Wed. 10/16 Thurs 10/17 Fri. 10/18	*(C 17)* **5.2** Biot-Savart Law *(C 17)* **5.2** Biot-Savart Law **T5** Quiver Plots	HW5
Mon.10/21 Wed.10/23 Thurs. 10/24	*(C 21.6-7,.9)* **1.3.4-1.3.5, 1.5.2-1.5.3, 5.3.1-.3.2** Div & Curl B *(C 21.6-7,.9)* **5.3.3-.3.4** Applications of Ampere's Law	HW6

From Last Time

$$\vec{F}_{mag} = \sum q_i \vec{v}_i \times \vec{B} = \int \rho\vec{v} \times \vec{B} d\tau = \int \vec{J} \times \vec{B} d\tau \qquad \vec{J} = \frac{d\vec{I}}{da_\perp}$$

$$\vec{F}_{mag} = \int \sigma\vec{v} \times \vec{B} dA = \int \vec{K} \times \vec{B} dA \qquad \vec{K} = \frac{d\vec{I}}{dl_\perp}$$

$$\vec{F}_{mag} = \int \lambda\vec{v} \times \vec{B} dl = \int I d\vec{l} \times \vec{B} \qquad \vec{I} = \frac{d\vec{q}}{dt} = \frac{dq}{dx}\frac{d\vec{x}}{dt} = \lambda\vec{v}$$

$$\vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}$$

When we started talking about Electric Interactions, we fairly breezed over the bit about relating the field to the force on a charge – that was rather trivial. What with the cross products involved in the Magnetic Interaction, it's not so trivial, so we've spent a little more time on relating the force to the field. Now we've gotten a general sense of how a charge, line-current, surface-current, or volume-current responds to a magnetic field.

We turn our attention to the *production* of the magnetic field.

<u>Summary</u>

Biot-Savart Law

A few days ago, I took us on a deep, dark digression into relativity so that we could understand the force a charged particle moving near a wire feels. The result was, we, in the lab frame, would expect it to feel a force to the tune of

$$\vec{F} = -\tfrac{\mu_o}{4\pi}\frac{2I_z}{\mathscr{r}} q v_{p.z}\hat{\mathscr{r}}$$

It seems a bit unwieldy, but to generalize this, preserving the fact that it's the parallel components of the current and particle velocity that determine the attraction/repulsion, we can write this as

$$\vec{F} = q\vec{v} \times \tfrac{\mu_o}{4\pi}\frac{2\vec{I} \times \hat{\mathscr{r}}}{\mathscr{r}}$$

Or, comparing with the form of the Lorentz Force Law introduce last time, we'd identify

$$\vec{B}(\vec{r}) = \tfrac{\mu_o}{4\pi}\frac{2\vec{I}\times\hat{\mathscr{r}}}{\mathscr{r}}.$$

This is for the rather special case of an infinitely long wire. We'd like to build on this to get the general case but, as is usually the case, it's straightforward to go from the general case to a specific one, but not always possible to go the other way around. It's not at all self-evident that the contribution of just a morsel of current carrying wire is

$$d\vec{B}(\vec{r}) = \tfrac{\mu_o}{4\pi}\frac{\vec{I}dl'\times\hat{\mathscr{r}}}{\mathscr{r}^2}.$$

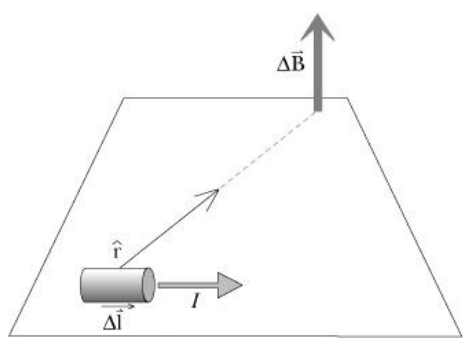
Later, we will at least see that integrating over all contributions of an infinitely long, straight wire returns the result we've already found.

that in its frame, it thinks there's an *electric* force to the tune of (expressed in terms of the parameters that we'd measure in the lab frame)

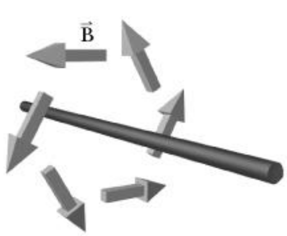
The magnetic field of a moving point charge is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}'\times\hat{\mathscr{r}}}{\mathscr{r}^2}, \tag{5.32}$$

where $d\vec{\ell}$ is a segment of length that points in the direction of the current and $\vec{\mathscr{r}}$ is a vector that points from the segment to the location $\vec{r}$ where the field is being calculated (as shown below).



