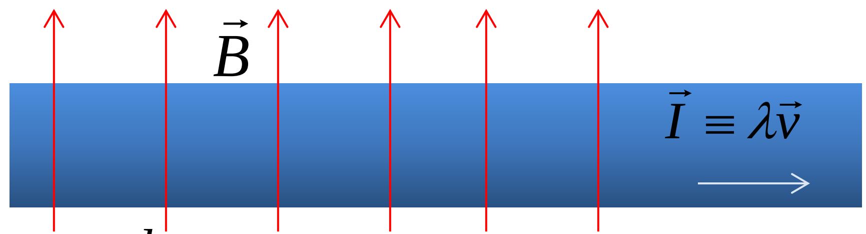
Wed. Thurs. Fri.	(C 17) 5.1.3 Lorentz Force Law: currents (C 17) 5.2 Biot-Savart Law	HW6
Mon. Tues. Wed. Fri.	(C 17) 5.2 Biot-Savart Law T5 Quiver Plots (C 21.6-7,.9) 1.3.4-1.3.5, 1.5.2-1.5.3, 5.3.1-.3.2 Div & Curl B (C 21.6-7,.9) 5.3.3-.3.4 Applications of Ampere's Law	HW7

Magnetic Force *on* Charge Distribution

1-D collection of moving charges: wire

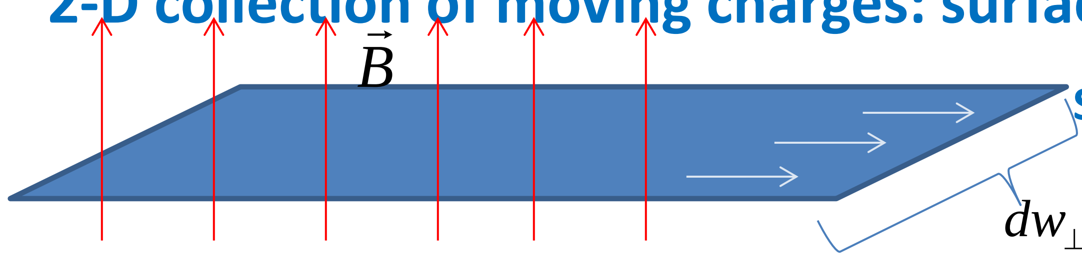


$$\vec{F} \Rightarrow \int dq\vec{v} \times \vec{B} = \int dl \frac{dq}{dl}\vec{v} \times \vec{B} = \int dl\lambda\vec{v} \times \vec{B} = \int dl\vec{I} \times \vec{B} \qquad \vec{I} \equiv \lambda\vec{v}$$

For charge flow confined to infinitesimally-thin wire must flow *along* wire, so $\hat{v} = d\hat{l}$ **and** $\vec{F} = \int dl\vec{I} \times \vec{B} = \int d\vec{l}\, I \times \vec{B}$

Note: *I* and *B* can vary along (be functions of) length.

2-D collection of moving charges: surface



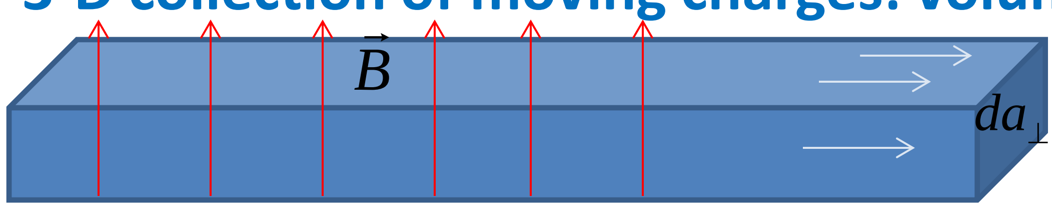
Surface Current Density

$$\vec{K} \equiv \sigma\vec{v}$$

$$\vec{K} = \frac{d\vec{I}}{dw_{\perp}}$$

$$\vec{F} = \int dq\vec{v} \times \vec{B} = \int da \frac{dq}{da}\vec{v} \times \vec{B} = \int da\sigma\vec{v} \times \vec{B} = \int da\vec{K} \times \vec{B}$$

3-D collection of moving charges: volume



Volume Current Density

$$\vec{J} \equiv \rho\vec{v}$$

$$\vec{J} = \frac{d\vec{I}}{da_{\perp}}$$

$$\vec{F} = \int dq\vec{v} \times \vec{B} = \int d\tau \frac{dq}{d\tau}\vec{v} \times \vec{B} = \int d\tau\rho\vec{v} \times \vec{B} = \int d\tau\vec{J} \times \vec{B}$$

Exercise:

a) Current I flows down wire of radius a. If it's uniformly distributed over the wire's cylindrical surface, then what is the surface current density *K*?

b) Current I flows down wire of radius a. If it's distributed throughout the volume such that $J \propto \frac{1}{s}$, what would be the expression for the volume current density?

Charge Continuity Equation

3-D collection of moving charges

Volume Current Density

$$\vec{J} \equiv \rho \vec{v}$$

$$\vec{J} = \frac{d\vec{I}}{da_\perp}$$

Current crossing through closed surface is rate of change of charge in enclosed volume:

$$\oint \vec{J} \cdot d\vec{a} = I = -\frac{dq}{dt}$$

Fundamental theorem for Divergences

$$\int \left(\vec{\nabla} \cdot \vec{J}\right) d\tau = I = -\frac{d}{dt} \int \rho \, d\tau$$

Equating Integrands

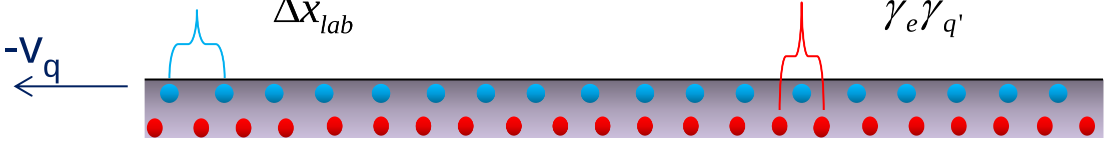
$$\vec{\nabla} \cdot \vec{J} = -\frac{d\rho}{dt}$$

A divergence of current means a depletion of charge

Biot-Savart Law Suggested

Charge's frame:

$\lambda_+' = \dfrac{\gamma_{q'} e}{\Delta x_{lab}}$ $\qquad$ $\lambda_- = \dfrac{-\gamma_{e'}}{\gamma_e \gamma_{q'}} \lambda_+'$



