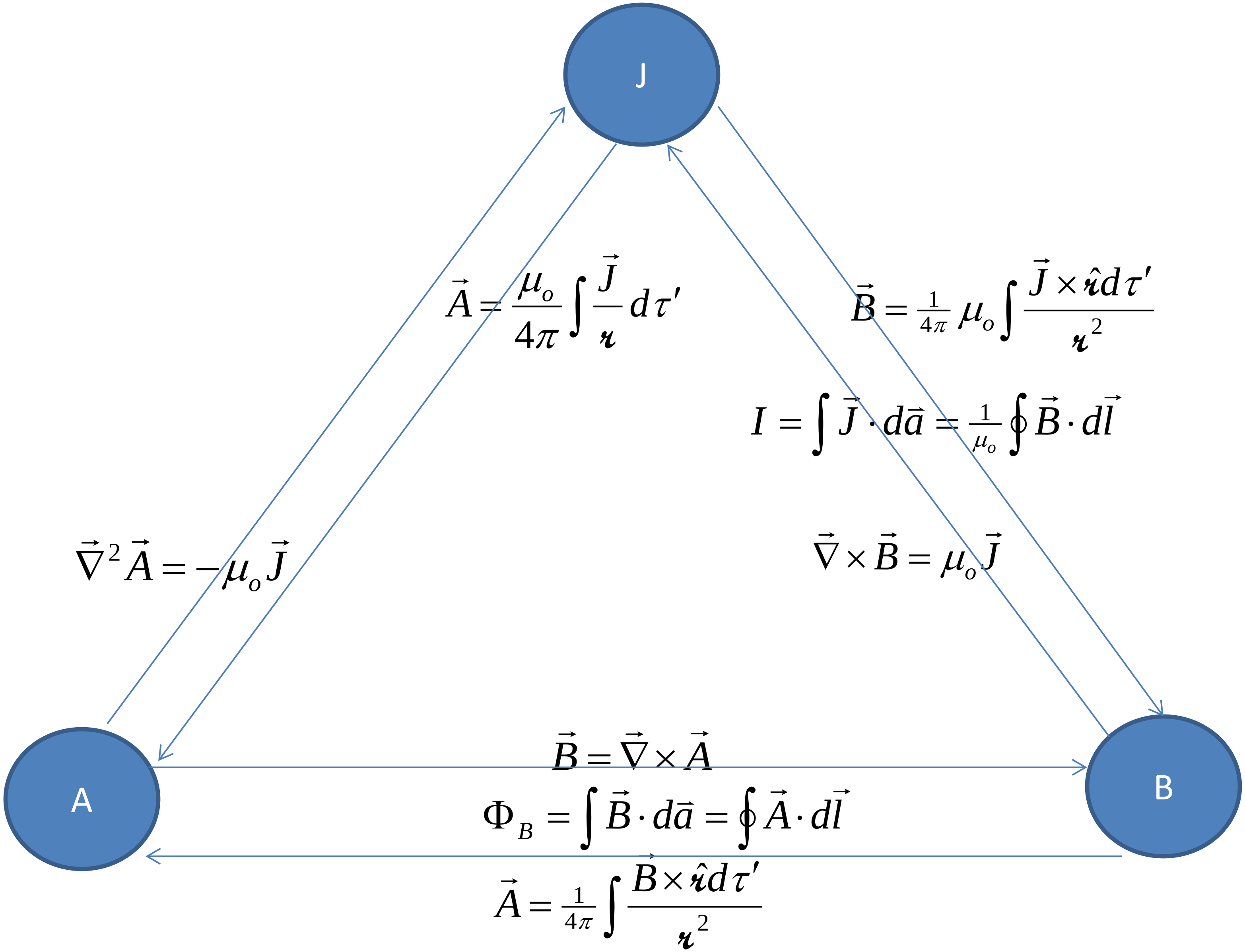
Wed. Thurs. Fri.	5.4.3 Multipole Expansion of the Vector Potential 6.1 Magnetization	HW8
Mon. Wed.	Review Exam 2 (Ch 3 & 5)	

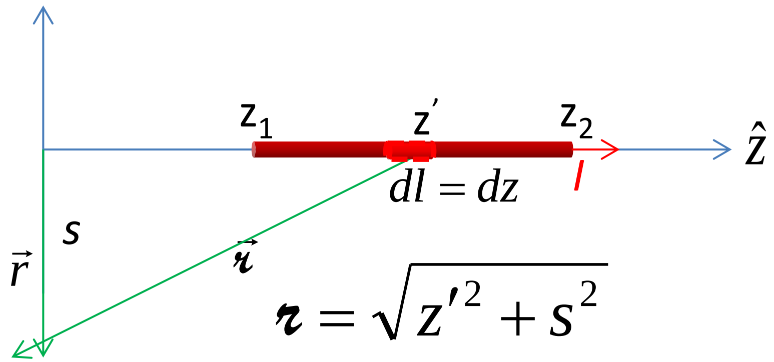
Relating Current, Potential, and Field



Finding A from J

$$\vec{A} = \frac{\mu_o}{4\pi}\int \frac{\vec{J}}{\mathcal{r}} d\tau' = \frac{\mu_o}{4\pi}\int \frac{\vec{I}}{\mathcal{r}} dl'$$

Find the vector potential for a current *I* along the *z* axis from z_1 to z_2.



$$\mathcal{r} = \sqrt{z'^2 + s^2}$$

$$\vec{A} = \frac{\mu_0}{4\pi}\int_{z_1}^{z_2} \frac{Idz}{\sqrt{z^2 + s^2}}\,\hat{z}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\left[\ln\left(z + \sqrt{z^2 + s^2}\right)\right]_{z_1}^{z_2}\hat{z}$$

$$\vec{A} = \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2 + \sqrt{z_2^2 + s^2}}{z_1 + \sqrt{z_1^2 + s^2}}\right]\hat{z}.$$

Finding J from Vector Potential

What current density would produce the vector potential $\vec{A} = k\,\hat{\phi}$ (where *k* is a constant) in cylindrical coordinates?

$$\vec{\nabla}^2\vec{A} = -\mu_o\vec{J} \quad \text{where} \quad \vec{\nabla}^2\vec{A} = \vec{\nabla}^2 A_x\hat{x} + \vec{\nabla}^2 A_y\hat{y} + \vec{\nabla}^2 A_z\hat{z}$$

So, convert to Cartesian $\vec{A} = k\,\langle -\sin\phi, \cos\phi, 0\rangle$

One component at a time

$$\vec{\nabla}^2 A_x = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial(-k\sin\phi)}{\partial s}\right) + \frac{1}{s^2}\frac{\partial(-k\sin\phi)}{\partial\phi^2} + \frac{\partial^2(-k\sin\phi)}{\partial z^2} = \ldots = k\frac{\sin\phi}{s^2}$$

$$\vec{\nabla}^2 A_y = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial(k\cos\phi)}{\partial s}\right) + \frac{1}{s^2}\frac{\partial(k\cos\phi)}{\partial\phi^2} + \frac{\partial^2(k\cos\phi)}{\partial z^2} = \ldots = -k\frac{\cos\phi}{s^2}$$

$$\vec{\nabla}^2\vec{A} = \left\langle k\frac{\sin\phi}{s^2}, -k\frac{\cos\phi}{s^2}, 0\right\rangle$$

$$\vec{\nabla}^2\vec{A} = -\frac{k}{s^2}\langle -\sin\phi, \cos\phi, 0\rangle = -\frac{k}{s^2}\hat{\phi} = -\mu_o\vec{J} \quad \text{so} \quad \vec{J} = \frac{1}{\mu_o}\frac{k}{s^2}\hat{\phi}$$