The constant is $\mu_0 = 4\pi\times 10^{-7}\,\mathrm{N/A^2}$. Recall that the units for magnetic field are teslas, 1 T = 1 N/(A•m) = 1 (N/C)/(m/s).



Superposition Principle

To find the total magnetic field, we add (or integrate) up the contributions from all of the segments of current.

If we know the answers for multiple shapes, we can add them up. Be sure to remember that you are adding vectors, so you must take their directions into account. Sketches help you to do this correctly.

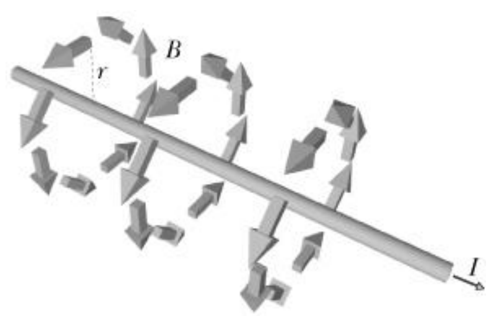
There are less easily solvable problems, so we'll review the examples in detail. Typically, we won't try to find the magnetic field everywhere.

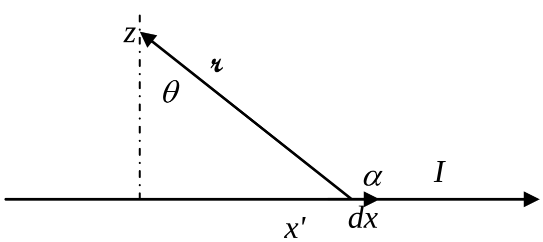
<u>Examples/Exercises</u>:

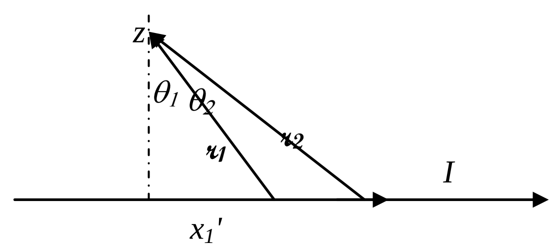
Example 5.5 – Straight Thin Wire

Though we can't derive the general expression from the special case that we'd reasoned from relativity, we can at least show that the general expression is *consistent* with it / can be used to derive irFind the magnetic field at a distance *s* from a straight wire carrying a steady current *I*.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^2},$$







The book does this in a very cute way, mostly in terms of angles; I'll first set it up the old fashioned way, in terms of x,y, z; then I'll relate it over to the way Griffith's set it up.

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \vec{\boldsymbol{\imath}}}{\boldsymbol{\imath}^3} = \frac{\mu_0}{4\pi} I \int \frac{dx'\hat{x} \times \vec{\boldsymbol{\imath}}}{\left(x'^2 + z^2\right)^{3/2}}, = \frac{\mu_0}{4\pi} I \int \frac{dx' z(-\hat{y})}{\left(x'^2 + z^2\right)^{3/2}}$$

$$= \frac{\mu_0}{4\pi} z(-\hat{y}) I \int \frac{dx'}{\left(x'^2 + z^2\right)^{3/2}} = \frac{\mu_0}{4\pi} z(-\hat{y}) I \left.\frac{x'}{z^2\left(x'^2 + z^2\right)^{1/2}}\right|_{x_i}^{x_f}$$

Now for doing it in terms of angles:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2} = \frac{\mu_0}{4\pi} I \int \frac{dx'\hat{x} \times \hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}$$

$$x' = z \tan\theta$$

$$dx' = zd\tan\theta = z\frac{1}{\cos^2\theta} d\theta \quad \text{so} \quad \text{similarly} \quad \boldsymbol{\imath} = \frac{z}{\cos\theta}$$

For that matter, $\left|d\vec{\ell}' \times \hat{\boldsymbol{\imath}}\right| = d\ell' \sin\alpha = d\ell' \sin\left(\theta + 90^\circ\right) = d\ell' \cos\theta$ while the right-hand-rule tells us it points in the –y direction.