● = ionic atomic core

● = electron

v_e' = electron velocity measured by charge

$$\vec{F}_{q\leftarrow wire}' = q\vec{E}_{wire}'$$

q

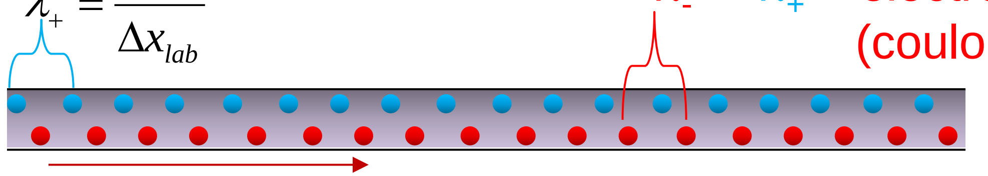
$$\vec{E}_{wire}' = \vec{E}_+' + \vec{E}_-' = \frac{\mu_o}{4\pi}\frac{2I\gamma_{q'}}{\boldsymbol{\imath}} v_q$$

Define $I = \lambda_+ v_e$ and $\mu_o \equiv \frac{1}{\varepsilon_o c^2}$

$$\vec{F}_{q\leftarrow wire}' = q v_q \frac{\mu_o}{4\pi}\frac{2I}{\boldsymbol{\imath}}\gamma_{q'}$$

Lab's frame:

$\lambda_+ = \dfrac{e}{\Delta x_{lab}}$ $\qquad$ $\lambda_- = -\lambda_+$ = electrons' charge density (coulombs/meter)



v_e = electron velocity measured by us in the "lab frame"

$$\vec{F}_{q\leftarrow wire} = q v_q \frac{\mu_o}{4\pi}\frac{2I}{\boldsymbol{\imath}}\hat{y} = q\vec{v}\times\frac{\mu_o}{4\pi}\frac{2I\times\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}}$$

q $\longrightarrow$ v_q

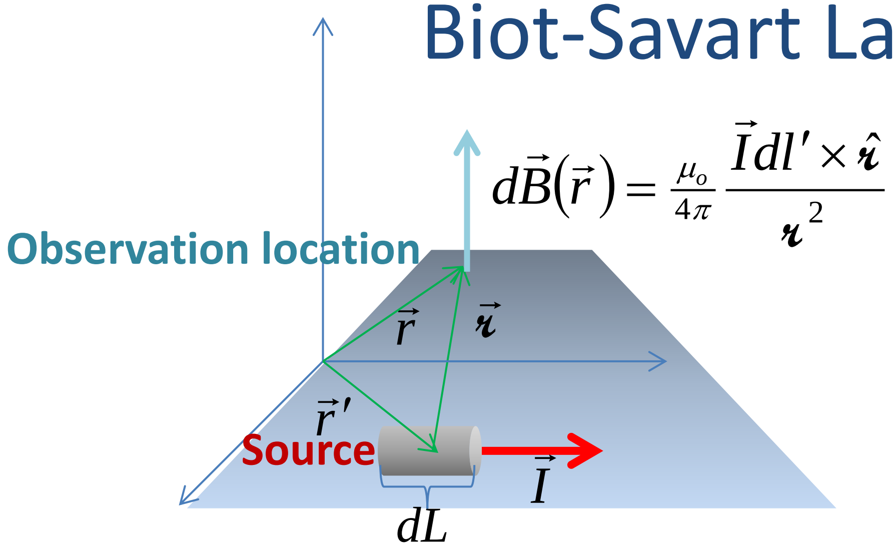
$$\vec{B} \equiv \frac{\mu_o}{4\pi}\frac{2I\times\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}}$$

Generally for a morsel of wire:

$$d\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi}\frac{\vec{I}dl'\times\hat{\boldsymbol{\imath}}}{\boldsymbol{\imath}^2}$$

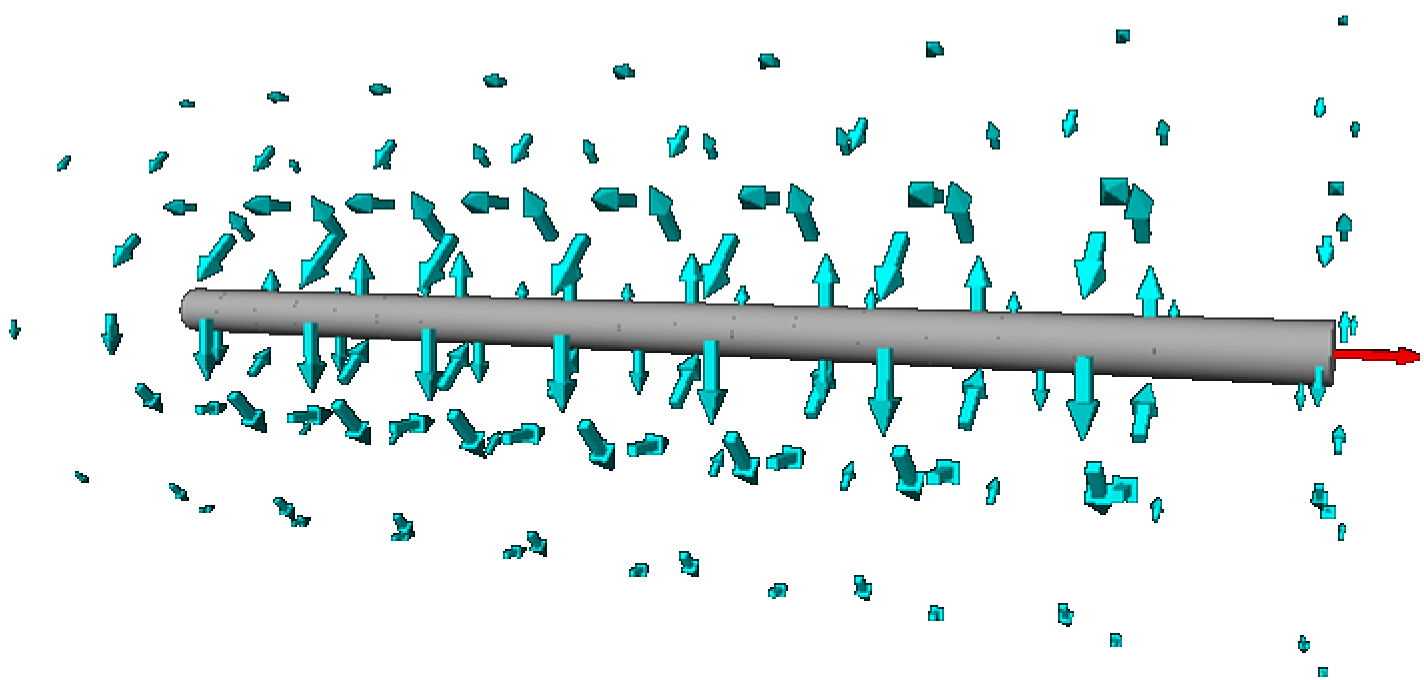
(we've not yet proven)