Finding J from Vector Potential

What current density would produce the vector potential $\vec{A} = k\,\hat{\phi}$ (where *k* is a constant) in cylindrical coordinates?

$$\vec{\nabla}^2 \vec{A} = -\mu_o \vec{J}$$

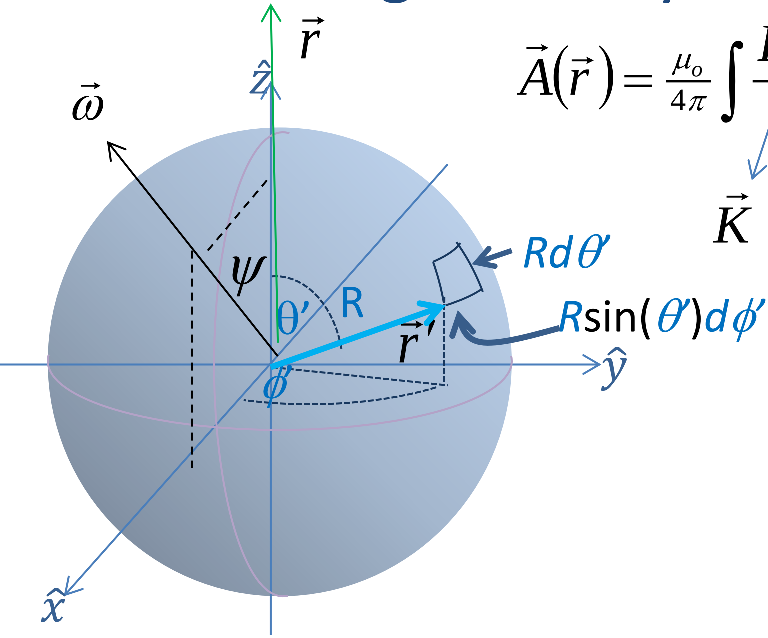
Alternatively,

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}\left(sA_\phi\right)\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}\left(ks\right)\hat{z} = \frac{k}{s}\,\hat{z}$$

and then

$$\vec{J} = \tfrac{1}{\mu_o}\vec{\nabla} \times \vec{B} = \frac{1}{\mu_0}\left(-\frac{\partial B_z}{\partial s}\right)\hat{\phi} = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \frac{k}{\mu_0 s^2}\,\hat{\phi}$$

Ex. 5.11: What's the magnetic potential of a sphere with surface charge density constant σ rotating at ω.



$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int \frac{\vec{K}da'}{\mathscr{r}}$$

$$da' = R^2 d\varphi' \sin\theta' d\theta'$$

$$\mathscr{r} = \sqrt{R^2 + r^2 - 2Rr\cos\theta'}$$

$$\vec{K} = \sigma\vec{v}$$

What is $\vec{v}$?

If rotating about z, it would simply be $R\sin\theta'\omega\hat{\phi}$.

If $\vec{\omega} = \omega\hat{z}$ this would have been $\vec{\omega}\times\vec{r}\,' = R\sin\theta'\omega\hat{\phi} = \vec{v}$

Generally, $\vec{\omega}\times\vec{r}\,' = \vec{v}$

$$\vec{\omega} = \omega\langle \sin\psi, 0, \cos\psi, 0\rangle$$

$$\vec{r}\,' = R\langle \sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta'\rangle$$

$$\vec{v} = \vec{\omega}\times\vec{r}\,' = \omega\langle \sin\psi, 0, \cos\psi, 0\rangle \times R\langle \sin\theta'\cos\phi', \sin\theta'\sin\phi', \cos\theta'\rangle$$

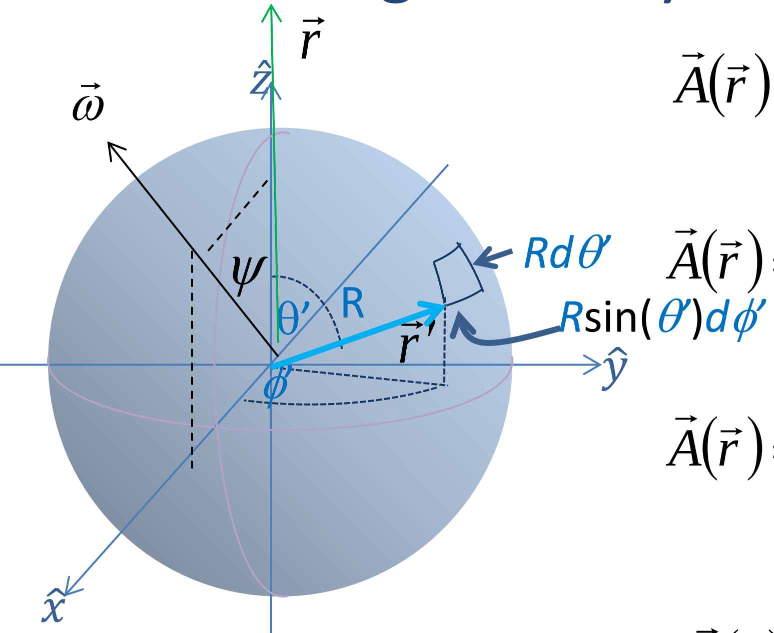
$$\vec{v} = \omega R\langle -\sin\theta'\cos\phi'\cos\psi, (\cos\psi\sin\theta'\cos\phi' - \sin\psi\cos\theta'), \sin\psi\sin\theta'\sin\phi'\rangle$$

For the four terms to v, there will be four integrals. All but one has a factor of

$$\int_0^{2\pi}\cos\phi' d\varphi' = 0 \quad \text{or} \quad \int_0^{2\pi}\sin\phi' d\varphi' = 0$$

leaving $\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_0^{\phi'=2\pi}\int_0^{\theta'=\pi}\frac{\sigma\omega R(-\sin\psi\cos\theta')R^2 d\phi'\sin\theta' d\theta'}{\sqrt{R^2 + r^2 - 2Rr\cos\theta'}}\hat{y}$

Ex. 5.11: What's the magnetic potential of a sphere with surface charge density constant σ rotating at ω.



$$\vec{A}(\vec{r}) = \frac{\mu_o}{4\pi}\int_0^{\phi'=2\pi}\int_0^{\theta'=\pi}\frac{\sigma\omega R(-\sin\psi\cos\theta')R^2 d\phi'\sin\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\hat{y}$$

$$\vec{A}(\vec{r}) = -\frac{\mu_o}{4\pi}2\pi\sigma\omega R^3\sin\psi\int_0^{\theta'=\pi}\frac{\cos\theta'\sin\theta' d\theta'}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\hat{y}$$

$$\vec{A}(\vec{r}) = -\frac{\mu_o}{2}\sigma\omega R^3\sin\psi\int_1^{\cos\theta'=0}\frac{\cos\theta' d(\cos\theta')}{\sqrt{R^2+r^2-2Rr\cos\theta'}}\hat{y}$$

