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| Fri.10/25 | (**1.6, 5.4.1-.4.2** Magnetic Vector Potential | |
| Mon., 10/28<br>Wed., 10/30<br>Thurs. 10/31 | **5.4.3** Multipole Expansion of the Vector Potential<br>**7.1.1-7.1.3** *Ohm's Law & Emf* | HW7 |

**Announcements:**

Test in 2 weeks!

**Last Time**

Using Ampere's Law

(no name) $\vec{\nabla}\cdot\vec{B}=0$ $\oint_S \vec{B}\cdot d\vec{a}=0$

Ampere's law: $\vec{\nabla}\times\vec{B}=\mu_0\vec{J}$ $\oint \vec{B}\cdot d\vec{\ell}=\mu_0 I_{enc}$

Key to using Ampere's law: Your answer's only as good as your assumptions so be confident of your field symmetry – use divergence and Biot-Savart Law to help convince yourself.

**<u>Boundary Conditions</u>** – Suppose that there is a surface current on a boundary. How do the magnetic field and vector potential above and below the boundary compare?

Consider at a very thin pillbox that extends across the surface. Because the sides are very small, there is no magnetic flux through them. We know that $\oint \vec{B}\cdot d\vec{a}=0$, so $B^{\perp}_{above}=B^{\perp}_{below}$ (they are in the same direction, too).

Consider an amperian loop perpendicular to the surface current (as shown below). Applying Ampere's law gives

$$\oint \vec{B}\cdot d\vec{\ell}=\mu_0 I_{enc},$$
$$B^{||}_{above}\ell - B^{||}_{below}\ell=\mu_0 K\ell,$$
$$B^{||}_{above} - B^{||}_{below}=\mu_0 K.$$

The components of the magnetic field parallel to the surface are perpendicular to the surface current.

**<u>Summary</u>**

*Magnetic Vector Potential*

We're going to define a Vector Potential and then play around with it a bit. As the article I suggested notes, in energy-based formulations of mechanics, such as the Hamiltonian or Lagrangian, it's the Vector Potential, rather than B that plays a prominent role – so it's got a lot of mathematical use.

*Recall Defining Scalar Potential*

When we introduced the Scalar Potential, a.k.a., the Electric Potential, we motivated it both *mathematically* and *physically*.

Our mathematical argument was that, since

$$\vec{\nabla}\times\vec{E}=0$$

And Vector Identity 10 tells us that generally

$$\vec{\nabla}\times\vec{\nabla}f=0$$

We could define an "f" to go along with E, specifically,

$$\vec{E}=-\vec{\nabla}V$$

Note that our defining equation is for the *gradient* of V, not for V itself. That means that there's an ambiguity in V itself – you can add any constant to it and still have a V that fits the definition.

Our physical argument was that, since $\vec{F}=-\vec{\nabla}U$ and $\vec{F}=q\vec{E}$, it made sense to factor a q out of U to define our new entity, V.

The ambiguity is here too since, the physically significant thing is the *change* in potential energy, or the change in voltage, not the specific value of voltage.

**V and U's *conditional* relation.** Note however, that *really* U is defined as a thing that's *shared* between two interacting parties. The definition of U is

$$\Delta U_{1,2}=-\int\vec{F}_{2\to1}\cdot d\vec{r}_{2\to1}=-\left(\int\vec{F}_{2\to1}\cdot d\vec{r}_2-\int\vec{F}_{2\to1}\cdot d\vec{r}_1\right)=-\left(\int\vec{F}_{2\to1}\cdot d\vec{r}_2+\int\vec{F}_{1\to2}\cdot d\vec{r}_1\right)=-\left(W_{2\to1}+W_{1\to2}\right)$$

, where that change in position is really the change in separation between the two objects – that can be changed *either* by moving object 1 *or* by moving object 2. In contrast, $\Delta V_{1,2}=-\int\vec{E}_{2\to1}\cdot d\vec{r}_1=-W_{2\to1}/q$. This equals U/q only if 2 remains stationary (i.e., no work gets done on it.) I point this out because something similar will happen with A.

Okay, let's turn our eye on B and the vector potential.