Putting all these together gives

$$\vec{B} = \frac{\mu_0}{4\pi} I \int \frac{\left|d\vec{\ell} \times \hat{\boldsymbol{\imath}}\right|}{\boldsymbol{\imath}^2}(-\hat{y}) = \frac{\mu_0 I}{4\pi} \int \left(\frac{\cos^2\theta}{s^2}\right)\left[\left(\frac{s}{\cos^2\theta} d\theta\right)\cos\theta\right](-\hat{y}) = \frac{\mu_0 I}{4\pi s} \int \cos\theta\, d\theta (-\hat{y})$$

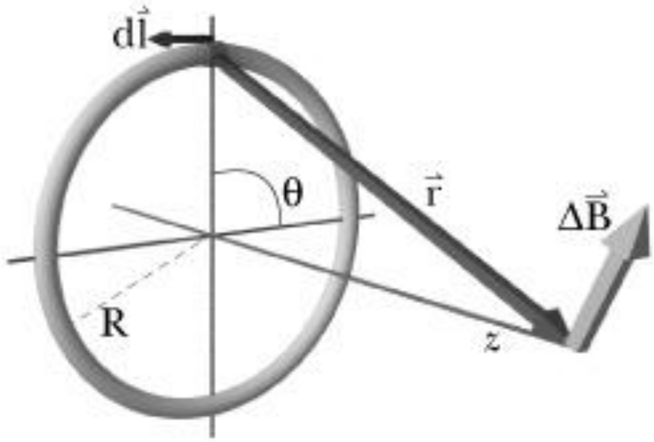
$$\vec{B} = \frac{\mu_0 I}{4\pi s}\left(\sin\theta_2 - \sin\theta_1\right)(-\hat{y})$$

For an infinite wire, $\theta_1 = -90^\circ = -\pi/2$ and $\theta_2 = 90^\circ = \pi/2$, so

$$B = \frac{\mu_0 I}{2\pi s}.$$

Example 5.6 – Circular Loop of Wire

Find the magnetic field due to a current carrying loop. The book does the deceptively simple case of a on-axis. We'll start out by setting up the more general case (and then chicken out) so you see more generally how it's done.



$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \vec{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^3},$$

If we put the observation location some distance along the x-axis and some distance along the z-axis, then

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}' = (x - x', 0 - y', z - 0) = (x - R\cos\theta, -R\sin\theta, z)$$

$$|\vec{\mathscr{r}}| = \left((x - R\cos\theta)^2 + (R\sin\theta)^2 + z^2\right)^{1/2} = \left(x^2 + R^2 - 2xR\cos\theta + z^2\right)^{1/2}$$

$$d\vec{\ell}' = Rd\vec{\theta} = Rd\theta(-\sin\theta, \cos\theta, 0)$$

$$d\vec{\ell}' \times \vec{\mathscr{r}} = Rd\theta\cos\theta z\hat{x} - Rd\theta\sin\theta z\hat{y} + (R^2 d\theta\sin^2\theta - Rxd\theta\cos\theta + R^2 d\theta\cos^2\theta)\hat{z}$$

$$d\vec{\ell}' \times \vec{\mathscr{r}} = Rd\theta\left(\cos\theta z\hat{x} - \sin\theta z\hat{y} + (R - x\cos\theta)\hat{z}\right)$$

So, we could put all this together and, presumably, integrate the three different terms.

Now, let's pull back and say we're on axis – then things get simpler since x = 0

$$|\vec{\mathscr{r}}| = \left(R^2 + z^2\right)^{1/2}$$

$$d\vec{\ell}' \times \vec{\mathscr{r}} = Rd\theta\cos\theta z\hat{x} - Rd\theta\sin\theta z\hat{y} + (R^2 d\theta\sin^2\theta + R^2 d\theta\cos^2\theta)\hat{z}$$

$$d\vec{\ell}' \times \vec{\mathscr{r}} = Rd\theta\left(\cos\theta z\hat{x} + \sin\theta z\hat{y} + R\hat{z}\right)$$

From the symmetry of the situation, we could see that there's going to be no net x or y field, just a z-field. Alternatively, we could set it up a little more

