Fri.	3.2 Images T4 Relaxation Method	
Mon. Wed. Thurs	3.4.1-.4.2 Multipole Expansion 3.4.3-.4.4 Multipole Expansion	 HW4

Poisson's & Laplace's Equations

$$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon_0$$

$$\vec{E} = -\vec{\nabla} V$$

Poisson's $$\nabla^2 V = -\rho / \varepsilon_0$$

Laplace's $$\nabla^2 V = 0$$ In "free space" where no charges are

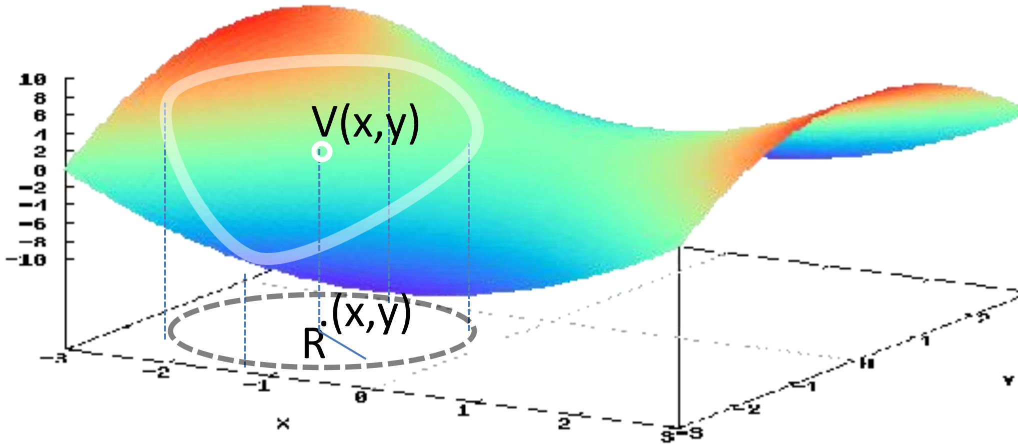
In Cartesian

$$\nabla^2 V = \vec{\nabla} \cdot \left(\vec{\nabla} V \right) = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

Properties of Laplace's

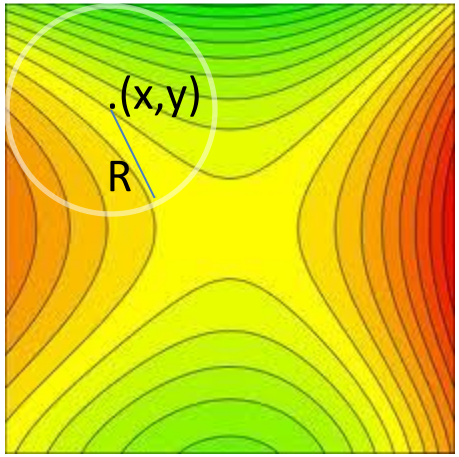
$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

- No local min or max; either flat (possible tipped plane) or saddle points



- V at mid point of range is average of V around edges (proof forthcoming)

- In 2D $V(x,y) = \frac{\oint V dl_{circle}}{2\pi R}$



- In 3D $V(x,y,z) = \frac{\oint V da_{sphere}}{4\pi R^2}$

Uniqueness Theorems (by hook or crook)

Voltages: (as with any differential equation) Regardless of *how* you've found it, if you've found one solution to Laplace / Poisson's equation that satisfies the boundary conditions, you've found the *only* solution.

Fields: Given a charge density in a cavity within a charged conductor, the field within the conductor is uniquely determined by the *inner* charge distribution and the conductor's charge amount.

Hooks and Crook: Interesting ways of finding V and E

Images: replace a problem with a simpler equivalent one (based on corollary of the first uniqueness theorem)

Relaxation: a computational method based on the potential at a point being the average of the values at the same distance (more about Next Time).

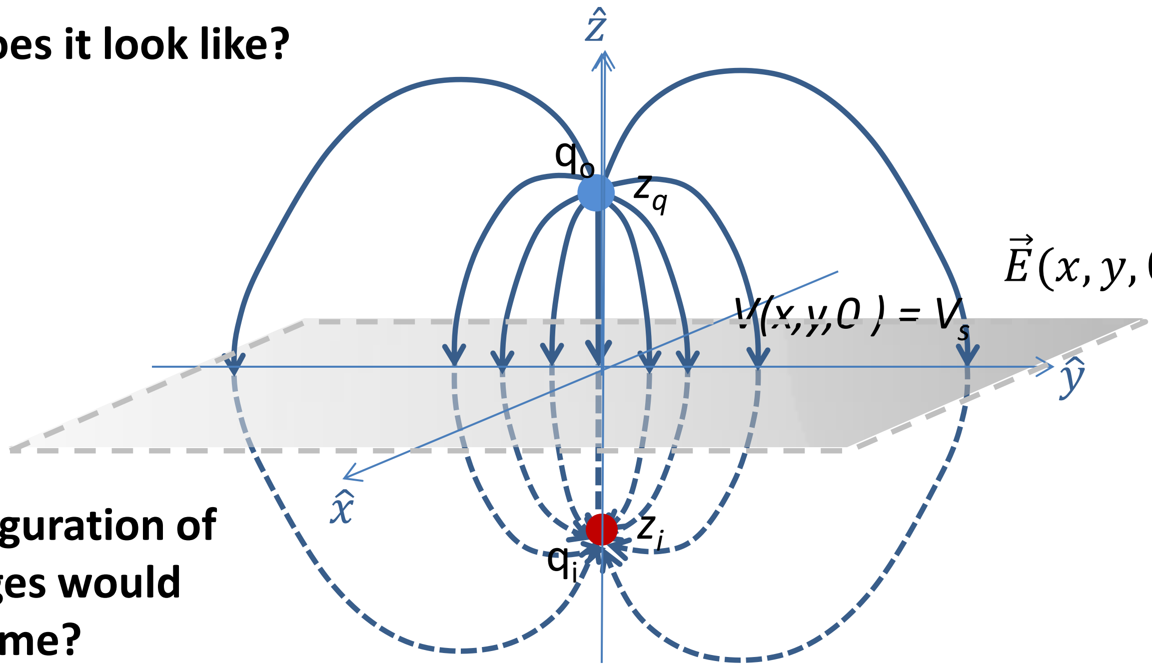
Multipole Expansion: a method for getting approximate answers for *V* far from a charge distribution (section 3.4)

Charge Images Reflected in Conductors

Images: replace a problem with a simpler equivalent one (based on corollary of the first uniqueness theorem – if your solution works on the boundary, it works everywhere)

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

What does it look like?



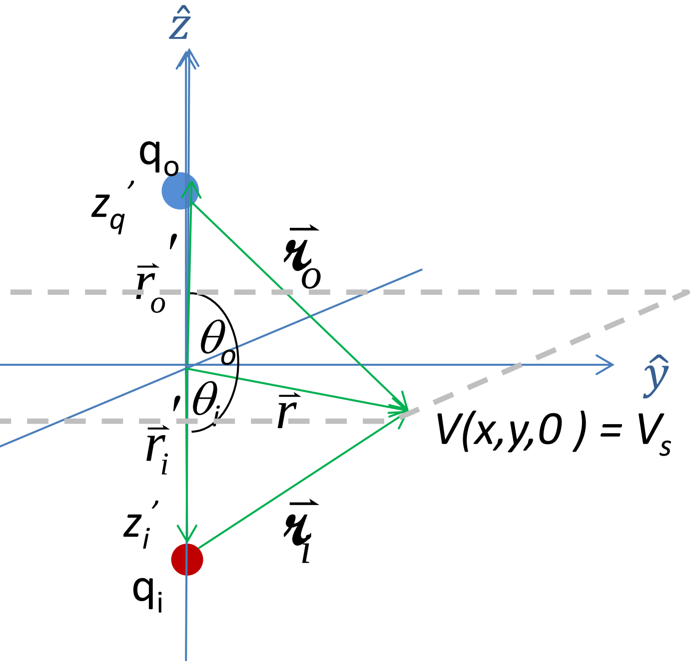
$\vec{E}(x, y, 0)$ points straight down.