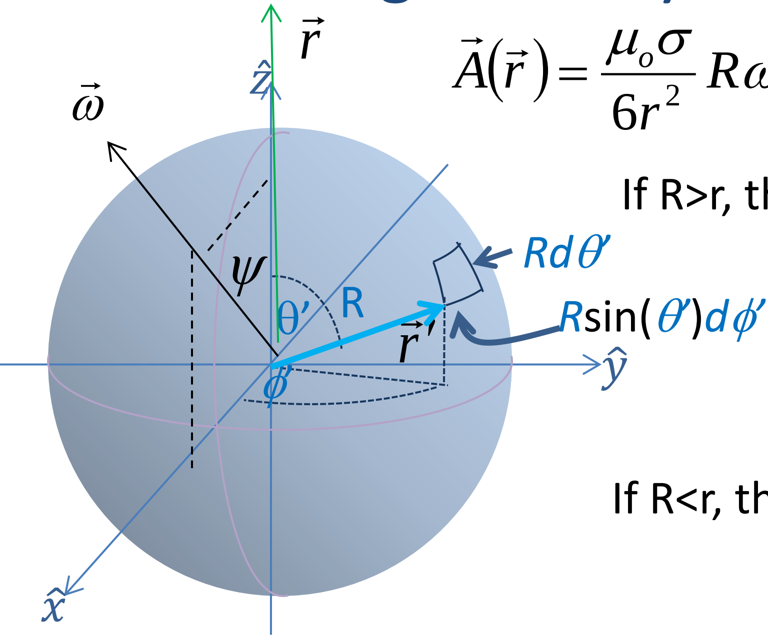
$$\vec{A}(\vec{r}) = -\frac{\mu_o}{2}\sigma\omega R^3\sin\psi\int_1^{\zeta=-1}\frac{\zeta d\zeta}{\sqrt{R^2+r^2-2Rr\zeta}}\hat{y}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o}{2}\sigma\omega R^3\sin\psi\left(\frac{R^2+r^2+Rr\zeta}{3R^2r^2}\sqrt{R^2+r^2-2Rr\zeta}\right)\Bigg|_1^{-1}\hat{y}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o\sigma}{6r^2}\omega R\sin\psi\left((R^2+r^2+Rr\zeta)\sqrt{R^2+r^2-2Rr\zeta}\right)\Big|_1^{-1}\hat{y}$$

$$\vec{A}(\vec{r}) = \frac{\mu_o\sigma}{6r^2}\omega R\sin\psi\left((R^2+r^2-Rr)\sqrt{(R+r)^2}-(R^2+r^2+Rr)\sqrt{(R-r)^2}\right)\hat{y}$$

Ex. 5.11: What's the magnetic potential of a sphere with surface charge density constant σ rotating at ω.



$$\vec{A}(\vec{r}) = \frac{\mu_o \sigma}{6r^2} R\omega \sin\psi \left(\left(R^2 + r^2 - Rr\right)(R+r) - \left(R^2 + r^2 + Rr\right)|R-r| \right)\hat{y}$$

If R>r, then $|R-r| = R-r$

$$\left(R^2 + r^2 - Rr\right)(R+r) - \left(R^2 + r^2 + Rr\right)(R-r) = 2r^3$$

$$\vec{A}(\vec{r}) = \frac{\mu_o \sigma}{3} R\omega r \sin\psi \hat{y}$$

If R<r, then $|R-r| = r-R$

$$\left(R^2 + r^2 - Rr\right)(R+r) - \left(R^2 + r^2 + Rr\right)(r-R) = 2R^3$$

$$\vec{A}(\vec{r}) = \frac{\mu_o \sigma}{3r^2} R^4 \omega \sin\psi \hat{y}$$

Recognizing that $\vec{\omega} \times \vec{r} = \omega r \sin\psi \hat{y}$ these can be written generally

$$\vec{A}(\vec{r}) = \begin{cases} \dfrac{\mu_o \sigma}{3} R \vec{\omega} \times \vec{r} & r < R \\ \dfrac{\mu_o \sigma}{3r^3} R^4 \vec{\omega} \times \vec{r} & r > R \end{cases}$$

Motivating Electric Potential, Physically

Akin to Potential *Energy*

Generally

$$W_{1\to2} \equiv \int_a^b \vec{F}_{1\to2} \cdot d\vec{\ell}$$

Object 2 is the "system", 1 is "external." Work done *by* object 1 when exerting force *on* object 2 which moves from *a* to *b*

$$\Delta P.E._{1,2} \equiv -\int_a^b \vec{F}_{1\to2} \cdot d\vec{\ell}$$

Objects 1 and 2 are the "system". Change in their potential as they interact while separating from *a* to *b*

Electrically

$$\vec{F}_{1\to2} = q_2 \vec{E}_1(\vec{r}_2)$$

Combining:

$$\Delta P.E._{1,2} = -\int_a^b q_2 \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell} = -q_2 \int_a^b \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell}$$

thus

$$\Delta V_1 \equiv \frac{\Delta P.E._{1,2}}{q_2} = -\int_a^b \vec{E}_1(\vec{r}_2) \cdot d\vec{\ell}$$

Physical Meaning of Vector Potential

Akin to potential *momentum*

From __future__ import time-varying electric

$$\vec{E} = -\vec{\nabla} V - \frac{\partial \vec{A}}{\partial t}$$

Consider your "system" a particle interacting with electric and magnetic fields
(*really* interacting with other charges via their electric and magnetic fields)

$$\frac{d}{dt}\vec{p} = \vec{F}_{net} = q\vec{v}\times\vec{B} + q\vec{E} = q\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + q\left(-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}\right) = q\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + q\left(-\vec{\nabla}V - \frac{d}{dt}\vec{A} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}\right)$$

for any vector

$$\frac{d}{dt}\vec{A} = \frac{\partial \vec{A}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

$$\frac{d}{dt}\vec{p} = q\left(\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \frac{d}{dt}\vec{A}\right) + q\left(-\vec{\nabla}V\right)$$

By Product rule (4)

$$\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) = \vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + \vec{A}\times\left(\vec{\nabla}\times\vec{v}\right) + \left(\vec{A}\cdot\vec{\nabla}\right)\vec{v} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$$

Derivative with respect to potential not source velocity

$$\frac{d}{dt}\underbrace{\left(\vec{p} + q\vec{A}\right)}_{"p"} = -\vec{\nabla}\underbrace{q\left(V - \vec{v}\cdot\vec{A}\right)}_{"U"}$$

Consider your "system" a particle *and* the fields.
The force is negative gradient the potential energy

if $-\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right) = 0$ then $\vec{p}_i + q\vec{A}_i = \vec{p}_f + q\vec{A}_f = const$

'potential momentum'

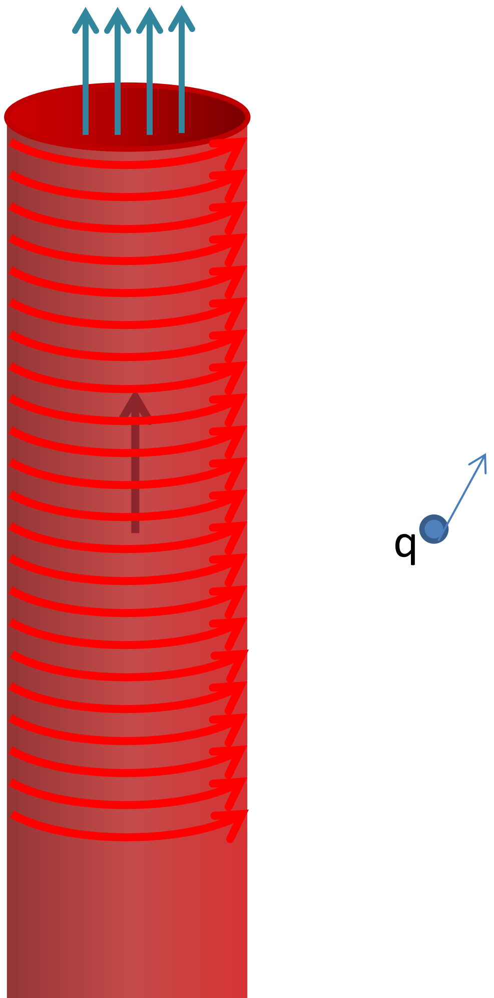
Finding Vector Potential

$$\oint \vec{A}\cdot d\vec{\ell} = \int \vec{B}\cdot d\vec{a} = \Phi$$

Charged particle outside a disappearing solenoid

$$\vec{A}_{initially} = \begin{cases} (\mu_0 n I s/2)\hat{\phi} & s < R, \\ (\mu_0 n I R^2/2s)\hat{\phi} & s > R. \end{cases}$$