Biot-Savart Law



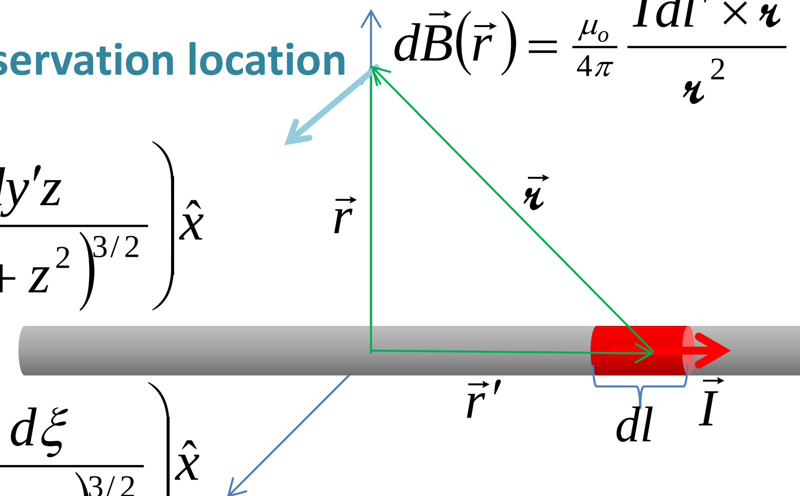
$$d\vec{B}(\vec{r}) = \frac{\mu_o}{4\pi} \frac{\vec{I}\, dl' \times \hat{\boldsymbol{\mathscr{r}}}}{\boldsymbol{\mathscr{r}}^2}$$

See 17_Bwire_with_r.py,   17_B_long_wire.py



Biot-Savart Law
Example 5.5: Field of Infinite Wire
(connecting to result of relativistic argument)

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \vec{\mathscr{r}}}{\mathscr{r}^3}$$



$$= \frac{\mu_0}{4\pi} I \int_{y'=-\infty}^{\infty} \frac{dy'\hat{y} \times (z\hat{z} - y'\hat{y})}{(y'^2 + z^2)^{3/2}} = \frac{\mu_0}{4\pi} I \left(\int_{y'=-\infty}^{\infty} \frac{dy' z}{(y'^2 + z^2)^{3/2}} \right) \hat{x}$$

$$= \frac{\mu_0}{4\pi} I \frac{z}{z^2} \left(\int_{\frac{y'}{z}=-\infty}^{\infty} \frac{d\left(\frac{y'}{z}\right)}{\left(\left(\frac{y'}{z}\right)^2 + 1\right)^{3/2}} \right) \hat{x} = \frac{\mu_0}{4\pi} \frac{I}{z} \left(\int_{\xi=-\infty}^{\infty} \frac{d\xi}{(\xi^2 + 1)^{3/2}} \right) \hat{x}$$

$$= \frac{\mu_0}{4\pi} \frac{I}{z} \frac{\xi}{(\xi^2 + 1)^{1/2}} \Bigg|_{-\infty}^{\infty} \hat{x} = \frac{\mu_0}{4\pi} \frac{I}{z} \left(\frac{\infty}{(\infty^2 + 1)^{1/2}} - \frac{-\infty}{(\infty^2 + 1)^{1/2}} \right) \hat{x}$$

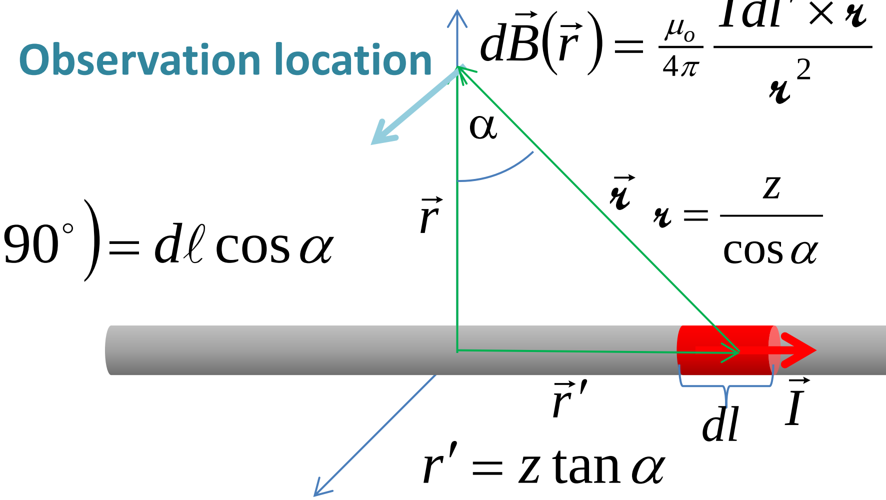
$$= \frac{\mu_0}{4\pi} \frac{I}{z} \left(2 \frac{\infty}{(\infty^2 + 1)^{1/2}} \right) \hat{x} = \frac{\mu_0}{4\pi} \frac{I}{z} \left(2 \frac{1}{\left(1 + \frac{1}{\infty^2}\right)^{1/2}} \right) \hat{x} = \frac{\mu_0}{4\pi} \frac{2I}{z} \hat{x}$$

Biot-Savart Law
Example 5.5: Field of Infinite Wire
(Book's approach)

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\vec{\ell}' \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

$$d\vec{\ell}' \times \hat{\mathfrak{r}} = \left|d\vec{\ell}' \times \hat{\mathfrak{r}}\right| \hat{x}$$

$$\left|d\vec{\ell}' \times \hat{\mathfrak{r}}\right| = |d\ell||\hat{\mathfrak{r}}| \sin\left(\alpha + 90^\circ\right) = d\ell \cos\alpha$$



$$d\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{I} dl' \times \hat{\mathfrak{r}}}{\mathfrak{r}^2}$$

$$\mathfrak{r} = \frac{z}{\cos\alpha}$$

$$r' = z \tan\alpha$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{d\ell \cos\alpha}{\mathfrak{r}^2} \hat{x}$$