What configuration of point charges would look the same?

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

In gory detail (for the experience), determine image's charge and location



For observation location in plane of the conductor

$\theta_o = \theta_i = 90°$

$\cos\theta_o = \cos\theta_i = 0$

$$\frac{q_o}{\sqrt{r^2 + z_o'^2}} + \frac{q_i}{\sqrt{r^2 + z_i'^2}} = 4\pi\varepsilon_o V_s$$

Must be true for *all* location in plane, so can choose easy-to-evaluate locations to determine values of q_i and z_i.

$r \to \infty$

$$\frac{q_o}{\sqrt{\infty^2 + z_o'^2}} + \frac{q_i}{\sqrt{\infty^2 + z_i'^2}} = 4\pi\varepsilon_o V_s$$

$$0 + 0 = 4\pi\varepsilon_o V_s$$

Apparently works at all only if $V_s = 0$.

$r \to 0$

$$\frac{q_o}{\sqrt{z_o'^2}} + \frac{q_i}{\sqrt{z_i'^2}} = 0$$

$$\frac{q_i}{|z_i'|} = -\frac{q_o}{|z_o'|}$$

Charge Images Reflected in Conductors

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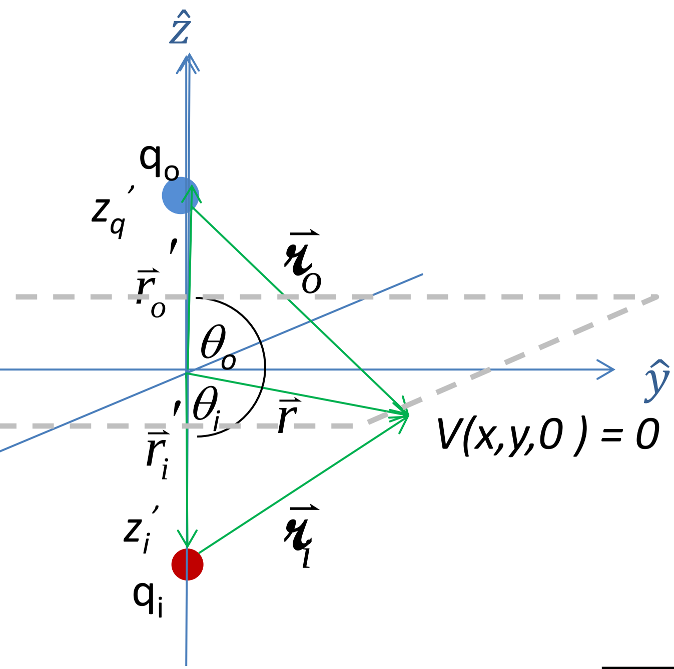
In gory detail (for the experience), determine image's charge and location



Return to $$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{q_i}{\sqrt{r^2+z_i'^2}}=4\pi\varepsilon_o V_s$$

Plug in what we've learned: $V_s = 0 \qquad q_i = -q_o\left|\frac{z_i'}{z_o'}\right|$

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{-q_0\left|\frac{z_i'}{z_o'}\right|}{\sqrt{r^2+z_i'^2}}=0$$

$$\frac{1}{\sqrt{r^2+z_o'^2}}=\left|\frac{z_i'}{z_o'}\right|\frac{1}{\sqrt{r^2+z_i'^2}}$$

$$\frac{1}{|z_o'|\sqrt{\left(\frac{r}{z_o'}\right)^2+1}}=\left|\frac{z_i'}{z_o'}\right|\frac{1}{|z_i'|\sqrt{\left(\frac{r}{z_i'}\right)^2+1}}$$

$$\frac{1}{\sqrt{\left(\frac{r}{z_o'}\right)^2+1}}=\frac{1}{\sqrt{\left(\frac{r}{z_i'}\right)^2+1}}$$

$$|z_o'|=|z_i'|$$

$$z_i'=-z_o'$$

$$q_i=-q_o\left|\frac{z_i'}{z_o'}\right|$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

Digression: Force between q_o and Surface

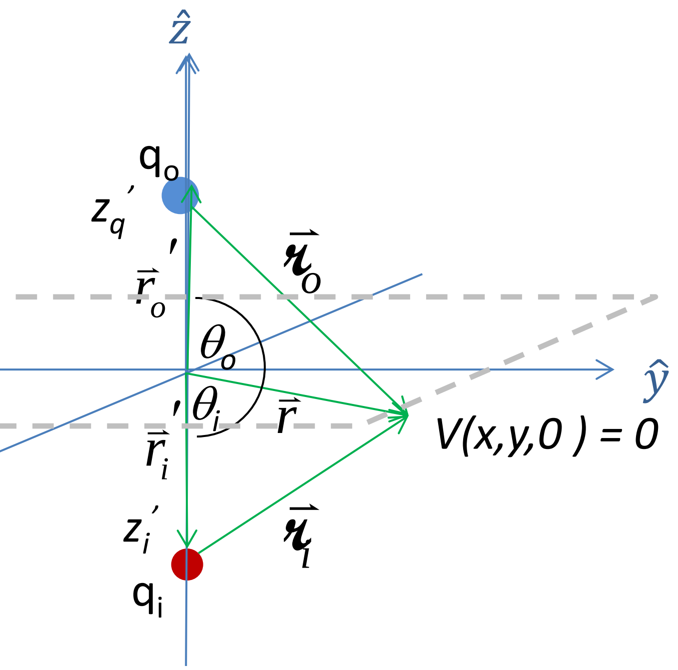


Image charge distance and magnitude

$$z_i' = -z_o' \qquad q_i = -q_o$$

Field everywhere above the plane is as if there were these two charges

$$\vec{F}_{i\to o} = \frac{1}{4\pi\varepsilon_o}\frac{q_i q_o}{(z_o - z_i)^2}\hat{z}$$

$$\vec{F}_{i\to o} = -\frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z_o)^2}\hat{z}$$

So,

$$\vec{F}_{o\to surface} = \frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z_o)^2}\hat{z}$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

Digression: Work of moving q_o into place (and arranging surface charge)

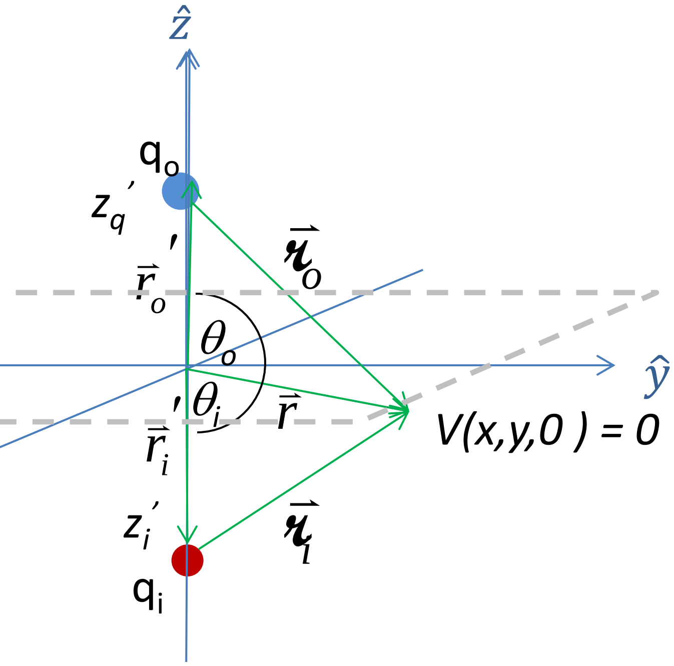


Image charge distance and magnitude

$$z_i' = -z_o' \qquad q_i = -q_o$$

Field everywhere above the plane is as if there were these two charges

$$\vec{F}_{surface\to o} = -\frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z_o)^2}\hat{z}$$

$$W = \int_\infty^{z_o} \vec{F}_{you\to o}\cdot d\vec{l} = -\int_\infty^{z_o} \vec{F}_{surface\to o}\cdot d\vec{l} = \int_{z_o}^{\infty} \vec{F}_{surface\to o}\cdot d\vec{l}$$