*Mathematical Motivation.*

$$\vec{\nabla}\cdot\vec{B}=0$$

Vector Identity 9 says that the divergence of a curl is 0,

$$\vec{\nabla}\cdot\vec{\nabla}\times\vec{A}=0$$

So, we're free to define an A to correspond with B:

$$\boxed{\vec{B}=\vec{\nabla}\times\vec{A}.}$$

*Physical Motivation*

Example of "generalized" energy and "generalized" momentum.

$$\vec{F}_{net} = \frac{d}{dt}\left(\vec{p}_1\right) = \vec{\nabla}_1 W = -\vec{\nabla}_1 U$$

Note: this uses the more general potential for E even when the fields are time-varying.

$$\frac{d}{dt}\left(m\vec{v}_1\right) = q\vec{v}\times\vec{B} + q\vec{E}$$

$$\frac{d}{dt}\left(m\vec{v}_1\right) = q\vec{v}\times\left(\vec{\nabla}\times\vec{A}\right) + q\left(-\vec{\nabla}V - \frac{\partial\vec{A}}{\partial t}\right)$$

$$\frac{d}{dt}\left(m\vec{v}_1\right) = q\left(\vec{\nabla}\left(\vec{v}\cdot\vec{A}\right) - \frac{d}{dt}\vec{A}\right) + q\left(-\vec{\nabla}V\right)$$

This uses $\frac{d}{dt}\vec{A} = \frac{\partial\vec{A}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{A}$ and Griffiths' product rule (4) where the deriviatives act only on the potential (so 0 for the terms with it acting on the *v*)

$$\frac{d}{dt}\left(m\vec{v}_1 + q\vec{A}\right) = -\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right)$$

So, the math works out so that *either* you could say that your system is a particle with the usual particle momentum and it's experiencing an external force of $q\vec{v}\times\vec{B} + q\vec{E}$

*Or* you could say that your system was some imaginary concoction who's momentum is $m\vec{v}_1 + q\vec{A}$ and who is subject to the work $-q\left(V - \vec{v}\cdot\vec{A}\right)$.

As always, if there is no external "force" on a system, then the time derivative of its "momentum" is 0 – i.e., that's a conserved quantity.

*If*

$$\frac{d}{dt}\left(m\vec{v}_1 + q\vec{A}\right) = -\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right) = 0$$

*Then*

$$\left(m\vec{v}_1 + q\vec{A}\right)_i = \left(m\vec{v}_1 + q\vec{A}\right)_f$$

In that way, just as the conservation of the total K+U of a system tells you how the "potential" energy sloshes over to be "kinetic" energy, this relation allows us to think about "potential" momentum and "kinetic" momentum.

While this might seem somewhat artificial, it's no more artificial than what we did to cook up the idea of "potential energy" to begin with – it's inherent to an *interaction*. Here, it's the interaction of the particle with the magnetic field.

You could think of this system as "particle plus fields", but don't forget that the fields themselves are inherently interacting with something else – their sources, so it isn't a "closed" system. Mathematically, it's *as good as* closed if $-\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right) = 0$.

Okay, so it's got a mathematical excuse for being defined, and there's some physical motivation. But still, do we *really* need this vector potential thing?

What why should we bother defining it?

(1) The same methods (see Ch. 3) that can be used to find the electric potential $V$ can be used to find each component of the magnetic vector potential $\vec{A}$ because they obey analogous equations.

(2) Electric potential $V$ is potential energy per charge and magnetic vector potential $\vec{A}$ can be thought of as momentum per charge.

In special relativity, the four-momentum $(E/c, \vec{p})$ includes both energy and momentum. The four-vector potential $(V/c, \vec{A})$ also has terms related to energy and momentum per charge.

(3) When it comes time to consider *time varying* currents, the most straight-forward place to begin is by looking at how that affects the potentials. From there we can develop expressions for the fields produced by time-varying currents.

(4) In quantum mechanics, potentials are used, not fields or forces. The term $Q(V - \vec{v} \cdot \vec{A})$ appears as a "velocity-dependent potential energy" in the Lagrangian. The four-momentum of a charged particle is replaced by the "canonical momentum":

$$p_i \rightarrow p_i + QA_i,$$

where $p_0 = E/c$ and $A_0 = V/c$.

The Aharonov-Bohm effect shows the use vector potential in a region where the magnetic field is zero. That is, the phase of the wave-function is effected by passing through a region of non-zero A *even if B=0 there*, such as outside a solenoid. So, for example, doing a double-slit experiment with a solenoid in the barrier that separates the two slits will result in an A-dependent new interference pattern.