$$dl = dr' = zd\left(\tan\alpha\right)$$

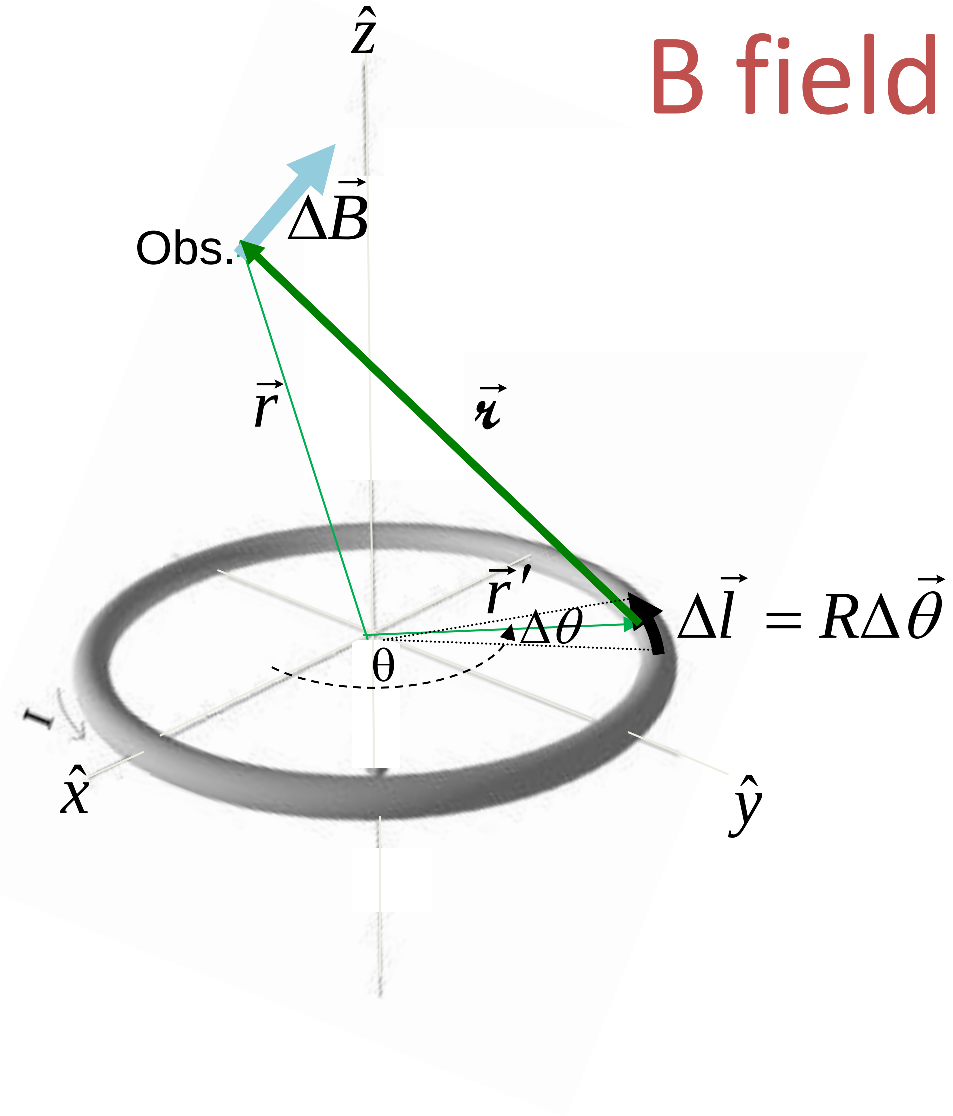
$$dl = z \frac{1}{\cos^2\alpha} d\alpha$$

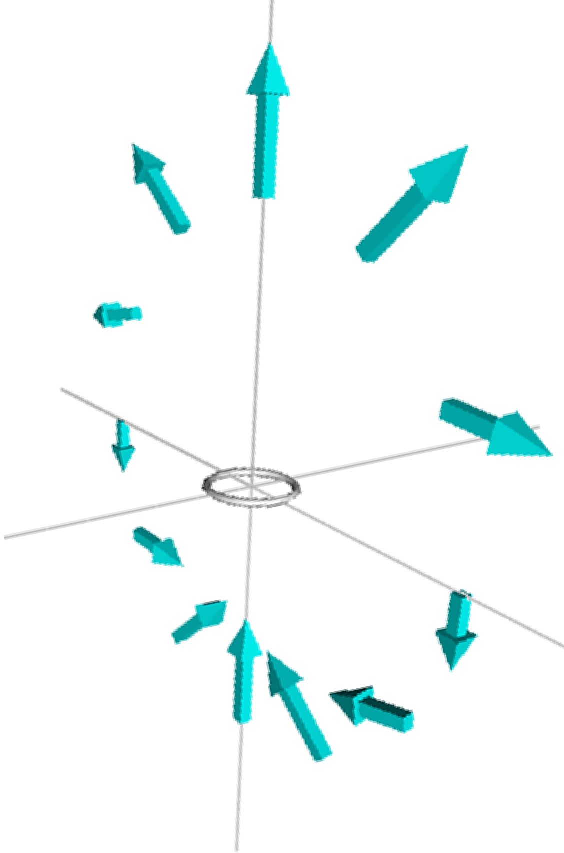
$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{z \frac{1}{\cos^2\alpha} d\alpha \cos\alpha}{\mathfrak{r}^2} \hat{x} = \frac{\mu_0}{4\pi} I \int \frac{r d\alpha}{\mathfrak{r}^2 \cos\alpha} \hat{x}$$

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} I \int \frac{(\cos\alpha)^2 z d\alpha}{z^2 \cos\alpha} \hat{x} = \frac{\mu_0}{4\pi} I \int \frac{(\cos\alpha) d\alpha}{z} \hat{x} = \frac{\mu_0}{4\pi} \frac{I}{z} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\left(\sin\alpha\right) \hat{x}$$