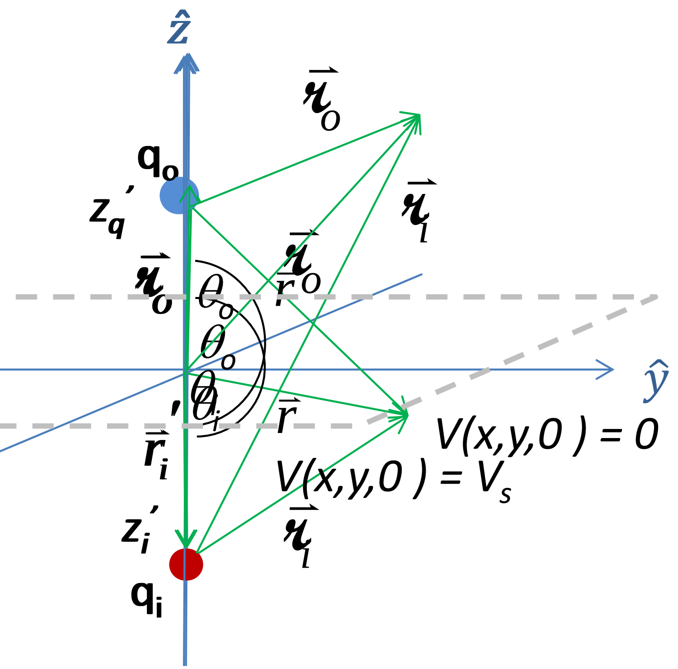
$$W = \int_{z_o}^{\infty} -\frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z)^2}\hat{z}\cdot d\vec{l} = \int_{z_o}^{\infty} -\frac{1}{4\pi\varepsilon_o}\frac{q_o^2}{(2z)^2}dz$$

$$W = -\frac{q_o^2}{16\pi\varepsilon_o}\int_{z_o}^{\infty}\frac{1}{z^2}dz = \frac{q_o^2}{16\pi\varepsilon_o}\frac{1}{z}\bigg|_{z_o}^{\infty} = -\frac{q_o^2}{16\pi\varepsilon_o}\frac{1}{z_o}$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

Return to determine image's charge and location



Return to

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{q_i}{\sqrt{r^2+z_i'^2}}=4\pi\varepsilon_o V_s$$

Plug in what we've learned: $z_i' = -z_o'$ $\quad q_i = -q_o$

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{-q_o}{\sqrt{r^2+z_o'^2}}=4\pi\varepsilon_o V_s$$

Works on the surface:

$$\frac{q_o}{\sqrt{r^2+z_o'^2}}+\frac{-q_o}{\sqrt{r^2+z_o'^2}}=0$$

Must work *everywhere* above the plane:

$$\frac{q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos\theta_o}}+\frac{q_i}{\sqrt{r^2+z_i'^2-2rz_i'\cos\theta_i}}=4\pi\varepsilon_o V(\vec{r})$$

becomes:

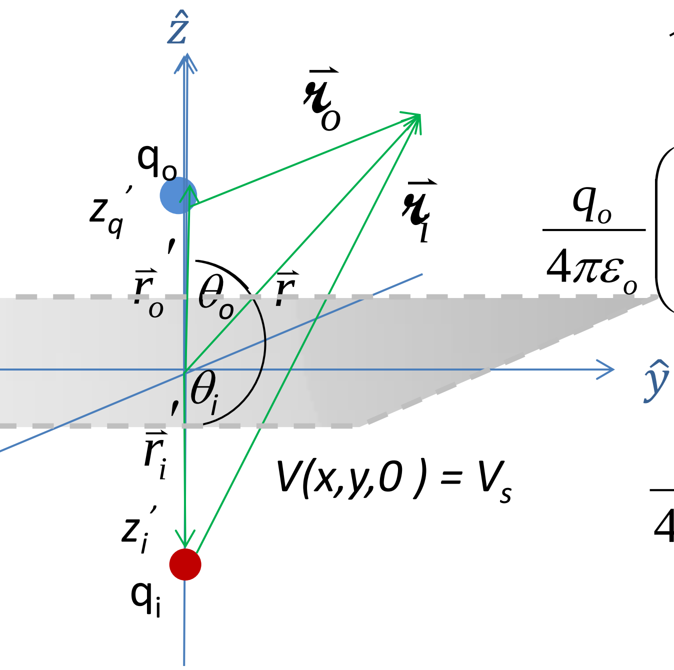
$$\frac{q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos\theta_o}}+\frac{-q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos(\pi-\theta_o)}}=4\pi\varepsilon_o V(\vec{r})$$

$$\frac{q_o}{\sqrt{r^2+z_o'^2-2rz_o'\cos\theta_o}}+\frac{-q_o}{\sqrt{r^2+z_o'^2+2rz_o'\cos(\theta_o)}}=4\pi\varepsilon_o V(\vec{r})$$

Charge Images Reflected in Conductors

Example: a charge q_o suspended distance z_o above a flat conducting surface that's held at voltage V_s. Find expression for V anywhere above the conducting surface.

Return to determine image's charge and location



Must work *everywhere* above the plane:

$$\frac{q_o}{\sqrt{r^2 + z_o'^2 - 2rz_o'\cos\theta_o}} + \frac{-q_o}{\sqrt{r^2 + z_o'^2 + 2rz_o'\cos(\theta_o)}} = 4\pi\varepsilon_o V(\vec{r})$$

If $V(x,y,0) = V_s \neq 0$ **simply add the constant offset.**

$$\frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{r^2 + z_o'^2 - 2rz_o'\cos\theta_o}} - \frac{1}{\sqrt{r^2 + z_o'^2 + 2rz_o'\cos(\theta_o)}}\right) + V_s = V(\vec{r})$$

or

$$\frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{x^2 + y^2 + (z - z_o')^2}} - \frac{1}{\sqrt{x^2 + y^2 + (z + z_o')^2}}\right) + V_s = V(\vec{r})$$

Even for the *real* system

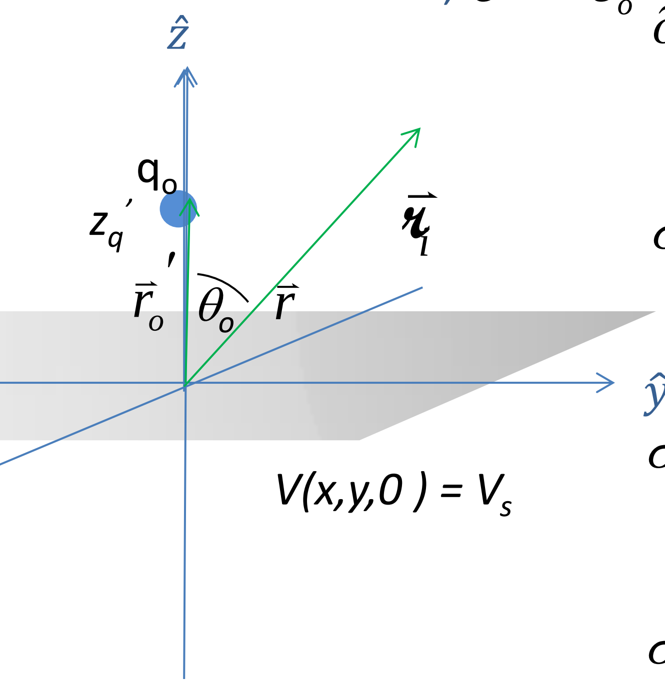
Surface Charge Density

$$\frac{q_o}{4\pi\varepsilon_o}\left(\frac{1}{\sqrt{x^2+y^2+(z-z_o')^2}}-\frac{1}{\sqrt{x^2+y^2+(z+z_o')^2}}\right)+V_s=V(\vec{r})$$