**Building Mathematical Tools**

Now that we've motivated this object both mathematically and physically, let's build up some tools for using it.

Plugging this into the differential version of Ampere's law gives

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}$$
$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J}$$
$$\vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Now, recall that with the scalar potential, a constant offset isn't of any physical significance since what's significant is the difference between two V values, or its gradient – in either case, the offset dies off.

There's a similar ambiguity here. It's the curl of A that has physical significance, so we're free to choose its divergence to be 0. Griffiths' makes a plausible argument. We'll just run with that.

We *can* choose

$$\vec{\nabla}\cdot\vec{A}=0,$$

This choice is known as the "Coulomb Gauge."

(mind you, just as we don't always choose the scalar potential's offset to be 0, we don't *always* make this choice.)

which allows that

$$\boxed{\nabla^2\vec{A}=-\mu_0\vec{J}.}$$

Now, what we *mean* by $\nabla^2$ of a vector is $\nabla^2$ of its individual components:

(1) This is like three copies of an equation analogous to Poisson's equation:

$$\nabla^2 V=-\rho/\varepsilon_0 \quad \Leftrightarrow \quad \begin{array}{l}\nabla^2 A_x=-\mu_0 J_x\\ \nabla^2 A_y=-\mu_0 J_y\\ \nabla^2 A_z=-\mu_0 J_z\end{array}$$

The electric potential $V$ and charge density $\rho$ are related in the same way as each component of the magnetic vector potential $\vec{A}$ and the current density $\vec{J}$ as long as $\vec{J}\rightarrow 0$ at infinity (localized current):

$$V(\vec{r})=\frac{1}{4\pi\varepsilon_0}\int\frac{\rho(\vec{r}')}{\mathscr{r}}d\tau' \quad \Leftrightarrow \quad \boxed{\vec{A}(\vec{r})=\frac{\mu_0}{4\pi}\int\frac{\vec{J}(\vec{r}')}{\mathscr{r}}d\tau'}.$$

The contribution of each segment of current to $\vec{A}$ is in the same direction as $\vec{J}$. If $\vec{J}$ has the same direction everywhere, that is the direction of $\vec{A}$.

(2) The relationship between $\vec{A}$ and $\vec{B}$ is the same as the more familiar relation between $\vec{B}$ and $\vec{J}$:

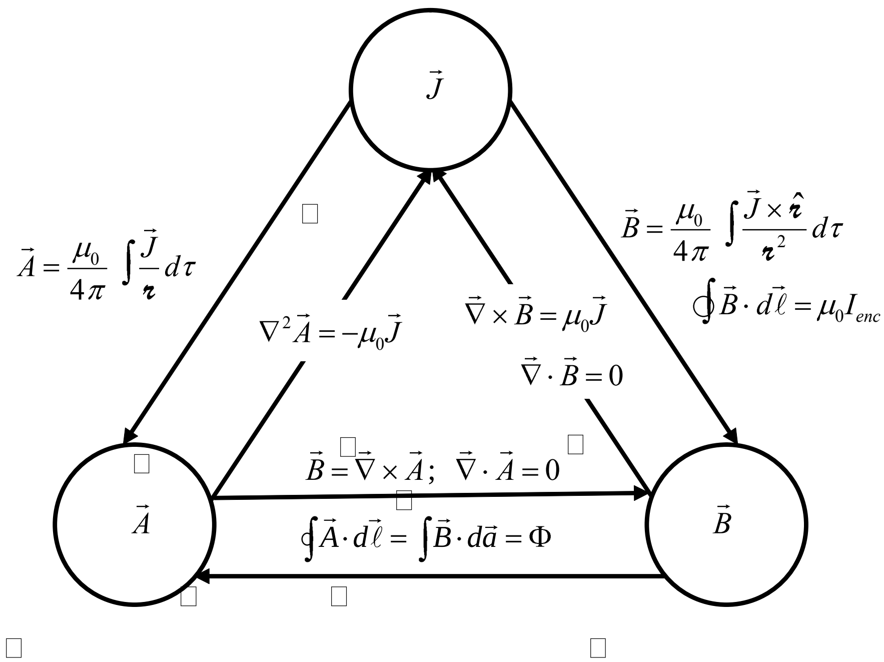
$$\begin{array}{c}\vec{\nabla}\cdot B=0\\ \vec{\nabla}\times\vec{B}=\mu_0\vec{J}\end{array} \quad \Leftrightarrow \quad \begin{array}{c}\vec{\nabla}\cdot\vec{A}=0\\ \vec{\nabla}\times\vec{A}=\vec{B}\end{array}$$