$$= \frac{\mu_0}{4\pi} \frac{2I}{z} \hat{x}$$

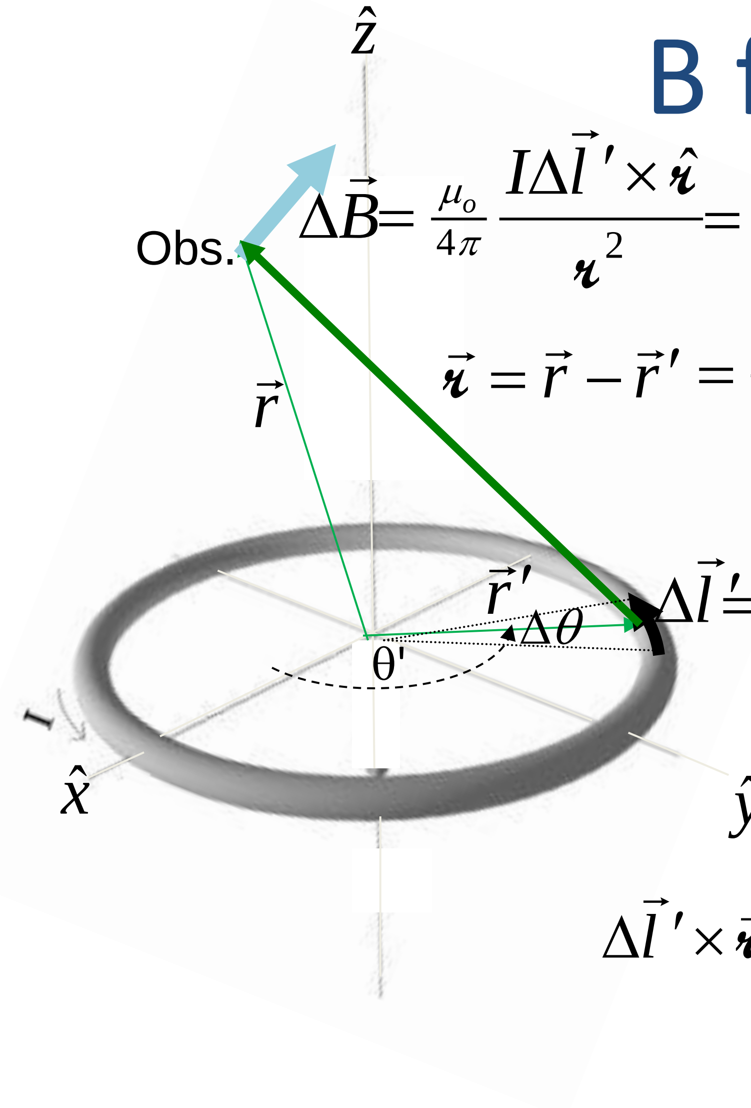
B field of loop





1. Visualize Representative segment's field
2. Expression for segment's field
3. Sum up field due to all segments
4. Evaluate results

B field of loop



$$\Delta\vec{B} = \frac{\mu_o}{4\pi}\frac{I\Delta\vec{l}\,'\times\hat{\mathscr{r}}}{\mathscr{r}^2} = \frac{\mu_o}{4\pi}\frac{I\Delta\vec{l}\,'\times\vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = \left\langle x - r'\cos\theta', y' - r'\sin\theta', z\right\rangle$$

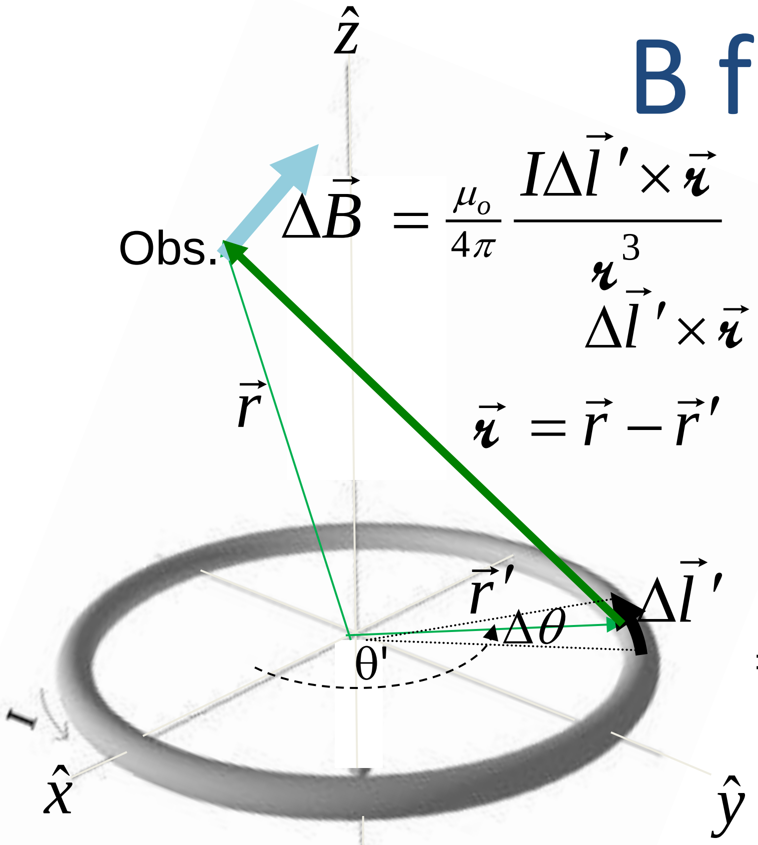
$$\Delta\vec{l}\,' = r'\Delta\vec{\theta}' = r'\Delta\theta'\left(\Delta\hat{\theta}\right) = r'\Delta\theta'\left\langle -\sin\theta', \cos\theta', 0\right\rangle$$

$$\Delta\vec{l}\,'\times\vec{\mathscr{r}} = \det\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -r'\Delta\theta'\sin\theta' & r'\Delta\theta'\cos\theta' & 0 \\ x - r'\cos\theta & y - r'\sin\theta' & z \end{vmatrix}$$

$$= r'\Delta\theta'\left\langle z\cos\theta', z\sin\theta', r' - y\sin\theta' - x\cos\theta'\right\rangle$$

1. Visualize Representative segment's field
2. Expression for segment's field
3. Sum up field due to all segments
4. Evaluate results

B field of loop



$$\Delta\vec{B} = \frac{\mu_o}{4\pi}\frac{I\Delta\vec{l}\,'\times\vec{\mathscr{r}}}{\mathscr{r}^3}$$

$$\Delta\vec{l}\,'\times\vec{\mathscr{r}} = r'\Delta\theta'\langle z\cos\theta', z\sin\theta', r' - y\sin\theta' - x\cos\theta'\rangle$$

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}\,' = \langle x - r'\cos\theta', y' - r'\sin\theta', z\rangle$$

$$= \left((x - r'\cos\theta)^2 + (y - r'\sin\theta)^2 + z^2\right)^{\frac{1}{2}}$$