From a Gaussian pill-box: $E_n=\frac{\sigma}{\varepsilon_o}$ *n* denotes component perpendicular to surface

In terms of V: $-\frac{\partial V}{\partial n}=E_n=\frac{\sigma}{\varepsilon_o}$ So, $\sigma=-\varepsilon_o\left.\frac{\partial V}{\partial n}\right|_{surface}$

In this case, $\sigma=-\varepsilon_o\left.\frac{\partial V}{\partial z}\right|_{z=0}=-\frac{q_o}{4\pi}\frac{\partial}{\partial z}\left.\left(\frac{1}{\sqrt{x^2+y^2+(z-z_o')^2}}-\frac{1}{\sqrt{x^2+y^2+(z+z_o')^2}}\right)\right|_{z=0}$



$$\sigma=-\frac{q_o}{4\pi}\left.\left(\frac{-1}{2}\frac{2(z-z_o')}{\left(x^2+y^2+(z-z_o')^2\right)^{\frac{3}{2}}}-\frac{-1}{2}\frac{2(z+z_o')}{\left(x^2+y^2+(z+z_o')^2\right)^{\frac{3}{2}}}\right)\right|_{z=0}$$

$$\sigma=-\frac{q_o}{4\pi}\left(\frac{(z_o')}{\left(x^2+y^2+(z_o')^2\right)^{\frac{3}{2}}}+\frac{(z_o')}{\left(x^2+y^2+(z_o')^2\right)^{\frac{3}{2}}}\right)$$

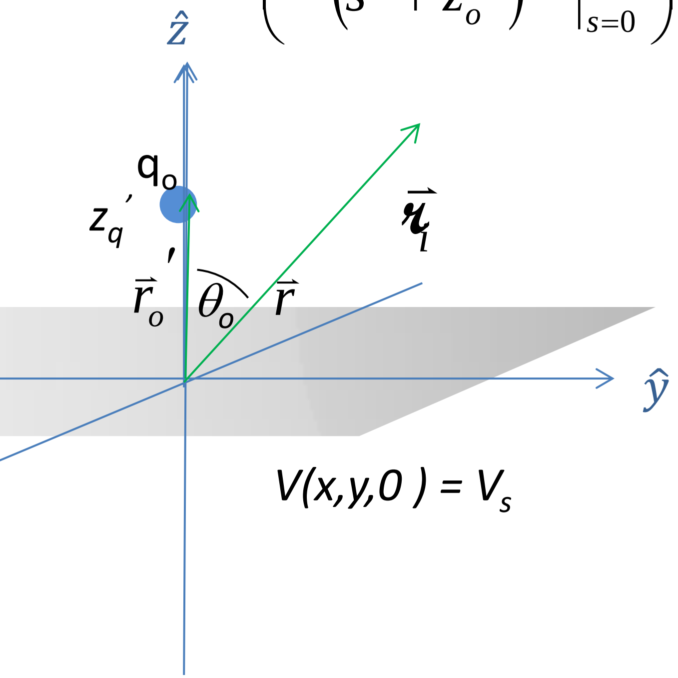
$$\sigma=-\frac{q_o}{2\pi}\frac{z_o'}{\left(x^2+y^2+z_o'^2\right)^{\frac{3}{2}}}$$

Surface Charge

$$q_{surf} = \int \sigma da$$

$$q_{surf} = \int -\tfrac{q_o}{4\pi}\frac{2z_o'}{\left(x^2+y^2+z_o'^2\right)^{3/2}}dxdy = -\tfrac{q_o}{4\pi}\int \frac{2z_o'}{\left(s^2+z_o'^2\right)^{3/2}} s\,d\phi\,ds = -q_o\int \frac{z_o'}{\left(s^2+z_o'^2\right)^{3/2}}\tfrac{1}{2}ds^2$$

$$q_{surf} = -q_o\left(-\frac{z_o'}{\left(s^2+z_o'^2\right)^{1/2}}\Bigg|_{s=0}^{s=\infty}\right) = q_o\left(0-\frac{z_o'}{\left(z_o'^2\right)^{1/2}}\right) = -q_o$$

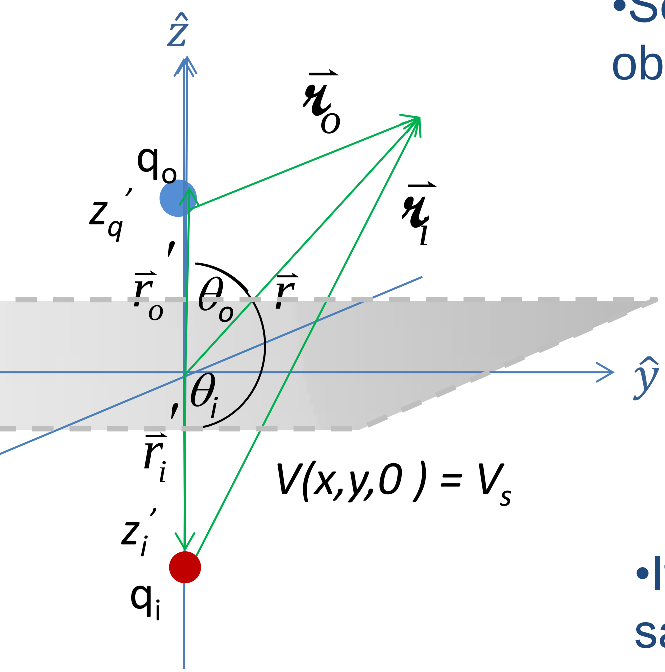


General Approach

- Draw picture

- Appeal to symmetry (and intuition about mirrors)
- Apply the condition $\sum \frac{q_o}{\mathscr{r}_o} + \sum \frac{q_i}{\mathscr{r}_i} = const$ on conductor

- See what you've got to do to remove dependence on the observation location on conductor.

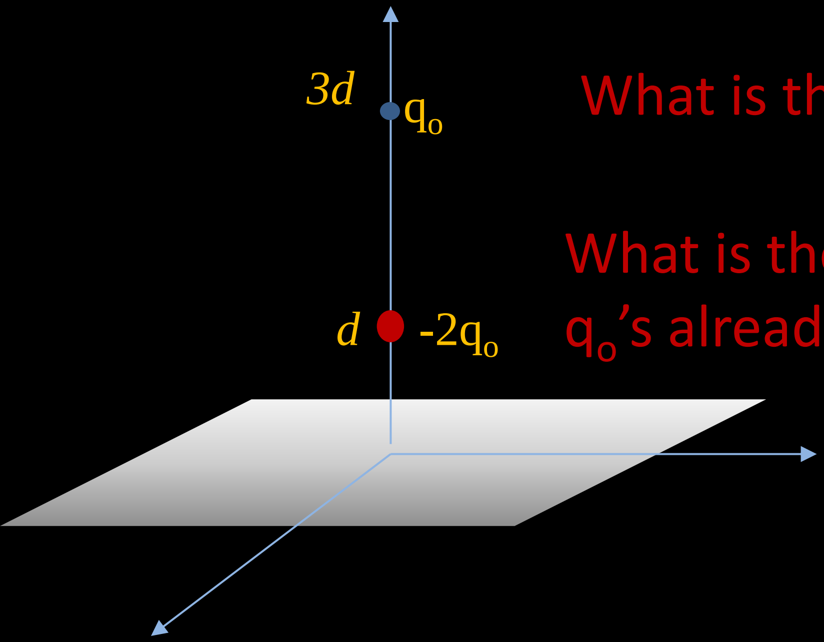


1)Mathematically, you've got 3 free parameters: the constant, the image charge's value, and the image charge's location.