(from the Stoke's curl theorem)

$$\oint\vec{B}\cdot d\vec{\ell}=\int\mu_0\vec{J}\cdot d\vec{a}=\mu_0 I_{enc} \quad \Leftrightarrow \quad \boxed{\oint\vec{A}\cdot d\vec{\ell}=\int\vec{B}\cdot d\vec{a}=\Phi,}$$

where $\Phi$ is the magnetic flux through the loop. We can think of $\Phi$ as the source of $\vec{A}$ in just the same way as $\mu_0 I_{enc}$ is the source of $\vec{B}$.

**Summary of Magnetostatics** – This is similar to the diagram that we had for electrostatics, but all three quantities are vectors. Also, we haven't come up with a relation for one link.



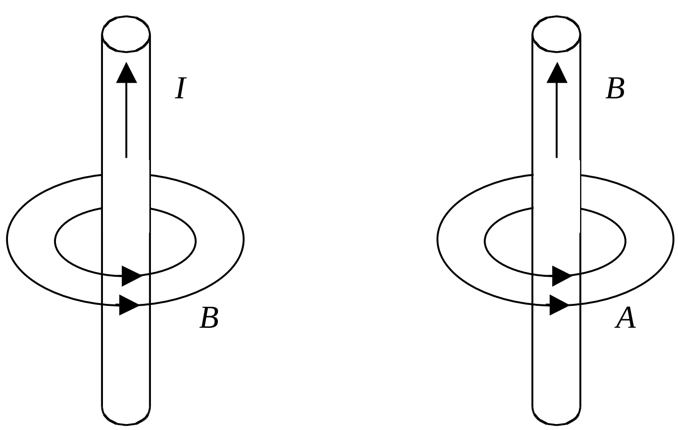

**Examples/Exercises**:

**Example 5.12 – Solenoid:** Find the vector potential of an infinite solenoid with *n* turns per length, radius *R*, and current *I*.

The solenoid produces a magnetic field of

$$\vec{B}_{solenoid} = \begin{cases} (\mu_0 n I)\hat{z} & s < R, \\ 0 & s > R. \end{cases}$$

We want the magnetic vector potential produced by this magnetic field. The analogous situations are shown below ( $\mu_0 I$ as a source of *B* and *B* as a source of *A*).



Outside of a wire carrying current $I$, the magnetic field is

$$\vec{B}_{wire} = \frac{\mu_0 I}{2\pi s}\hat{\phi},$$

so for the solenoid on the right we know that the vector potential is ( $\mu_0 I \to \Phi$):

$$\vec{A} = \frac{\Phi}{2\pi s}\hat{\phi}.$$

The magnetic flux is

$$\Phi = \begin{cases} B_{solenoid}\left(\pi s^2\right) = \left(\mu_0 n I\right)\left(\pi s^2\right) & s < R, \\ B_{solenoid}\left(\pi R^2\right) = \left(\mu_0 n I\right)\left(\pi R^2\right) & s > R. \end{cases}$$

The magnetic vector potential is

$$\vec{A} = \begin{cases} \left(\mu_0 n I s/2\right)\hat{\phi} & s < R, \\ \left(\mu_0 n I R^2/2s\right)\hat{\phi} & s > R. \end{cases}$$

Check the curl and divergence (in cylindrical coordinates). The only non-zero term in the curl is

$$\vec{\nabla} \times \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}\left(sA_\phi\right)\hat{z} = \left\{ \begin{matrix} \left(\mu_0 n I\right)\hat{z} & s < R \\ 0 & s > R \end{matrix} \right\} = \vec{B}_{solenoid}$$

The divergence is

$$\vec{\nabla} \cdot \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}\left(sA_s\right) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = 0,$$

because $A_\phi$ doesn't depend on $\phi$.