$$\vec{A}_{finally} = 0$$



$$\frac{d}{dt}\left(m\vec{v} + q\vec{A}\right) = -\vec{\nabla} q\left(V - \vec{v}\cdot\vec{A}\right) = 0$$

initially

$$m\vec{v}_i + q\vec{A}_i = m\vec{v}_f + q\vec{A}_f$$

$$q\left(\mu_0 n I R^2/2s\right)\hat{\phi} = m\vec{v}_f$$

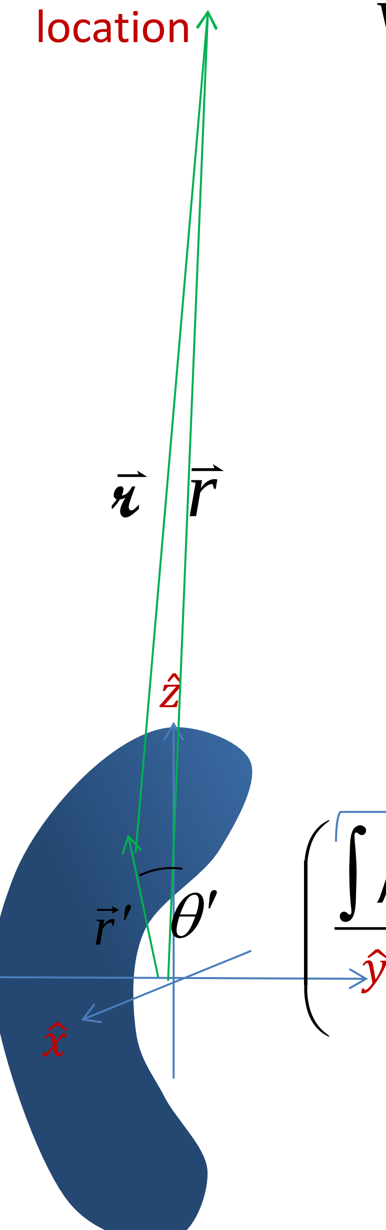
$$\frac{q}{m}\left(\mu_0 n I R^2/2s\right)\hat{\phi} = \vec{v}_f$$

$$-\hat{x}$$

Memory Lane: Multi-pole Expansion of *Scalar* Potential

Continuous *charge* distribution

Observation location



$$V(r) = \frac{1}{4\pi\varepsilon_o}\int \frac{dq}{\mathscr{r}}$$

$$\frac{1}{\mathscr{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

nth Legendre polynomial

$$P_n(\cos\theta')$$

$$P_0 = 1$$

$$P_1(u) = u$$

$$P_2(u) = \frac{(3u^2 - 1)}{2}$$

$$V(r) = \frac{1}{4\pi\varepsilon_o}\int\left(\left(\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')\right)\rho(\vec{r}')d\tau'\right)$$

Re-ordering sums

$$V(r) = \frac{1}{4\pi\varepsilon_o}\sum_{n=0}^{\infty}\left(\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\rho(\vec{r}')d\tau'\right)$$

$$\left(\frac{\int\rho(\vec{r}')d\tau'}{r} + \frac{\int r'\cos\theta'\rho(\vec{r}')d\tau'}{r^2} + \frac{\int r'^2\left(3(\cos\theta')^2 - 1\right)\rho(\vec{r}')d\tau'}{2r^3} + \ldots\right.$$

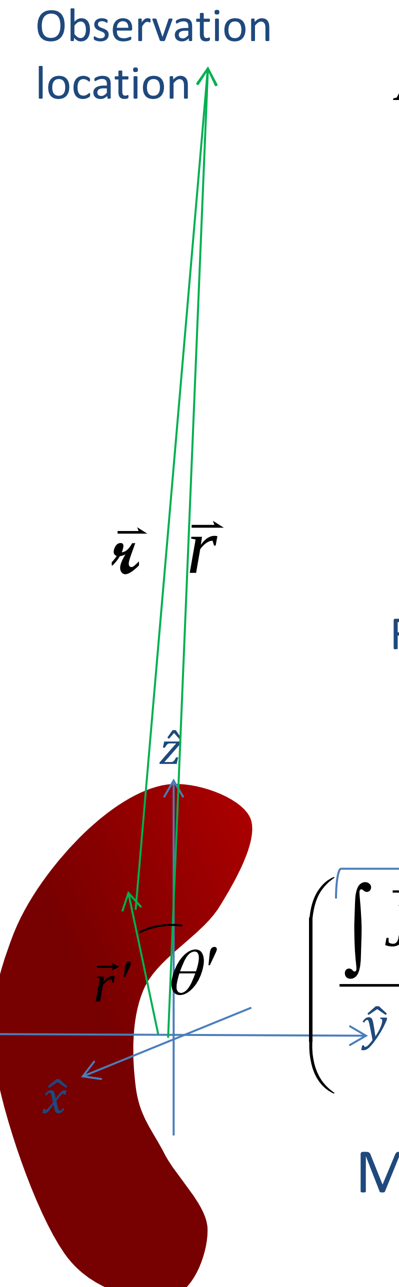
$$V(r) = \frac{1}{4\pi\varepsilon_o}\frac{Q}{r} + \frac{1}{4\pi\varepsilon_o}\frac{\vec{p}\cdot\hat{r}}{r^2} + \ldots$$

Electric Dipole Moment

$$\vec{p} \equiv \int \vec{r}'\rho(\vec{r}')d\tau'$$

Multi-pole Expansion of *Vector* Potential

Continuous *current* distribution



$$\vec{A}(r) = \frac{\mu_o}{4\pi}\int \frac{dq\vec{v}}{\mathcal{r}}$$

$$\frac{1}{\mathcal{r}} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

n^{th} Legendre polynomial

$$P_n(\cos\theta')$$

$$P_0 = 1$$

$$P_1(u) = u$$

$$P_2(u) = \frac{(3u^2-1)}{2}$$

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\int\left(\left(\frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta')\right)\vec{J}(\vec{r}')d\tau'\right)$$

Re-ordering sums

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\sum_{n=0}^{\infty}\left(\frac{1}{r^{n+1}}\int r'^n P_n(\cos\theta')\vec{J}(\vec{r}')d\tau'\right)$$

$$\left(\frac{\int \vec{J}(\vec{r}')d\tau'}{r} + \frac{\int r'\cos\theta'\vec{J}(\vec{r}')d\tau'}{r^2} + \frac{\int r'^2\left(3(\cos\theta')^2-1\right)\vec{J}(\vec{r}')d\tau'}{2r^3} + \ldots\right.$$