2) Since the relation should be true for *all* observation locations on the conducting surface, choose easy ones to help you determine the three parameter's values.

- If you've got a solution that works for the boundary and satisfies Poisson's equation, then you've got *the* solution.

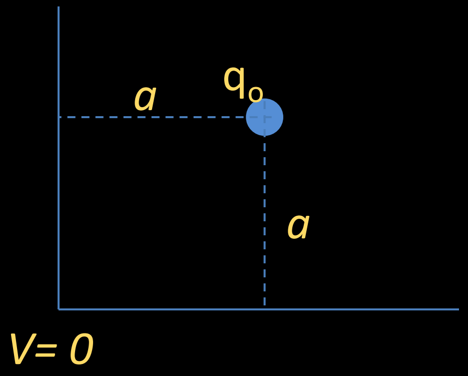
Exercise: where and what are the image charges?



What is the force on charge $-2q_o$?

What is the work of bringing it into position, once q_o's already there?

Exercise: charge and bent conductor, what's the equivalent charge distribution?

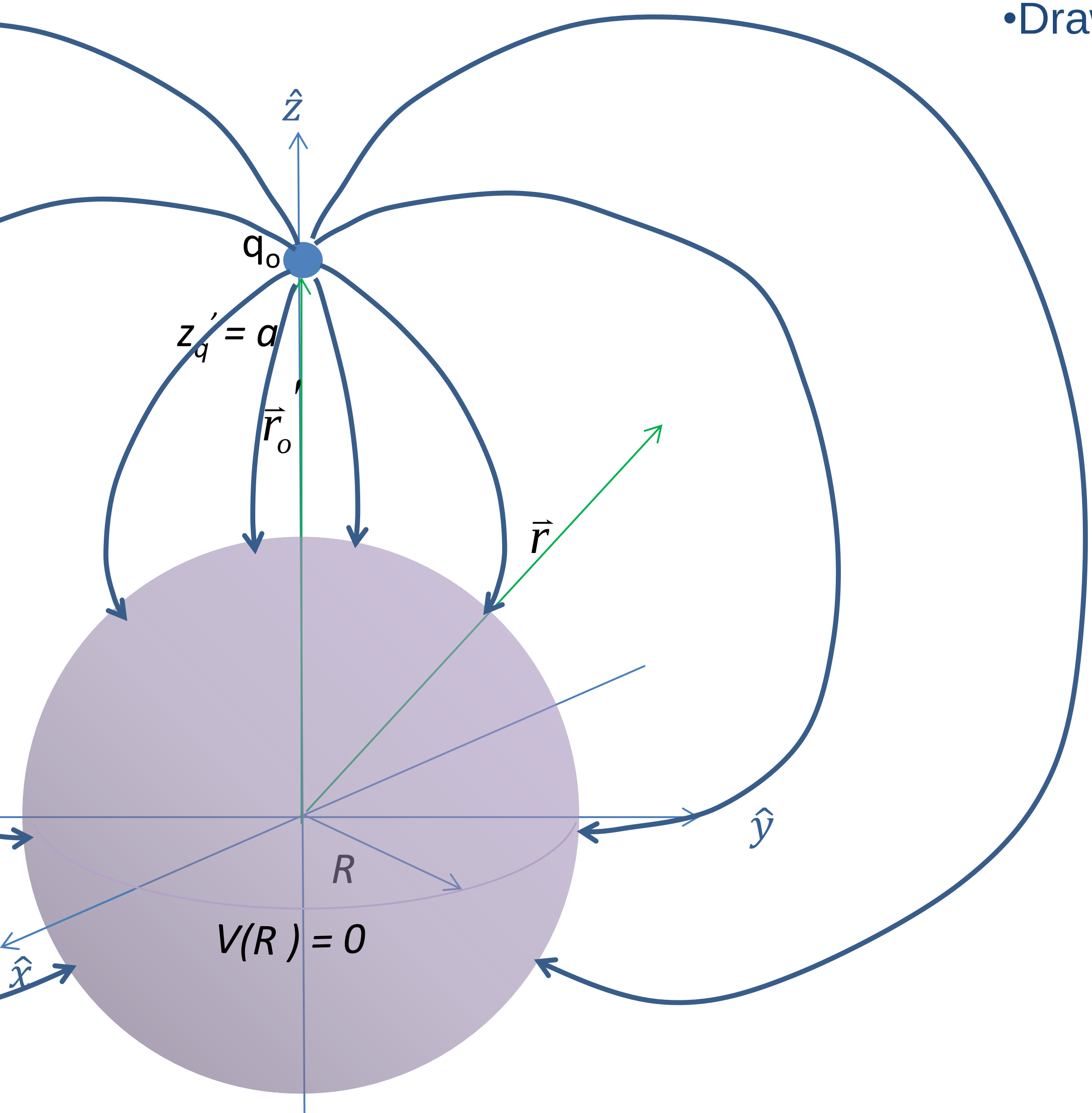


What's the potential expression in front of the conductor?

What's the charge density on each face of the conductor?

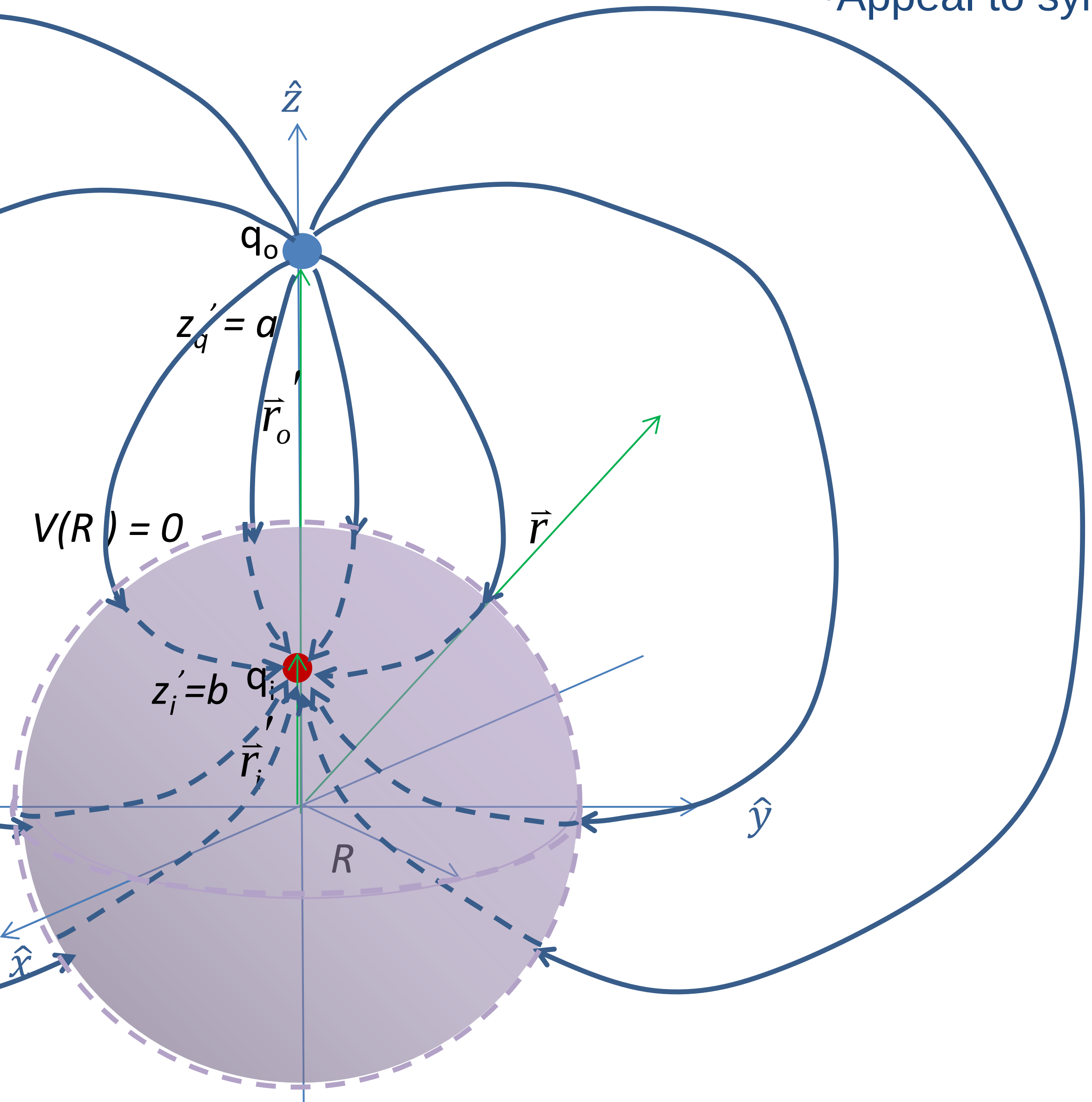
Abbreviated Example: Charge outside grounded sphere