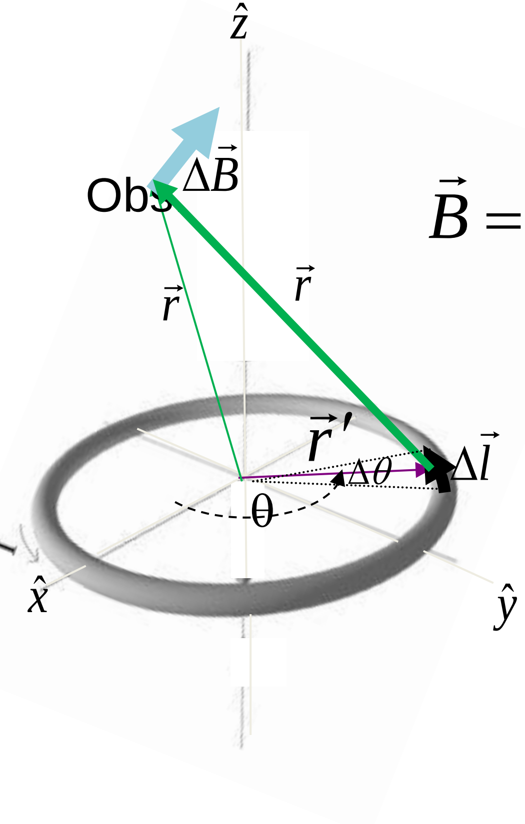
$$= \left(x^2 + y^2 + z^2 + r'^2 - 2r'(x\cos\theta' + y\sin\theta')\right)^{\frac{1}{2}}$$

$$= \left(r^2 + r'^2 - 2r'(x\cos\theta' + y\sin\theta')\right)^{\frac{1}{2}}$$

$$\Delta\vec{B} = \frac{\mu_o}{4\pi}\frac{Ir'\Delta\theta'\langle z\cos\theta', z\sin\theta', r' - y\sin\theta' - x\cos\theta'\rangle}{\left(r^2 + r'^2 - 2r'(x\cos\theta' + y\sin\theta')\right)^{\frac{3}{2}}}$$

1. Visualize Representative segment's field
2. Expression for segment's field
3. Sum up field due to all segments
4. Evaluate results

B field of loop



$$\vec{B} = \sum_{loop} \Delta\vec{B} = \sum_{\theta=0}^{\theta=2\pi} \Delta\vec{B}$$

where

$$\Delta\vec{B} = \frac{\mu_o}{4\pi} \frac{Ir'\Delta\theta'\langle z\cos\theta', z\sin\theta', r' - y\sin\theta' - x\cos\theta'\rangle}{\left(r^2 + r'^2 - 2r'(x\cos\theta' + y\sin\theta')\right)^{\frac{3}{2}}}$$

By components

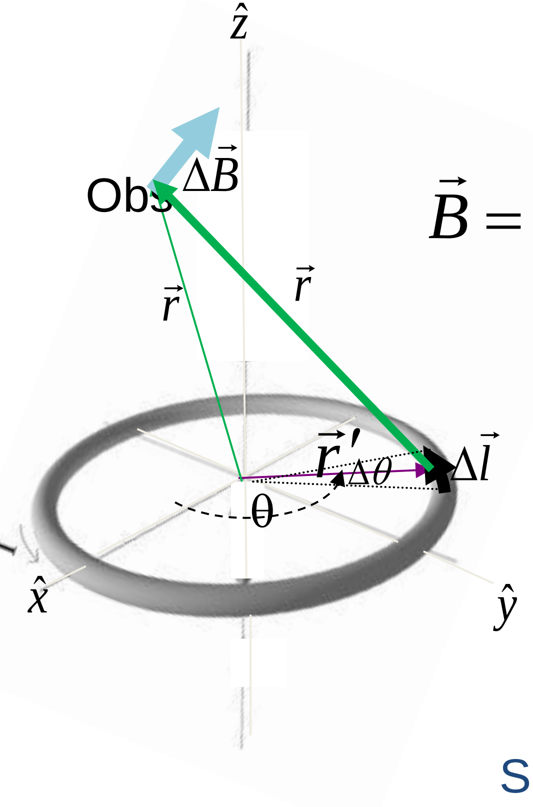
$$B_x = \sum_{\theta=0}^{\theta=2\pi} \Delta\vec{B}_x$$

$$B_x = \sum_{\theta=0}^{\theta=2\pi} \frac{\mu_o}{4\pi} \frac{Ir'z\cos\theta\Delta\theta}{\left(r^2 + r'^2 - 2r'(x\cos\theta' + y\sin\theta')\right)^{\frac{3}{2}}}$$

$$B_x = \frac{\mu_o}{4\pi} Ir'z \int_{\theta'=0}^{\theta'=2\pi} \frac{\cos\theta' d\theta'}{\left(r^2 + r'^2 - 2r'(x\cos\theta' + y\sin\theta')\right)^{\frac{3}{2}}}$$

1. Visualize Representative segment's field
2. Expression for segment's field
3. Sum up field due to all segments
4. Evaluate results

B field of loop



$$\vec{B} = \sum_{loop} \Delta\vec{B} = \sum_{\theta=0}^{\theta=2\pi} \Delta\vec{B}$$

where

$$\Delta\vec{B} = \frac{\mu_o}{4\pi} \frac{Ir'\Delta\theta'\langle z\cos\theta', z\sin\theta', r' - y\sin\theta' - x\cos\theta'\rangle}{\left(r^2 + r'^2 - 2r'\left(x\cos\theta' + y\sin\theta'\right)\right)^{\frac{3}{2}}}$$

By components

$$B_x = \frac{\mu_o}{4\pi} Ir'z \int_{\theta'=0}^{\theta'=2\pi} \frac{\cos\theta' d\theta'}{\left(r^2 + r'^2 - 2r'\left(x\cos\theta' + y\sin\theta'\right)\right)^{\frac{3}{2}}}$$

Similarly

$$B_y = \frac{\mu_o}{4\pi} Ir'z \int_{\theta'=0}^{\theta'=2\pi} \frac{\sin\theta' d\theta'}{\left(r^2 + r'^2 - 2r'\left(x\cos\theta' + y\sin\theta'\right)\right)^{\frac{3}{2}}}$$

$$B_z = \frac{\mu_o}{4\pi} Ir' \int_{\theta'=0}^{\theta'=2\pi} \frac{\left(r' - y\sin\theta' - x\cos\theta'\right)d\theta'}{\left(r^2 + r'^2 - 2r'\left(x\cos\theta' + y\sin\theta'\right)\right)^{\frac{3}{2}}}$$