Monopole term

Dipole term

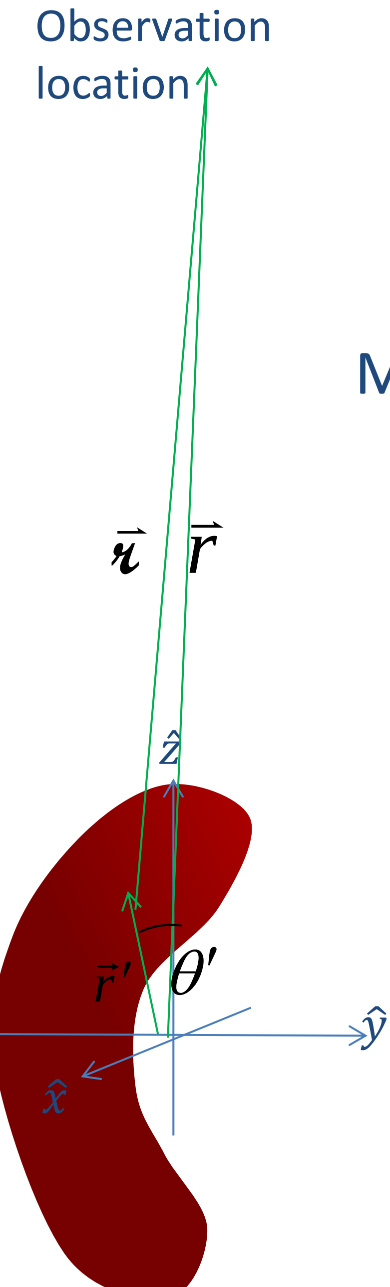
Multi-pole Expansion of *Vector* Potential

Continuous *current* distribution

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\int \frac{dq\vec{v}}{\mathscr{r}} = \frac{\mu_o}{4\pi}\left(\frac{\int \vec{J}(\vec{r}')d\tau'}{r} + \frac{\int r'\cos\theta' \vec{J}(\vec{r}')d\tau'}{r^2} + \dots \right)$$

Monopole term

Dipole term



Monopole's integral

$$\int \vec{J}(\vec{r}')d\tau' = \oint I(\vec{r}')d\vec{l}\,' = \oint \frac{dq}{dt}d\vec{l}\,' = \oint dq\frac{d\vec{l}\,'}{dt} = \oint dq\vec{v}\,' = 0$$

If we were to divide by Q, we'd have the charge-averaged velocity.

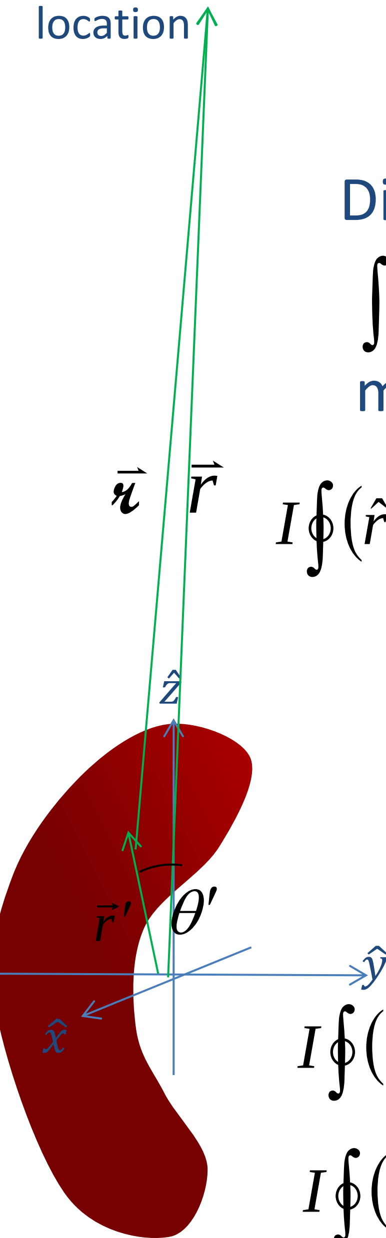
If the source is stationary, the current is steady, then the *average* velocity is just 0.

Multi-pole Expansion of *Vector* Potential

Continuous *current* distribution

$$\vec{A}(r)=\frac{\mu_o}{4\pi}\int\frac{dq\vec{v}}{\mathscr{r}}=\frac{\mu_o}{4\pi}\left(\frac{\int r'\cos\theta'\vec{J}\left(\vec{r}\,'\right)d\tau'}{r^2}+\dots\right)$$

Dipole term



Dipole's integral

Steady current

$$\int r'\cos\theta'\vec{J}\left(\vec{r}\,'\right)d\tau'=\int\left(\hat{r}\cdot\vec{r}\,'\right)\vec{J}\left(\vec{r}\,'\right)d\tau'=\oint\left(\hat{r}\cdot\vec{r}\,'\right)Id\vec{l}\,'=I\oint\left(\hat{r}\cdot\vec{r}\,'\right)d\vec{l}\,'$$

mathland

Stokes' to Pr. 1.61*e* using Rule's 7 and then 1:

$$I\oint\left(\hat{r}\cdot\vec{r}\,'\right)d\vec{l}\,'=-I\int\vec{\nabla}_{r'}\left(\hat{r}\cdot\vec{r}\,'\right)\times d\vec{a}\,'$$

curlless

Derivative w/ respect to source not observation location

Product rule (4)

$$\vec{\nabla}_{r'}\left(\hat{r}\cdot\vec{r}\,'\right)=\hat{r}\times\left(\vec{\nabla}_{r'}\times\vec{r}\,'\right)+\vec{r}\,'\times\left(\vec{\nabla}_{r'}\times\hat{r}\right)+\left(\hat{r}\cdot\vec{\nabla}_{r'}\right)\vec{r}\,'+\left(\vec{r}\,'\cdot\vec{\nabla}_{r'}\right)\hat{r}$$

$$\left(\left(\hat{x}+\hat{y}+\hat{z}\right)\cdot\left(\frac{\partial}{\partial x'}\hat{x}+\frac{\partial}{\partial y'}\hat{y}+\frac{\partial}{\partial z'}\hat{z}\right)\right)\left(x'\hat{x}+y'\hat{y}+z'\hat{z}\right)$$

$$\left(\left(\frac{\partial}{\partial x'}+\frac{\partial}{\partial y'}+\frac{\partial}{\partial z'}\right)\right)\left(x'\hat{x}+y'\hat{y}+z'\hat{z}\right)$$

$$I\oint\left(\hat{r}\cdot\vec{r}\,'\right)d\vec{l}\,'=-I\int\hat{r}\times d\vec{a}\,'$$

$$I\oint\left(\hat{r}\cdot\vec{r}\,'\right)d\vec{l}\,'=-I\hat{r}\times\int d\vec{a}\,'=-I\hat{r}\times\vec{a}\,'=I\vec{a}\,'\times\hat{r}$$

$$\left(\hat{x}+\hat{y}+\hat{z}\right)=\hat{r}$$

Multi-pole Expansion of *Vector* Potential

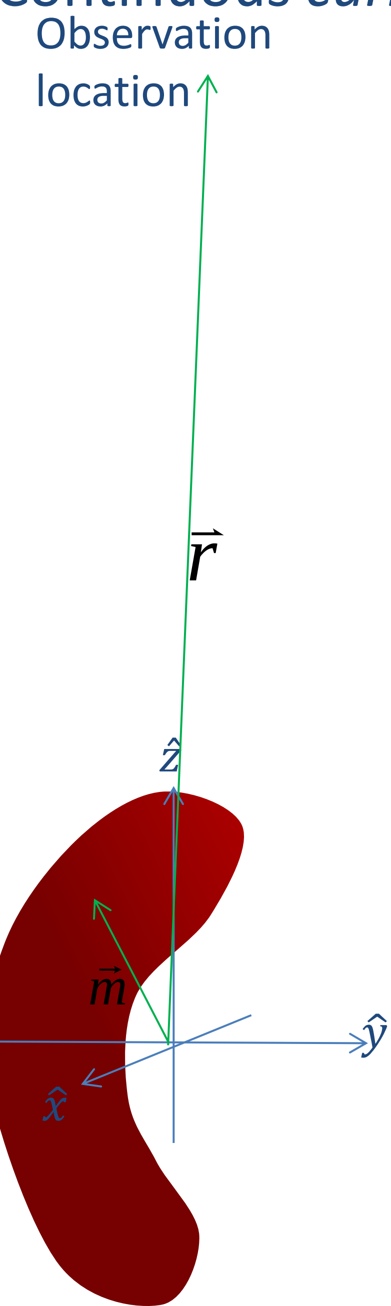
Continuous *current* distribution

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\int \frac{dq\vec{v}}{\mathscr{r}} = \frac{\mu_o}{4\pi}\left(\frac{I\vec{a}'\times\hat{r}}{r^2} + \ldots\right)$$