- Draw picture



Abbreviated Example: Charge outside grounded sphere

- Appeal to symmetry (and intuition about mirrors)



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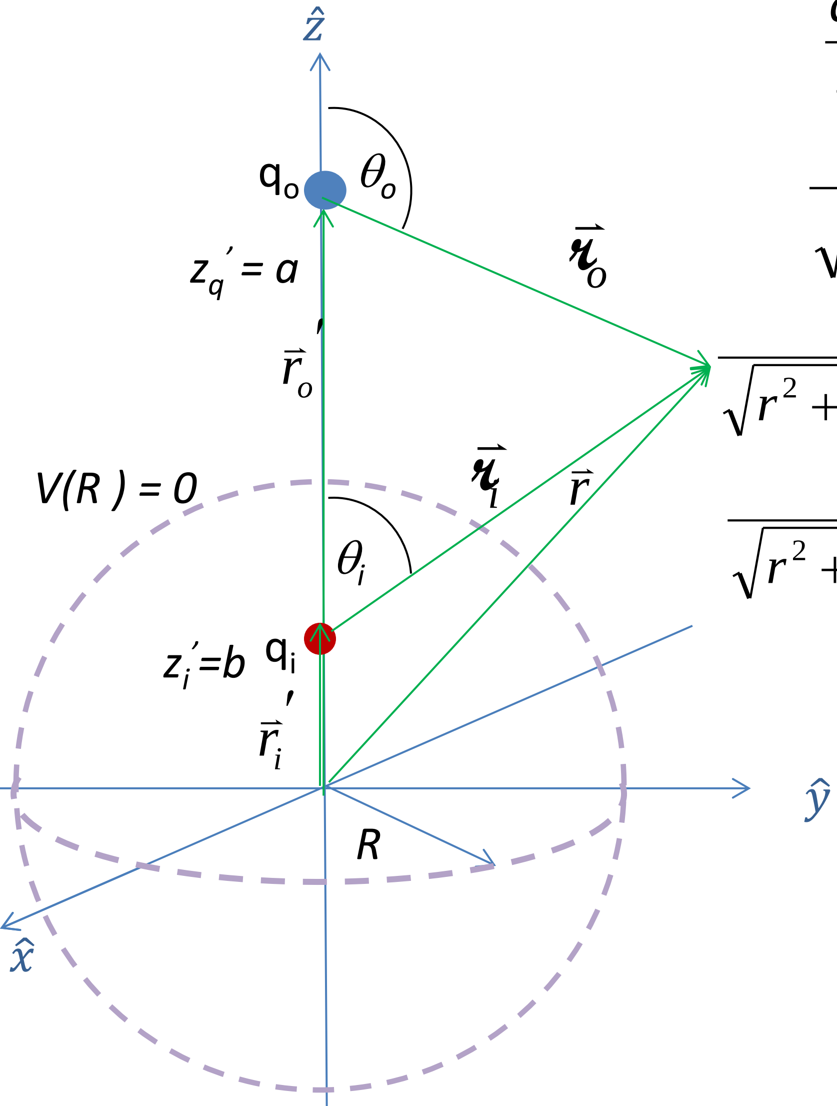


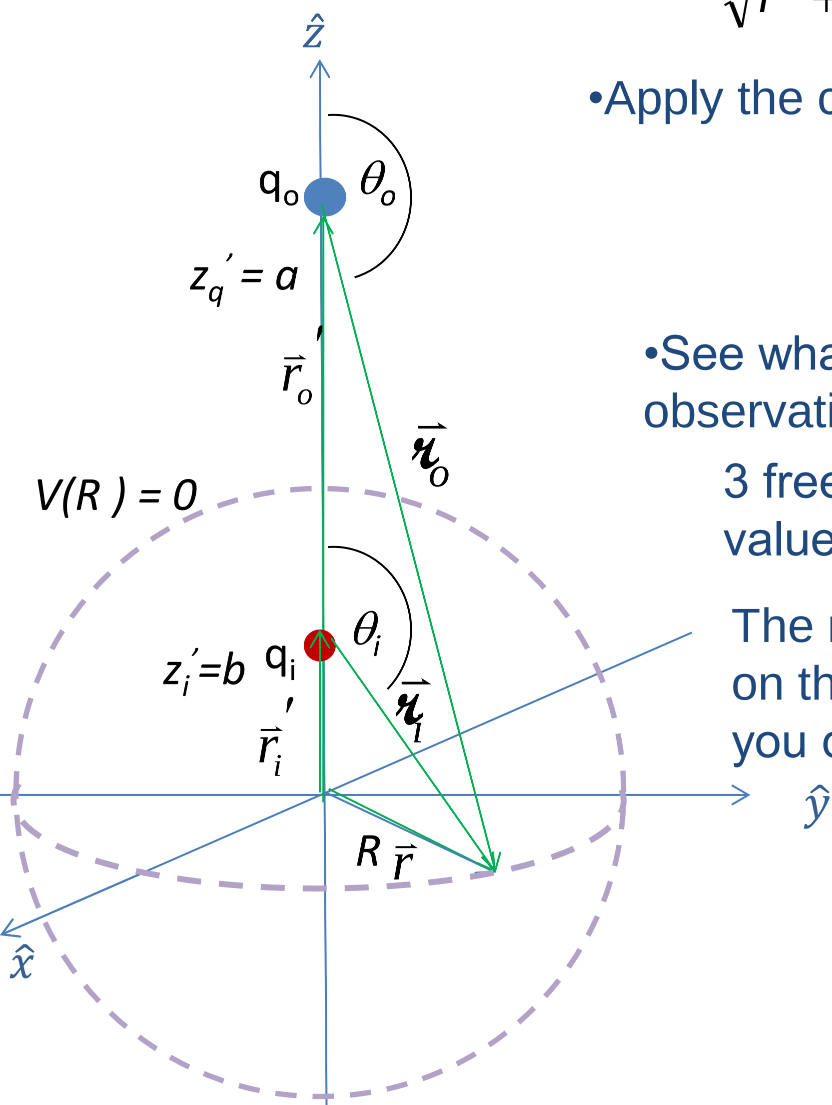
$$\frac{q_o}{\mathscr{r}_o} + \frac{q_i}{\mathscr{r}_i} = 4\pi\varepsilon_o V(\vec{r})$$

$$\frac{q_o}{\sqrt{(\vec{r} - \vec{r}_o')^2}} + \frac{q_i}{\sqrt{(\vec{r} - \vec{r}_i')^2}} = 4\pi\varepsilon_o V(\vec{r})$$

$$\frac{q_o}{\sqrt{r^2 + r_o'^2 - 2rr_o' \cos\theta_o}} + \frac{q_i}{\sqrt{r^2 + r_i'^2 - 2rr_i' \cos\theta_i}} = 4\pi\varepsilon_o V(\vec{r})$$

$$\frac{q_o}{\sqrt{r^2 + a^2 - 2ra \cos\theta_o}} + \frac{q_i}{\sqrt{r^2 + b^2 - 2rb \cos\theta_i}} = 4\pi\varepsilon_o V(\vec{r})$$

Abbreviated Example: Charge outside grounded sphere



$$\frac{q_o}{\sqrt{r^2 + a^2 - 2ra\cos\theta_o}} + \frac{q_i}{\sqrt{r^2 + b^2 - 2rb\cos\theta_i}} = 4\pi\varepsilon_o V(\vec{r})$$

- Apply the condition $\sum \frac{q_o}{\mathscr{r}_o} + \sum \frac{q_i}{\mathscr{r}_i} = const$ on conductor

$$\frac{q_o}{\sqrt{R^2 + a^2 - 2Ra\cos\theta_o}} + \frac{q_i}{\sqrt{R^2 + b^2 - 2Rb\cos\theta_i}} = 0$$

- See what you've got to do to remove dependence on the observation location on conductor.