1. Visualize Representative segment's field
2. Expression for segment's field
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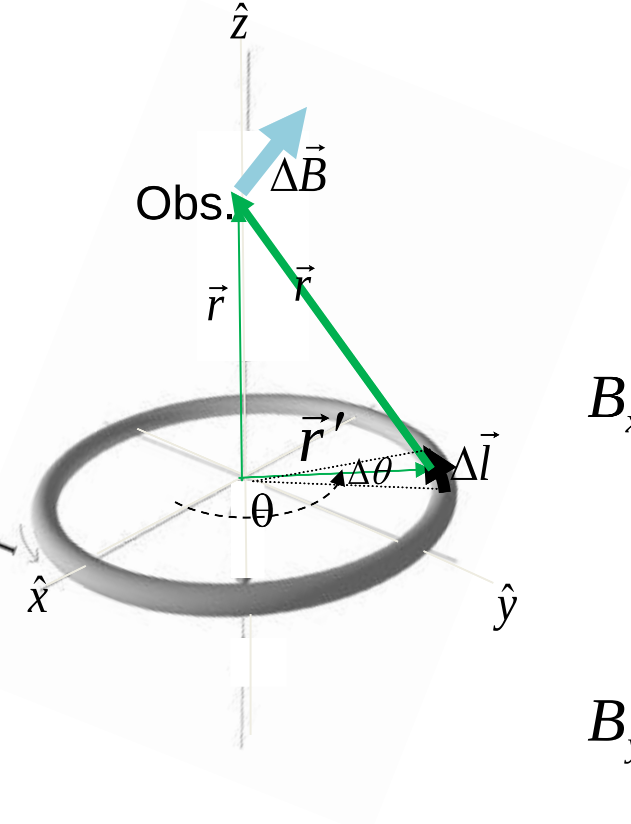
4. Evaluate results

B field of loop



Simple case for point on z-axis: $\vec{r}_o = \langle 0,0,z_o \rangle$

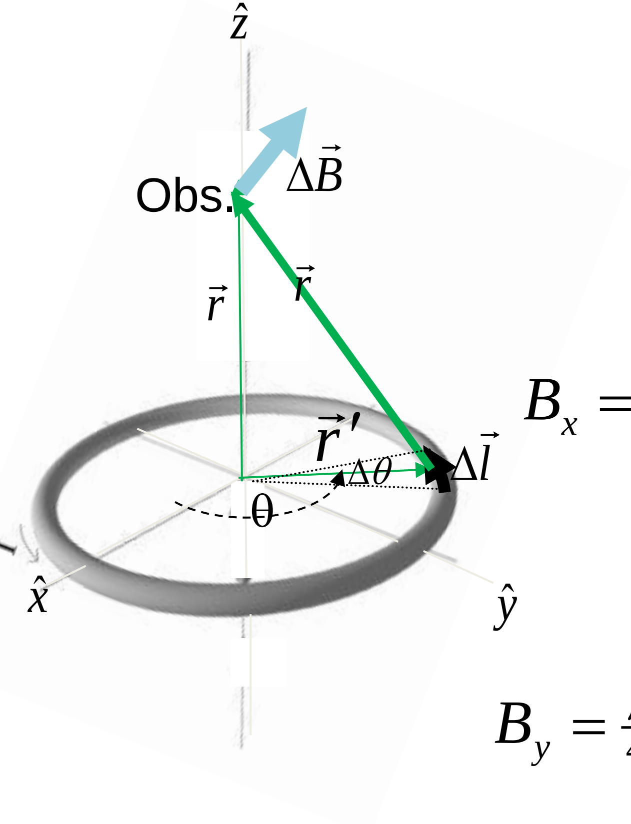
$$B_x = \frac{\mu_o}{4\pi} I r' z \int_{\theta'=0}^{\theta'=2\pi} \frac{\cos\theta' d\theta'}{\left(r^2 + r'^2 \cancel{- 2r'\left(x\cos\theta' + y\sin\theta'\right)}\right)^{\frac{3}{2}}}$$

$$B_y = \frac{\mu_o}{4\pi} I r' z \int_{\theta'=0}^{\theta'=2\pi} \frac{\sin\theta' d\theta'}{\left(r^2 + r'^2 \cancel{- 2r'\left(x\cos\theta' + y\sin\theta'\right)}\right)^{\frac{3}{2}}}$$

$$B_z = \frac{\mu_o}{4\pi} I r' \int_{\theta'=0}^{\theta'=2\pi} \frac{\left(r' \cancel{- y\sin\theta' - x\cos\theta'}\right) d\theta'}{\left(r^2 + r'^2 \cancel{- 2r'\left(x\cos\theta' + y\sin\theta'\right)}\right)^{\frac{3}{2}}}$$

1. Visualize Representative segment's field
2. Expression for segment's field
3. Sum up field due to all segments
4. Evaluate results

B field of loop



Simple case for point on z-axis: $\vec{r} = \langle 0,0,z \rangle$

$$B_x = \frac{\mu_o}{4\pi} Ir'z \int_{\theta'=0}^{\theta'=2\pi} \frac{\cos\theta' d\theta'}{\left(r^2 + r'^2\right)^{\frac{3}{2}}} = \frac{\mu_o}{4\pi} \frac{Ir'z}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} \int_{\theta'=0}^{\theta'=2\pi} \cos\theta' d\theta' = 0$$

$$B_y = \frac{\mu_o}{4\pi} Ir'z \int_{\theta'=0}^{\theta'=2\pi} \frac{\sin\theta' d\theta'}{\left(r^2 + r'^2\right)^{\frac{3}{2}}} = \frac{\mu_o}{4\pi} \frac{Ir'z}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} \int_{\theta=0}^{\theta=2\pi} \sin\theta d\theta = 0$$

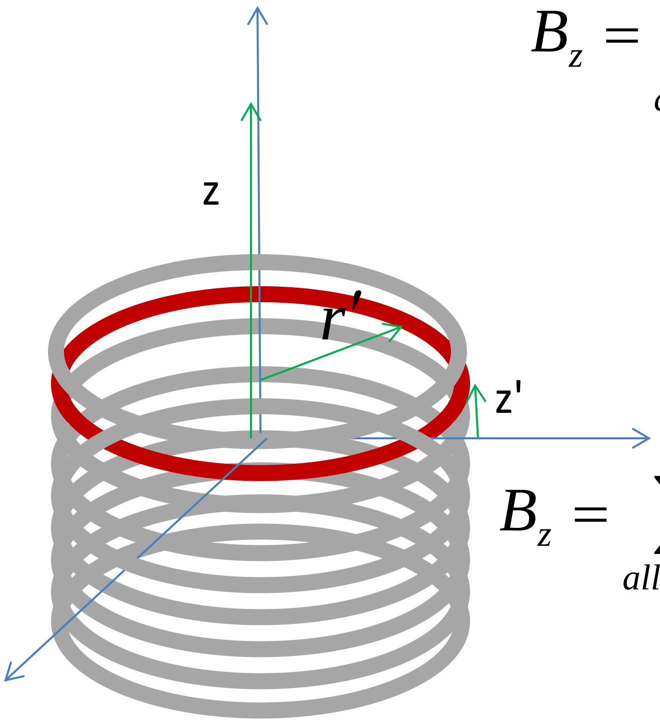
$$B_z = \frac{\mu_o}{4\pi} Ir'^2 \int_{\theta'=0}^{\theta'=2\pi} \frac{d\theta'}{\left(r^2 + r'^2\right)^{\frac{3}{2}}} = \frac{\mu_o}{4\pi} \frac{Ir'^2 z}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} \int_{\theta=0}^{\theta=2\pi} d\theta$$

$$= \frac{\mu_o}{4\pi} \frac{Ir'^2}{\left(z^2 + r'^2\right)^{\frac{3}{2}}} 2\pi$$