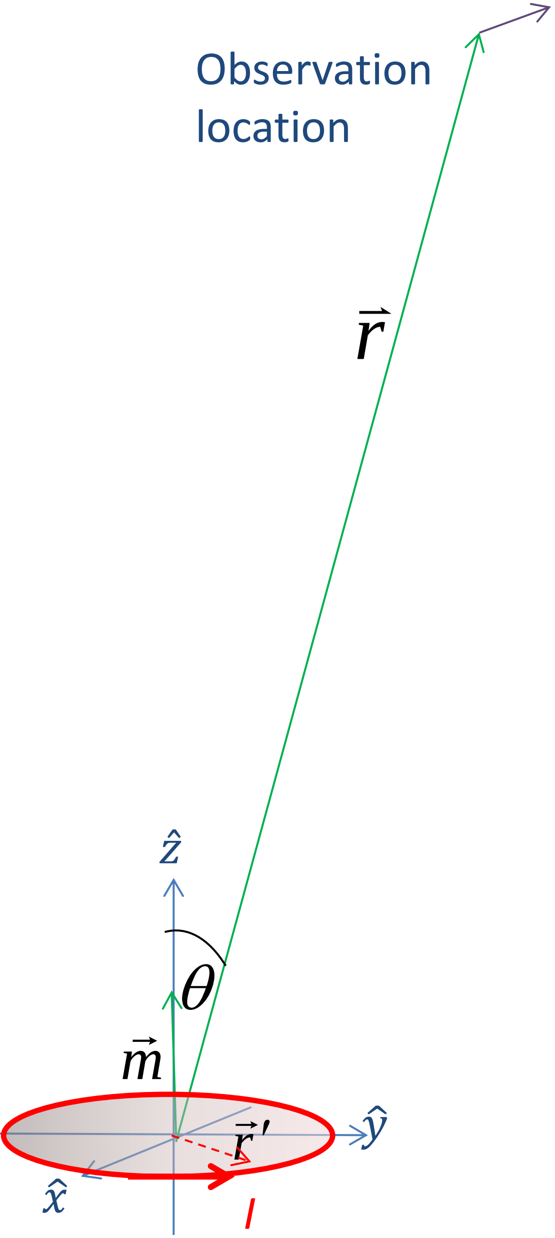
Magnetic Dipole Moment

$$\vec{m} \equiv I\vec{a}'$$

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{\vec{m}\times\hat{r}}{r^2} + \ldots\right)$$



Dipole term for a loop



$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{\vec{m}\times\hat{r}}{r^2} + \dots\right)$$

Magnetic Dipole Moment

$$\vec{m} \equiv I\vec{a}'$$

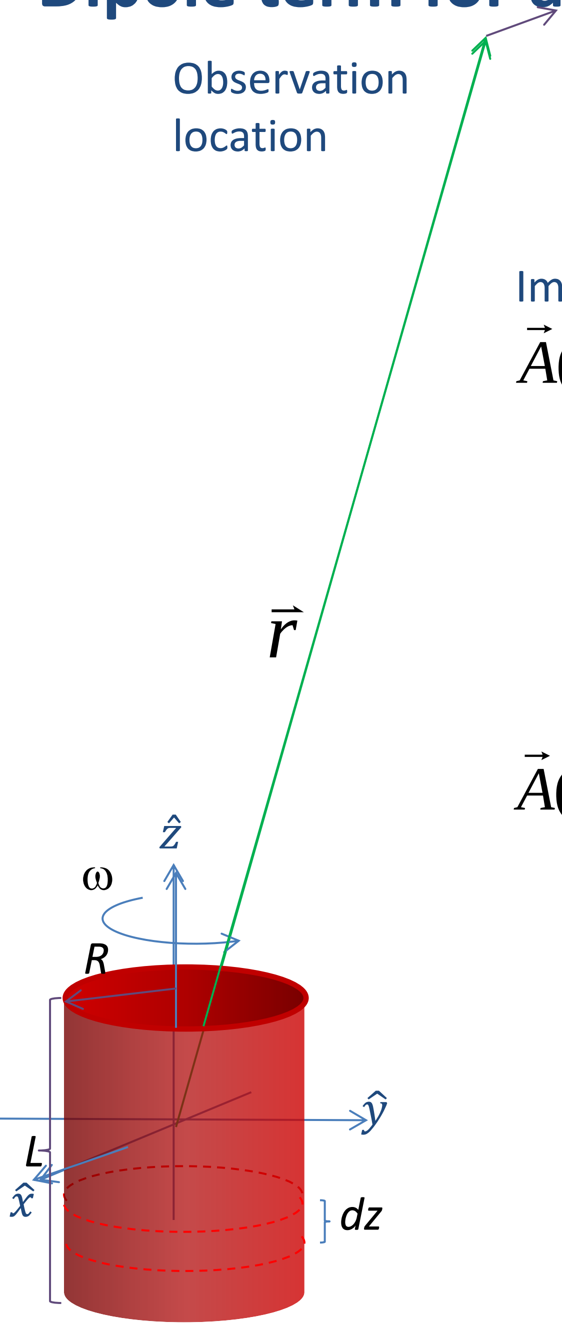
$$\vec{m} = I\left(\pi R^2 \hat{z}\right)$$

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{I\pi R^2 \hat{z}\times\hat{r}}{r^2} + \dots\right)$$

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{I\pi R^2 \sin\theta}{r^2}\hat{\phi} + \dots\right)$$

Same direction as current

Dipole term for a cylindrical shell spinning at ω with surface charge density σ



$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{\vec{m}\times\hat{r}}{r^2} + \ldots\right)$$

Imagine as a stack of differentially-thin current loops

$$\vec{A}(r)_{shell} = \sum d\vec{A}(r)_{loop}$$

$$d\vec{A}(r)_{loop} = \frac{\mu_o}{4\pi}\left(\frac{d\vec{m}_{loop}\times\hat{r}}{r^2} + \ldots\right)$$