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \vec{\mathscr{r}}}{\mathscr{r}^3} = \frac{\mu_0}{4\pi} I \int \frac{R d\theta (\cos\theta z\hat{x} + \sin\theta z\hat{y} + R\hat{z})}{(R^2 + z^2)^{3/2}}$$

$$= \frac{\mu_0}{4\pi} \frac{R}{(R^2 + z^2)^{3/2}} I \int d\theta (\cos\theta z\hat{x} + \sin\theta z\hat{y} + R\hat{z})$$

$$= 0\hat{x} + 0\hat{y} + \frac{\mu_0}{4\pi} \frac{IR^2 2\pi}{(R^2 + z^2)^{3/2}} \hat{z}$$

$$= \frac{\mu_0}{2} \frac{IR^2}{(R^2 + z^2)^{3/2}} \hat{z}$$

At the center of the loop ($z = 0$),

$$B_z = \mu_0 I/2R.$$

Much like the 1/R dependence of the field due to an infinite wire.

Far from the loop (z>>R), it looks like

$= \dfrac{\mu_0}{2}\dfrac{IR^2}{z^3}\hat{z}$, so it drops off like $1/z^3$ much like the electric field of an electric dipole.

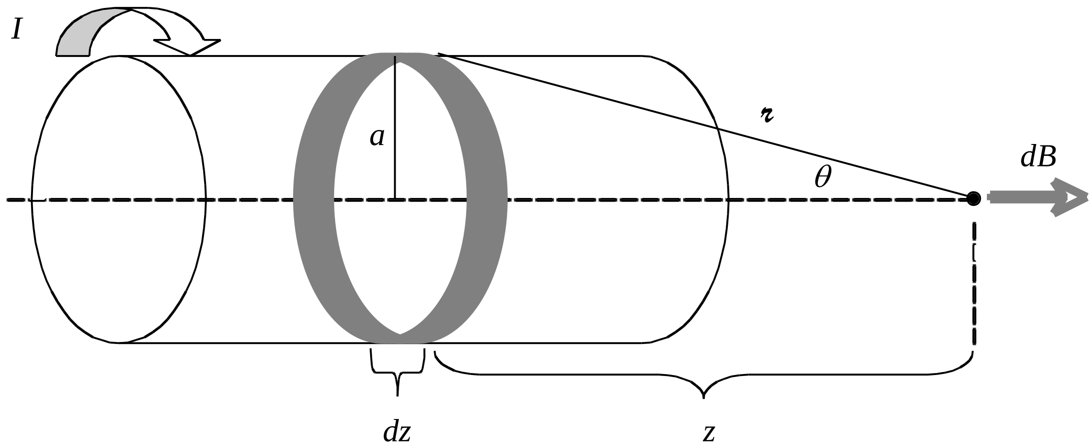
That's no coincidence. The fields look qualitatively the same. Much like an electric dipole has one + charge and one – charge; the magnetic dipole has one "north pole" and one "south pole" between which the fields run.

Problem 5.11 – Solenoid

Find the magnetic field at point *P* on the axis of a tightly wound solenoid consisting of *n* turns per length wrapped around a cylindrical tube of radius *a* and carrying current *I*. Express the answer in terms of θ_1 and θ_2 (as shown below).

QuickTime™ and a
TIFF (Uncompressed) decompressor
are needed to see this picture.

Consider a thin ring of width *dz* which is a distance *z* from point *P* (as shown below).



We’ll build on the result we already have for a ring, now we’ll say that we have a stack of rings, so the current that we were dealing with is just a differential bit compared to the total current circulating the solenoid’s surface. Then,

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{2}\frac{(dI)R^2}{\left(R^2+z^2\right)^{3/2}}\hat{z}$$

More generally, since not *all* the rings can live in the z=0 plane, this is

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{2}\frac{(dI)R^2}{\left(R^2+(z-z')^2\right)^{3/2}}\hat{z}$$

If, for convenience, we put the observation location at the origin, we can drop z.