1. Visualize Representative segment's field
2. Expression for segment's field
3. Sum up field due to all segments
4. Evaluate results

B field of Solenoid

Simple case for point on z-axis: $\vec{r} = \langle 0,0,z \rangle$



$$B_z = \sum_{all.rings} dB_z$$

$$dB_z = \frac{\mu_o}{4\pi} \frac{I r'^2}{\left((z-z')^2 + r'^2\right)^{\frac{3}{2}}} 2\pi$$

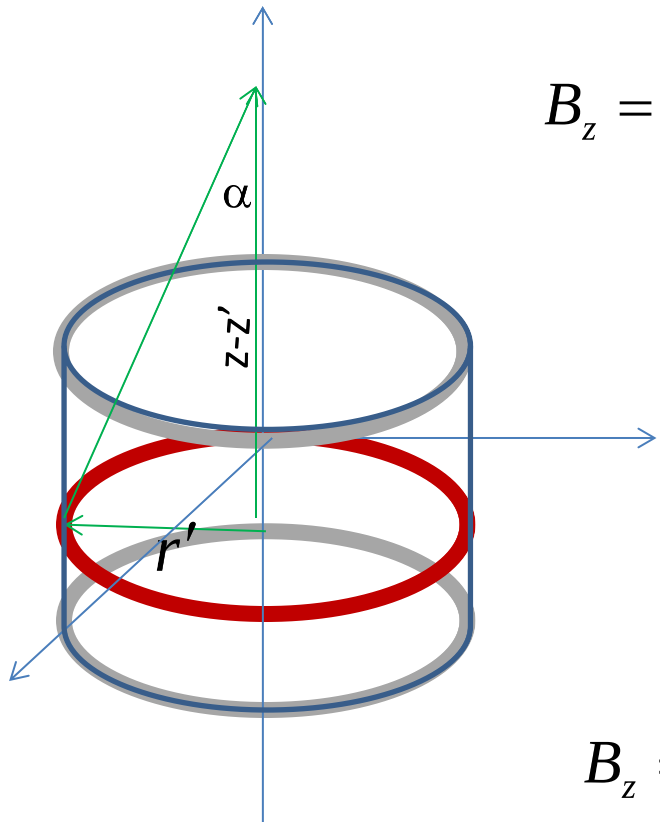
N rings over length L, or $n = \frac{N}{L} = \frac{\Delta N}{\Delta z'}$

$$B_z = \sum_{all.rings} \frac{\mu_o}{4\pi} \frac{I r'^2 n \Delta z'}{\left((z-z')^2 + r'^2\right)^{\frac{3}{2}}} 2\pi$$

$$B_z = \frac{\mu_o}{2} I r'^2 n \int_{z'=z_{bottom}}^{z_{top}} \frac{dz'}{\left((z-z')^2 + r'^2\right)^{\frac{3}{2}}}$$

B field of Solenoid

Simple case for point on z-axis: $\vec{r} = \langle 0,0,z \rangle$



$$B_z = \frac{\mu_o}{2} I r'^2 n \int_{z'=z_{bottom}}^{z_{top}} \frac{dz'}{\left((z-z')^2 + r'^2\right)^{\frac{3}{2}}}$$

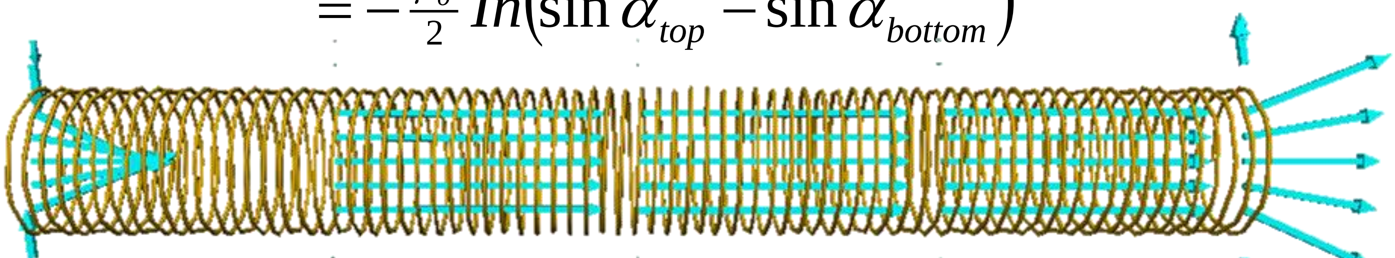
$$\left((z-z')^2 + r'^2\right)^{\frac{1}{2}} = \frac{r'}{\sin\alpha}$$

$$z - z' = r' \cot\alpha$$

$$dz' = -r' d(\cot\alpha) = -\frac{r'}{\sin^2\alpha} d\alpha$$

$$B_z = -\frac{\mu_o}{2} I r'^2 n \int_{\alpha_{bottom}}^{\alpha_{top}} \frac{\sin^3\alpha}{r'^3} \frac{r' d\alpha}{\sin^2\alpha} = -\frac{\mu_o}{2} In \int_{\alpha_{bottom}}^{\alpha_{top}} \sin\alpha d\alpha$$

$$= -\frac{\mu_o}{2} In\left(\sin\alpha_{top} - \sin\alpha_{bottom}\right)$$



Wed.	(C 17) 5.1.3 Lorentz Force Law: currents	
Thurs.		HW6
Fri.	(C 17) 5.2 Biot-Savart Law	
Mon.	(C 17) 5.2 Biot-Savart Law T5 Quiver Plots	
Tues.		HW7
Wed.	(C 21.6-7,.9) 1.3.4-1.3.5, 1.5.2-1.5.3, 5.3.1-.3.2 Div & Curl B	
Fri.	(C 21.6-7,.9) 5.3.3-.3.4 Applications of Ampere's Law	