$$d\vec{m}_{loop} = dI\left(\pi R^2\hat{z}\right)$$

$$dI_{loop} = Kdz = \sigma v dz$$

$v = \omega R$

$$dI_{loop} = \sigma R\omega dz$$

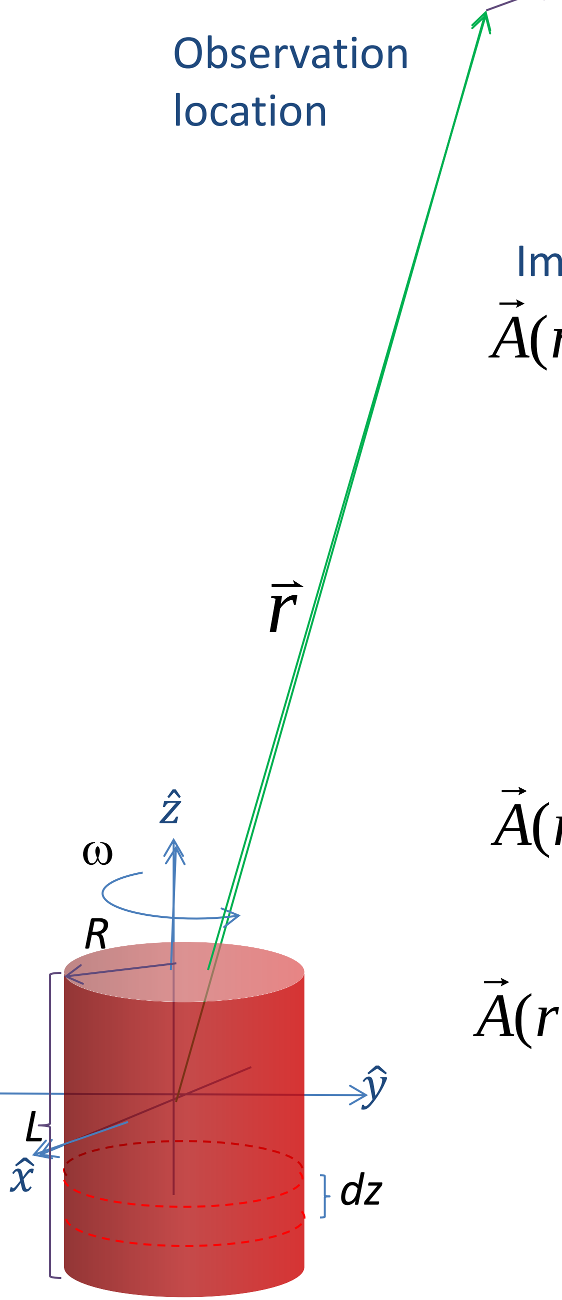
$$d\vec{m}_{loop} = \sigma R\omega dz\left(\pi R^2\hat{z}\right)$$

$$d\vec{A}(r)_{loop} = \frac{\mu_o}{4\pi}\left(\frac{\left(\sigma\omega\pi R^3 dz\right)\hat{z}\times\hat{r}}{r^2} + \ldots\right)$$

$$d\vec{A}(r)_{loop} = \frac{\mu_o}{4}\left(\frac{\sigma\omega R^3 dz\sin\theta}{r^2}\hat{\phi} + \ldots\right)$$

$$\vec{A}(r)_{shell} \approx \frac{\mu_o}{4}\frac{\sigma\omega R^3 L\sin\theta}{r^2}\hat{\phi}$$

Dipole term for a *solid* cylinder spinning at ω with *volume* charge density *ρ*



$$\vec{A}(r) = \tfrac{\mu_o}{4\pi}\left(\frac{\vec{m}\times\hat{r}}{r^2} + \ldots\right)$$

Imagine as a collection of concentric cylinders

$$\vec{A}(r)_{cylinder} = \sum d\vec{A}(r)_{shell}$$

$$d\vec{A}(r)_{shell} \approx \tfrac{\mu_o}{4}\frac{\sigma\omega r'^3 L\sin\theta}{r^2}\hat{\phi}$$

$$d\vec{A}(r)_{shell} \approx \tfrac{\mu_o}{4}\frac{\rho dr'\omega r'^3 L\sin\theta}{r^2}\hat{\phi}$$

$$\vec{A}(r)_{cylinder} \approx \tfrac{\mu_o}{4}\int_0^R \frac{\rho dr'\omega r'^3 L\sin\theta}{r^2}\hat{\phi}$$

$$\vec{A}(r)_{cylinder} \approx \tfrac{\mu_o}{4}\frac{\rho\omega L\sin\theta}{r^2}\int_0^R r'^3 dr'\hat{\phi} = \tfrac{\mu_o}{4}\frac{\rho\omega R^4 L\sin\theta}{4r^2}\hat{\phi}$$

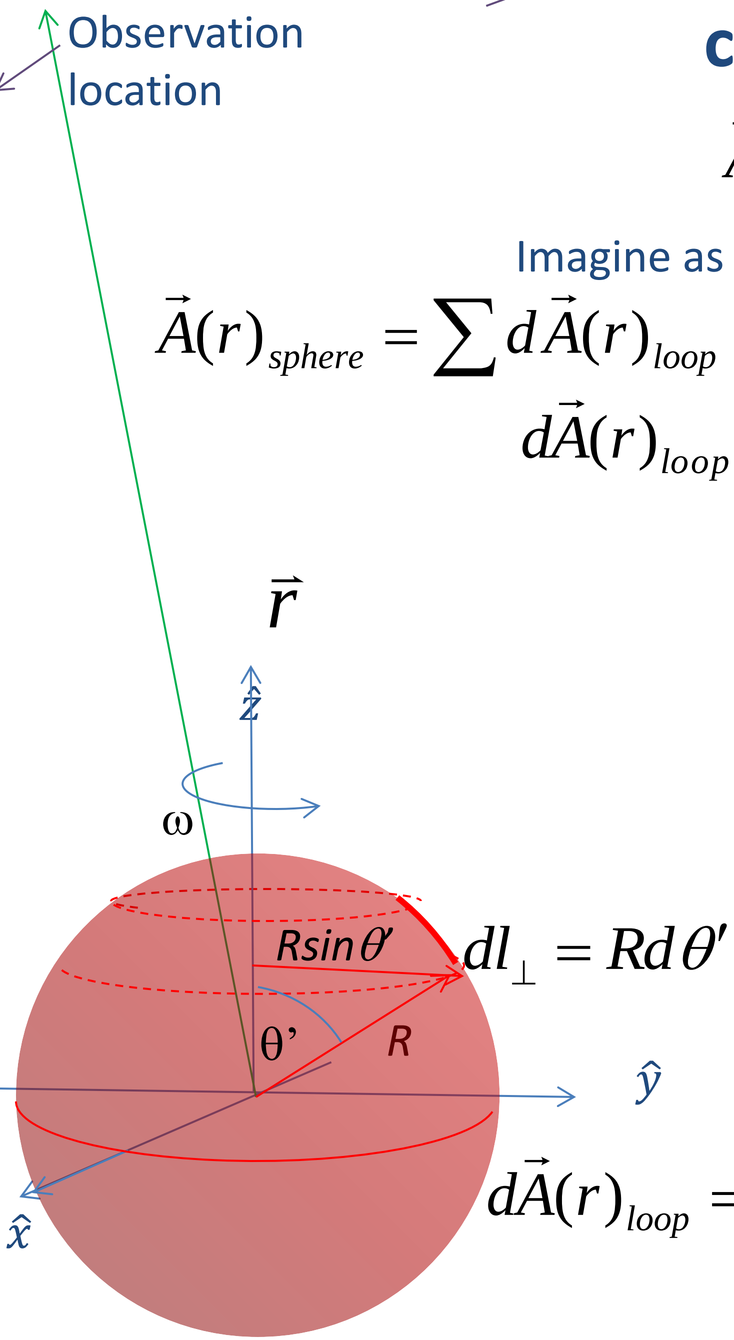
Dipole term for a spherical shell spinning at ω with *surface* charge density σ

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{\vec{m}\times\hat{r}}{r^2} + \ldots\right)$$

Imagine as a collection of coaxial loops

$$\vec{A}(r)_{sphere} = \sum d\vec{A}(r)_{loop}$$

$$d\vec{A}(r)_{loop} = \frac{\mu_o}{4\pi}\left(\frac{d\vec{m}_{loop}\times\hat{r}}{r^2} + \ldots\right)$$



$$d\vec{m}_{loop} = dI_{loop}\left(a'_{loop}\,\hat{z}\right)$$

$$a'_{loop} = \pi\left(R\sin\theta'\right)^2$$

$$dI_{loop} = K dl_\perp$$

$$K = \sigma v = \sigma\left(R\sin\theta'\right)\omega$$

$$dl_\perp = Rd\theta'$$

$$d\vec{A}(r)_{loop} = \frac{\mu_o}{4\pi}\left(\frac{\sigma\left(R\sin\theta'\right)\omega R d\theta' \pi\left(R\sin\theta'\right)^2 \hat{z}\times\hat{r}}{r^2} + \ldots\right)$$