3 free parameters: the constant, the image charge's value, and the image charge's location.

The relation should be true for *all* observation locations on the conducting surface, so choose easy ones to help you determine the three parameter's values.

Abbreviated Example: Charge outside grounded sphere

$$\frac{q_o}{\sqrt{r^2+a^2-2ra\cos\theta_o}}+\frac{q_i}{\sqrt{r^2+b^2-2rb\cos\theta_i}}=4\pi\varepsilon_o V(\vec{r})$$

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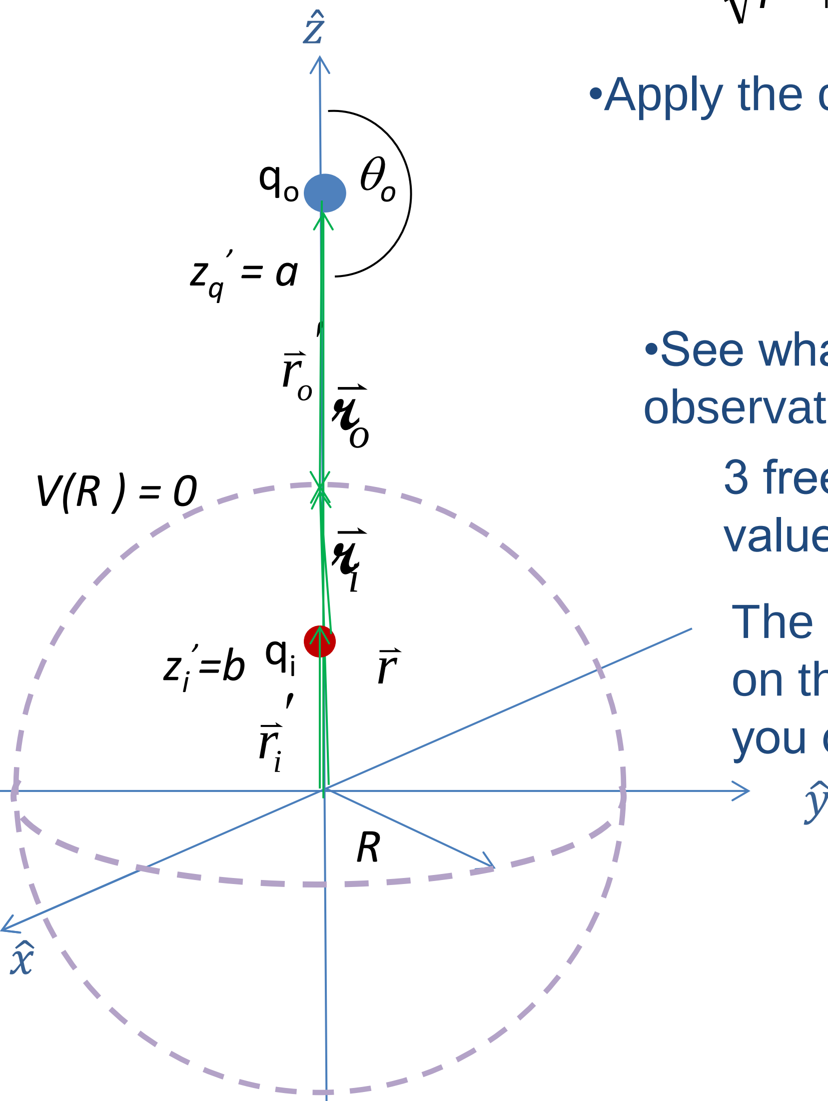
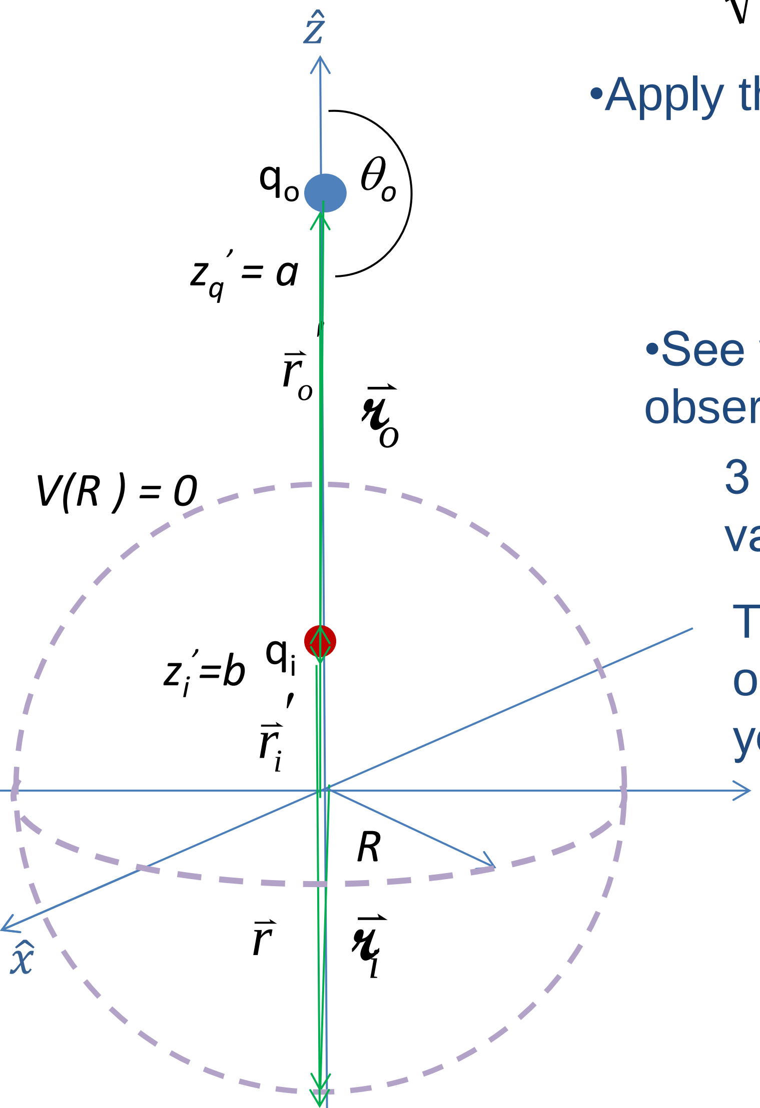


Top of sphere: $\frac{q_o}{(a-R)}+\frac{q_i}{(R-b)}=0$

Abbreviated Example: Charge outside grounded sphere

$$\frac{q_o}{\sqrt{r^2+a^2-2ra\cos\theta_o}}+\frac{q_i}{\sqrt{r^2+b^2-2rb\cos\theta_i}}=4\pi\varepsilon_o V(\vec{r})$$