**Interaction with charged particle.**

Recall that $$\frac{d}{dt}\left(m\vec{v}_1 + q\vec{A}\right) = -\vec{\nabla}q\left(V - \vec{v}\cdot\vec{A}\right)$$

Say we've got a charged particle *sitting outside* a solenoid. In the absence of an electric field, or a velocity, the right-hand-side of the equation is 0, which means that so must be the left-hand side, and so

$$\left(m\vec{v}_1 + q\vec{A}\right)_i = \left(m\vec{v}_1 + q\vec{A}\right)_f$$

$$\Delta m\vec{v} = -\Delta q\vec{A}$$

If the solenoid is suddenly shut off (the paper suggests, to avoid any faraday effect, we could have the Solenoid be made of two concentric and oppositely charged and oppositely spinning cylinders that suddenly stop spinning) then

$$\left(m\vec{v}_1 + q\vec{A}\right)_i = \left(m\vec{v}_1 + q\vec{A}\right)_f$$
$$\left(0 + q\vec{A}\right) = \left(m\vec{v}_f + 0\right)$$
$$\frac{-q\mu_o nIR^2}{2s}\hat{\phi} = m\vec{v}_f$$
$$\vec{v}_f = \frac{-q\mu_o nIR^2}{2ms}\hat{\phi}$$

Apparently the charge gets a kick in the tangential direction!

> **Note:** If you want a *causal* tool, look to forces – they're our mathematical model of pushes and pushes; but if you just want to know how things *turn out* look to energy – it doesn't necessarily tell you how you got from state a to state b, but it does tell you that you've got to end up there – that's it's mathematical power and its conceptual weakness. In this particular case, the kick 'comes from' an electric field that's produced by the decelerating charges when the current is shut off. We'll see how accelerating/decelerating charges generate particular fields later.

**Problem 5.25 (a) – Long, Thin Wire:** Find the vector potential of a thin wire carrying current *I*.

Let's say that the current is in the +z direction. We know that the current will be in the same direction and by symmetry its size can only depend on the distance from the axis, so $\vec{A} = A(s)\hat{z}$ (in cylindrical coordinates). We also know that the magnetic field produced by the wire is $\vec{B} = (\mu_0 I/2\pi s)\hat{\phi}$. The definition of the magnetic vector potential, $\vec{B} = \vec{\nabla}\times\vec{A}$, gives us a differential equation:

$$\left(\frac{\mu_0 I}{2\pi s}\right)\hat{\phi} = -\left(\frac{\partial A}{\partial s}\right)\hat{\phi}.$$

Integrating this equation gives

$$A = -\int\left(\frac{\mu_0 I}{2\pi s}\right)ds = -\frac{\mu_0 I}{2\pi}(\ln s + C) = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{a}\right) \quad \text{and} \quad \vec{A} = -\frac{\mu_0 I}{2\pi}\ln\left(\frac{s}{a}\right)\hat{z},$$

where $C = -\ln a$ to make the units look better.

Check the curl and divergence (in cylindrical coordinates). The only non-zero term in the curl is

$$\vec{\nabla}\times\vec{A} = -\frac{\partial A_z}{\partial s}\hat{\phi} = \frac{\mu_0 I}{2\pi s}\hat{\phi} = \vec{B}.$$

The divergence is

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z} = 0,$$

because $A_z$ doesn't depend on *z*.

**Problem 5.23:** What current density would produce the vector potential $\vec{A} = k\,\hat{\phi}$ (where $k$ is a constant) in cylindrical coordinates?

The associated magnetic field is

$$\vec{B} = \vec{\nabla} \times \vec{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)\hat{z} = \frac{1}{s}\frac{\partial}{\partial s}(ks)\hat{z} = \frac{k}{s}\hat{z}.$$

The current density can be found using Ampere's law (differential form):

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J},$$

$$\vec{J} = \frac{1}{\mu_0}\vec{\nabla} \times \vec{B} = \frac{1}{\mu_0}\left(-\frac{\partial B_z}{\partial s}\right)\hat{\phi} = \frac{1}{\mu_0}\left[-\frac{\partial}{\partial s}\left(\frac{k}{s}\right)\right]\hat{\phi} = \frac{k}{\mu_0 s^2}\hat{\phi}.$$

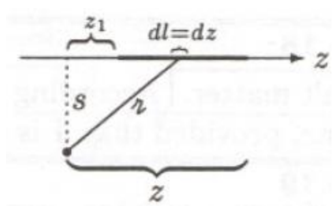
It circles the $z$ axis like the vector potential. It would be difficult to use $\nabla^2 \vec{A} = -\mu_0 \vec{J}$ directly because the unit vectors have derivatives in cylindrical coordinates.