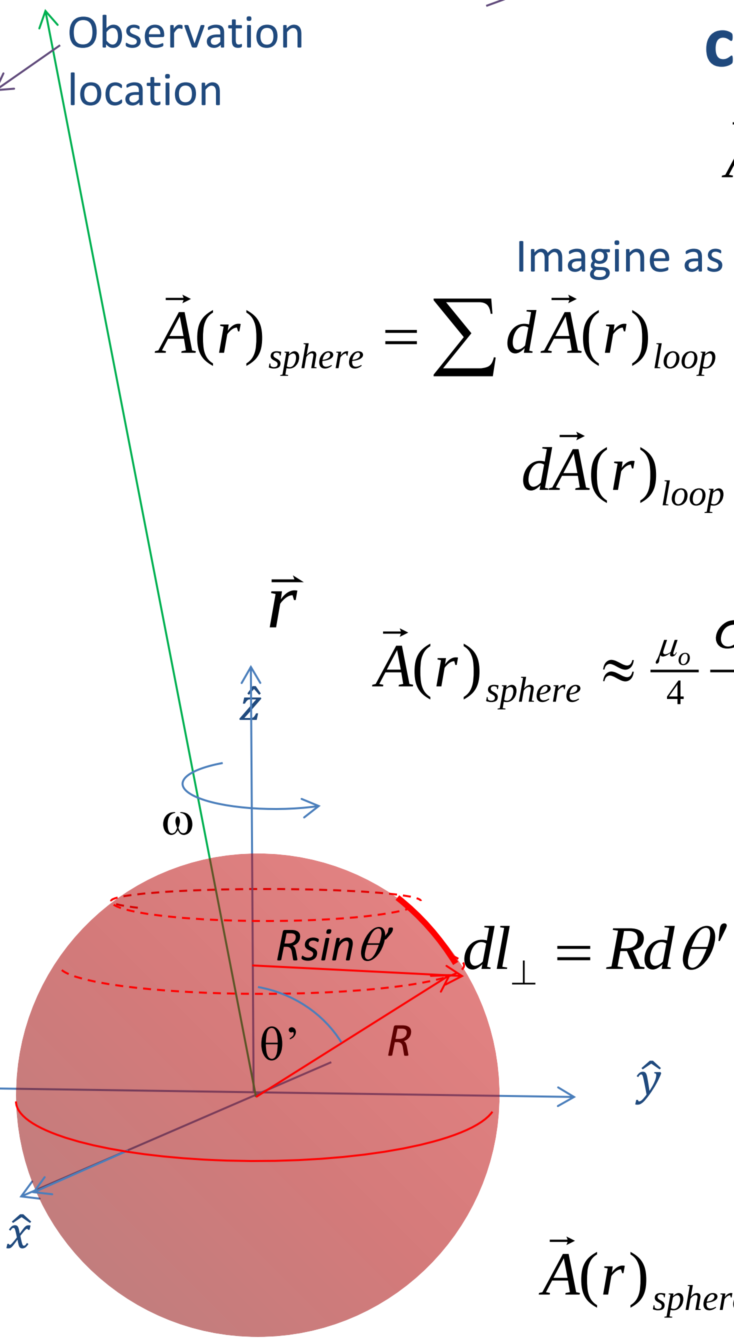
Dipole term for a spherical shell spinning at ω with *surface* charge density *σ*

$$\vec{A}(r) = \frac{\mu_o}{4\pi}\left(\frac{\vec{m}\times\hat{r}}{r^2} + \ldots\right)$$

Imagine as a collection of coaxial loops

$$\vec{A}(r)_{sphere} = \sum d\vec{A}(r)_{loop}$$

$$d\vec{A}(r)_{loop} = \frac{\mu_o}{4\pi}\left(\sigma\omega\pi\frac{\left(R^4\sin^3\theta'\right)d\theta'\hat{z}\times\hat{r}}{r^2} + \ldots\right)$$



$$\vec{A}(r)_{sphere} \approx \frac{\mu_o}{4}\frac{\sigma\omega R^4}{r^2}\int_0^\pi \sin^3\theta' d\theta'\left(\hat{\phi}\sin\theta\right)$$

$$\int_0^\pi \sin^3\theta' d\theta' = -\int_0^\pi \sin^2\theta' d\cos\theta'$$

$$-\int_1^{-1}\left(1-\cos^2\theta'\right)d\cos\theta'$$

$$-\left(-1-\tfrac{1}{3}(-1)^3 - \left(1-\tfrac{1}{3}(1)^3\right)\right) = \tfrac{8}{3}$$

$$\vec{A}(r)_{sphere} \approx \frac{\mu_o}{3}\frac{2\sigma\omega R^4}{r^2}\sin\theta\hat{\phi}$$

Which is actually the exact solution!

Wed. Thurs. Fri.	5.4.3 Multipole Expansion of the Vector Potential 6.1 Magnetization	HW8
Mon. Wed.	Review Exam 2 (Ch 3 & 5)	

Finding J from Vector Potential

What current density would produce the vector potential $\vec{A} = k\,\hat{\phi}$ (where *k* is a constant) in cylindrical coordinates?

$$\vec{\nabla}^2 \vec{A} = -\mu_o \vec{J} \quad \text{where} \quad \vec{\nabla}^2 \vec{A} = \vec{\nabla}^2 A_x \hat{x} + \vec{\nabla}^2 A_y \hat{y} + \vec{\nabla}^2 A_z \hat{z}$$

So, convert to Cartesian $\vec{A} = k\left\langle -\sin\phi, \cos\phi, 0\right\rangle = k\left\langle -\frac{y}{\left(x^2+y^2\right)^{1/2}}, \frac{x}{\left(x^2+y^2\right)^{1/2}}, 0\right\rangle$

One component at a time

$$\vec{\nabla}^2 A_x = -k\left(\frac{\partial^2}{\partial x^2}\frac{y}{\sqrt{x^2+y^2}} + \frac{\partial^2}{\partial y^2}\frac{y}{\sqrt{x^2+y^2}}\right)$$

$$\vec{\nabla}^2 A_x = -k\left(\frac{\partial}{\partial x}\frac{-xy}{\left(x^2+y^2\right)^{3/2}} + \frac{\partial}{\partial y}\left(\frac{1}{\left(x^2+y^2\right)^{1/2}} + \frac{-y^2}{\left(x^2+y^2\right)^{3/2}}\right)\right) = \ldots = k\frac{y}{\left(x^2+y^2\right)^{3/2}}$$

similarly

$$\vec{\nabla}^2 A_y = -k\frac{x}{\left(x^2+y^2\right)^{3/2}}$$

$$\vec{\nabla}^2 \vec{A} = -k\left\langle -\frac{y}{\left(x^2+y^2\right)^{3/2}}, \frac{x}{\left(x^2+y^2\right)^{3/2}}, 0\right\rangle$$

$$\vec{\nabla}^2 \vec{A} = -\frac{k}{s^2}\left\langle -\frac{y}{\left(x^2+y^2\right)^{1/2}}, \frac{x}{\left(x^2+y^2\right)^{1/2}}, 0\right\rangle = -\frac{k}{s^2}\hat{\phi} = -\mu_o \vec{J}$$

so

$$\vec{J} = \frac{1}{\mu_o}\frac{k}{s^2}\hat{\phi}$$