Now, how do we translate this bit of current into a bit of length of the tube?

$$K = \frac{dI}{dl_\perp} = \frac{dI}{dz'} = \frac{NI}{L} \Rightarrow dI = \frac{NI}{L}dz'$$

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{2}\frac{NI}{L}\frac{(dz')R^2}{\left(R^2+z'^2\right)^{3/2}}\hat{z}$$

so the total for the whole solenoid is

$$B = \frac{\mu_0 INR^2}{2L}\int_{z_1}^{z_2}\frac{dz'}{\left(R^2+z'^2\right)^{3/2}}.$$

(if we put the observation location at the origin for conevenience, then

From the diagram, we have three relations for the angles:

$$\sin\theta = \frac{R}{\sqrt{R^2+z'^2}}, \quad \cos\theta = \frac{z'}{\sqrt{R^2+z'^2}}, \quad \text{and} \quad \tan\theta = \frac{R}{z'}.$$

There are (at least) two methods of solution:

(1) Integrate, then put in terms of the angles

$$B = \frac{\mu_0 INR^2}{2L}\left[\frac{z'}{R^2\left(R^2+z'^2\right)^{1/2}}\right]_{z_1}^{z_2} = \frac{\mu_0 In}{2}\left[\frac{z_2'}{\left(R^2+z_2'^2\right)^{1/2}} - \frac{z_1'}{\left(R^2+z_1'^2\right)^{1/2}}\right]$$

$$= \frac{\mu_0 IN}{2L}\left(\cos\theta_2 - \cos\theta_1\right)$$

(2) Put in terms of the angles, then integrate

We know that $z' = R\cot\theta$, so

$$\frac{dz'}{d\theta} = -R\csc^2\theta = -\frac{R}{\sin^2\theta}, \quad \text{so} \quad dz' = -\frac{R}{\sin^2\theta}d\theta .$$

The integrand can be written as

$$\frac{1}{\left(R^2+z'^2\right)^{3/2}} = \frac{\sin^3\theta}{R^3} .$$

The integral becomes

$$B = \frac{\mu_0 INR^2}{2L}\int_{\theta_1}^{\theta_2}\left(\frac{\sin^3\theta}{R^3}\right)\left(-\frac{R}{\sin\theta}d\theta\right) = \frac{\mu_0 IN}{2L}\left[\cos\theta\right]_{\theta_1}^{\theta_2}$$

$$= \frac{\mu_0 IN}{2L}\left(\cos\theta_2 - \cos\theta_1\right)$$

<u>Special Case</u>: Long (~infinite) solenoid

In this case, $\theta_2 = 0$ and $\theta_1 = -180^\circ$, so $B = \mu_0 IN/L$. We've shown that this is true on the axis. With Ampere's law, we'll show that it's true everywhere inside a long solenoid!

Spinning Disk

Another direction in which we could expand upon our ring solution is radially – say we want to know the on-axis field due to a flat coil of current.

$$d\vec{B}(\vec{r}) = \frac{\mu_0}{2}\frac{(dI)r'^2}{\left(r'^2+z^2\right)^{3/2}}\hat{z}$$

Where

$$K = \frac{dI}{dl_\perp} = \frac{dI}{dr'} = \frac{NI}{R} \Rightarrow dI = \frac{NI}{R}dr'$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2}\frac{NI}{R}\int_0^R \frac{r'^2 dr'}{\left(r'^2+z^2\right)^{3/2}}\hat{z}$$

Preview

On Friday, we'll do more examples and learn how to make diagrams of vector fields (like $\vec{E}$ and $\vec{B}$).

You can also ask questions about HW 5 & 6.

"Can we go over converting the cross products in these integrals to sines and cosines? Also why does a single moving charge not constitute a current?"
Sam

"Could we also go over the relativistic transformations of the Electric and Magnetic fields in section 12.3.2? Do we need this to do problem 28 in the homework?"
Casey McGrath

"Can we go over converting the cross products in these integrals to sines and cosines? Also why does a single moving charge not constitute a current?"
Sam

"Can you walk through an example using the Bio-Savart law, I didn't really follow how Griffiths did the examples in this section."
Jessica

At least for me, I didn't really understand how he got the magnitude dl' sin(alpha) = dl' cos(theta). I mean, I can derive it trigonometrically and see it is true, but don't really see how he goes from the cross product to that statement. I'd appreciate especially talking about how he calculates the (dl')x(r_hat) values for different cases.
Casey McGrath

"I'd also like to go over the two examples Griffiths used in this section. Can you also give us examples of what kinds of configurations we are going to encounter when using the Biot-Savart Law for surfaces and volumes?"
Spencer

Yeah Griffiths kind of made some initial assumptions on Example 5.6 and zipped through the rest of it. I'd really like it if we could go through that a bit more slowly.
Rachael Hach

"Griffith's just gives us the Biot-Savart law right away - is he going to explain how it was derived at any point?"
[Casey McGrath]