**Problem 5.22:** Find the vector potential for a current $I$ along the $z$ axis from $z_1$ to $z_2$.

Integrate up the contribution of each segment of the current using

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{r}\,')}{\boldsymbol{\mathscr{r}}}\,d\tau' \rightarrow \frac{\mu_0}{4\pi}\int \frac{I\,d\vec{\ell}}{\boldsymbol{\mathscr{r}}}.$$

If a point is a distance $s$ from the $z$ axis, then the separation is $\boldsymbol{\mathscr{r}} = \sqrt{z^2 + s^2}$.




This gives (in cylindrical coordinates)

$$\vec{A} = \frac{\mu_0 I}{4\pi}\int_{z_1}^{z_2} \frac{dz}{\sqrt{z^2 + s^2}}\hat{z} = \frac{\mu_0 I}{4\pi}\left[\ln\left(z + \sqrt{z^2 + s^2}\right)\right]_1^2 \hat{z},$$

$$= \frac{\mu_0 I}{4\pi}\ln\left[\frac{z_2 + \sqrt{z_2^2 + s^2}}{z_1 + \sqrt{z_1^2 + s^2}}\right]\hat{z}.$$

**Return to Boundary Conditions (with A in hand)** – Suppose that there is a surface current on a boundary. How do the magnetic field and vector potential above and below the boundary compare?

Consider at a very thin pillbox that extends across the surface. Because the sides are very small, there is no magnetic flux through them. We know that $\oint \vec{B} \cdot d\vec{a} = 0$, so $B_{above}^{\perp} = B_{below}^{\perp}$ (they are in the same direction, too).

Consider an amperian loop perpendicular to the surface current (as shown below). Applying Ampere's law gives

$$\oint \vec{B} \cdot d\vec{\ell} = \mu_0 I_{enc},$$
$$B_{above}^{||}\ell - B_{below}^{||}\ell = \mu_0 K\ell,$$
$$B_{above}^{||} - B_{below}^{||} = \mu_0 K.$$

The components of the magnetic field parallel to the surface are perpendicular to the surface current.

The magnetic vector potential is continuous across the boundary, $\vec{A}_{above} = \vec{A}_{below}$, but it's "derivative" isn't because $\vec{\nabla} \times \vec{A} = \vec{B}$.

**Preview**

Next time, we'll talk about boundary conditions and summarize magnetostatics.

"I think I mostly understood the argument that Griffith's was making using the lambdas to prove the Delta dot A = 0, but I don't see how he makes the connection all of the sudden to "read off the solution" in eqn. 5.65." Casey McGrath

To me, it seems that that argument previous to this equation is under an entirely different circumstance - we purposely assumed Delta dot A was not = 0 to derive that (unnumbered) eqn. for lambda. Casey McGrath

"What is he accomplishing at the top of page 244 with the lambdas? I don't really follow what is happening at that point."Freeman,

"In ex 5.11 I don't follow griffith's reasoning as to why he orients the axis the way he does." Jessica

Im also having trouble following this example. Connor W,

"Will knowing about the discontinuities along a surface charge for the B field give us some other important tools in the future like it did for the (similar) E field discontinuities? There we were able to calculate the induced surface charge..."Casey McGrath

"So it seems like Griffiths is making a point of making us understand that A in magnetostatics is in many ways analogous to the electric potential V in electrostatics. Could we go over the similarities and dissimilarities of A and V."Ben Kid

Yes I too feel that he draws many mathematical parallels between the two, but conceptually I'm not seeing anything as easy as voltage to visualize and think of applications for. Could we talk a little bit about it? Thanks!Rachael Hach

"Why is it unlikely that we will ever need an equation for A in terms of B? Couldn't Griffiths have used this in Example 5.11 along with Ampere's Law and equations 5.22 and 5.23?" Spencer

Could we at least derive the equation and talk about it in class? Casey McGrath

"I'm still confused as to what A actually is and how its analogous to potential. Can we go over conceptually what A represents?"Sam

"Can we do an example like 5.12 but could we have a real world example instead? I'd like to have something physical to connect the concepts to."Casey P,