- Apply the condition $\sum \frac{q_o}{\mathscr{r}_o}+\sum \frac{q_i}{\mathscr{r}_i}=const$ on conductor

$$\frac{q_o}{\sqrt{R^2+a^2-2Ra\cos\theta_o}}+\frac{q_i}{\sqrt{R^2+b^2-2Rb\cos\theta_i}}=0$$

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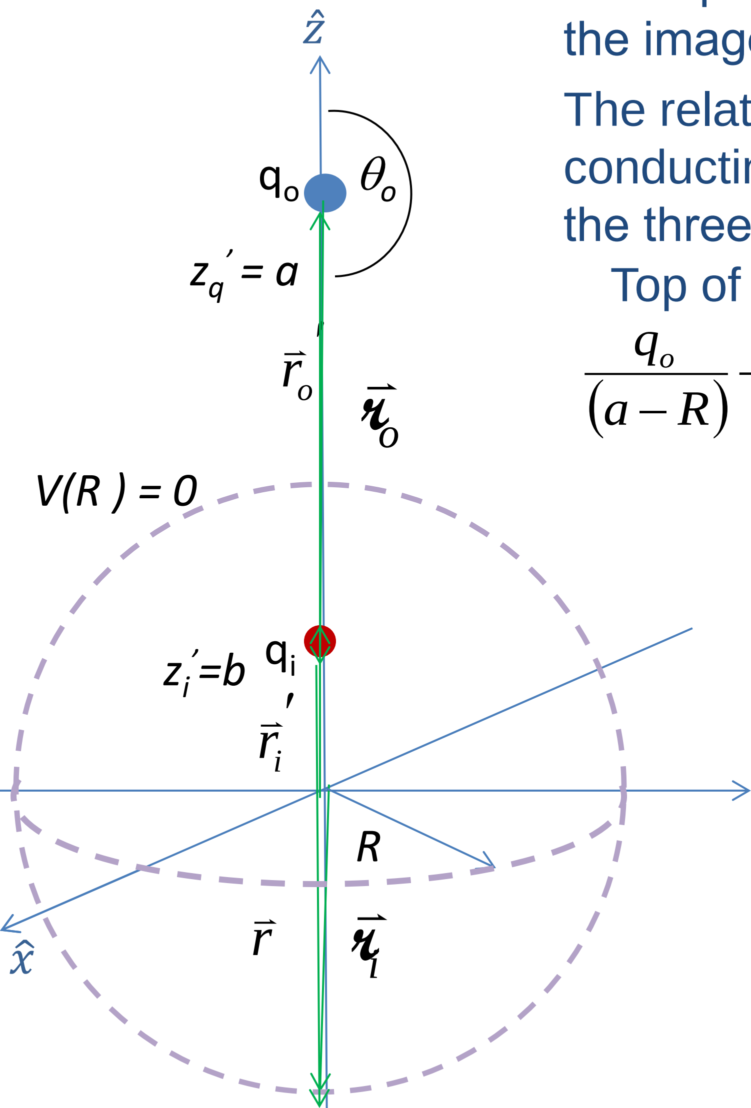
The relation should be true for *all* observation locations on the conducting surface, so choose easy ones to help you determine the three parameter's values.

Top of sphere: $$\frac{q_o}{(a-R)}+\frac{q_i}{(R-b)}=0$$

Bottom of sphere: $$\frac{q_o}{(a+R)}+\frac{q_i}{(R+b)}=0$$

Abbreviated Example: Charge outside grounded sphere

- See what you've got to do to remove dependence on the observation location on conductor.



3 free parameters: the constant, the image charge's value, and the image charge's location.

The relation should be true for *all* observation locations on the conducting surface, so choose easy ones to help you determine the three parameter's values.

Top of sphere:

$$\frac{q_o}{(a-R)} + \frac{q_i}{(R-b)} = 0$$

Bottom of sphere:

$$\frac{q_o}{(a+R)} + \frac{q_i}{(R+b)} = 0$$

Two equations, two unknowns (qi, b)

$$-q_o\frac{(R-b)}{(a-R)} = q_i = -q_o\frac{(R+b)}{(a+R)}$$

$$(R-b)(a+R) = (R+b)(a-R)$$

$$\left(Ra - ab - Rb + R^2\right) = \left(Ra - R^2 + ab - Rb\right)$$

$$\left(Ra - ab - Rb + R^2\right) = \left(Ra - R^2 + ab - Rb\right)$$

$$\left(-ab + R^2\right) = \left(-R^2 + ab\right)$$

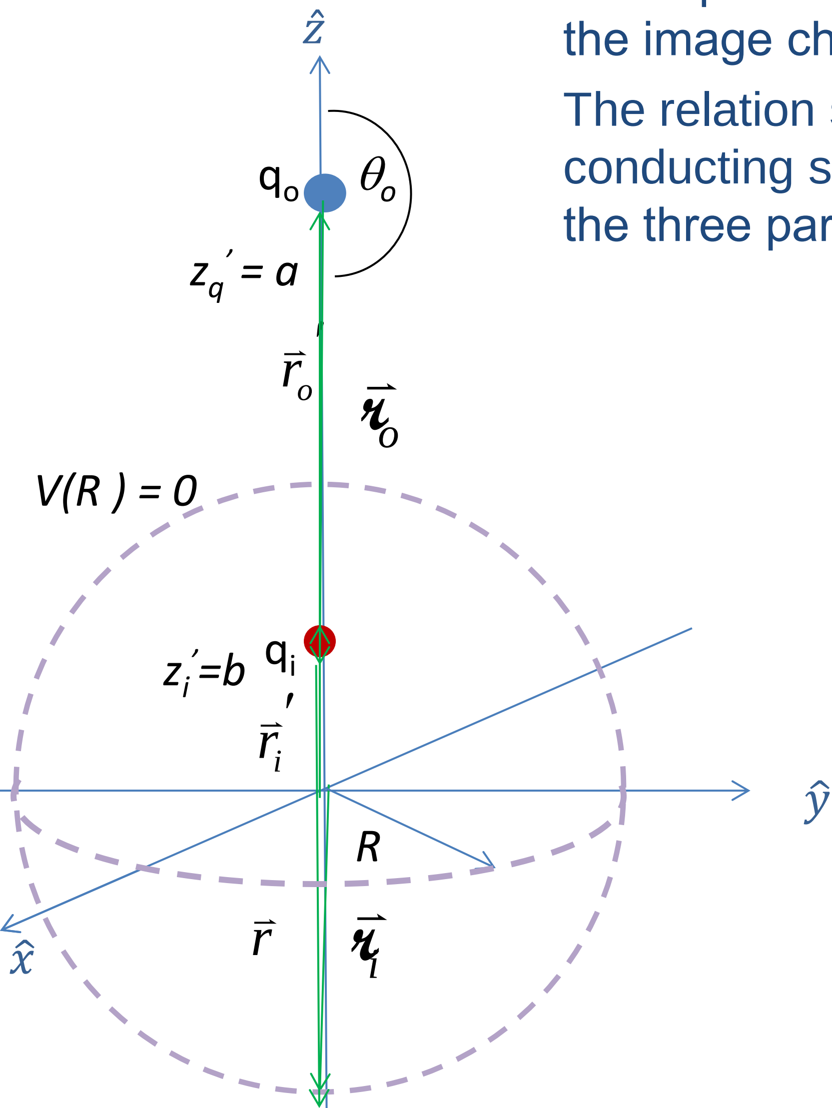
$$b = R^2/a$$

Abbreviated Example: Charge outside grounded sphere

- See what you've got to do to remove dependence on the observation location on conductor.

3 free parameters: the constant, the image charge's value, and the image charge's location.

The relation should be true for *all* observation locations on the conducting surface, so choose easy ones to help you determine the three parameter's values.



$$b = R^2/a$$

$$-q_o\frac{(R-b)}{(a-R)} = q_i = -q_o\frac{(R+b)}{(a+R)}$$

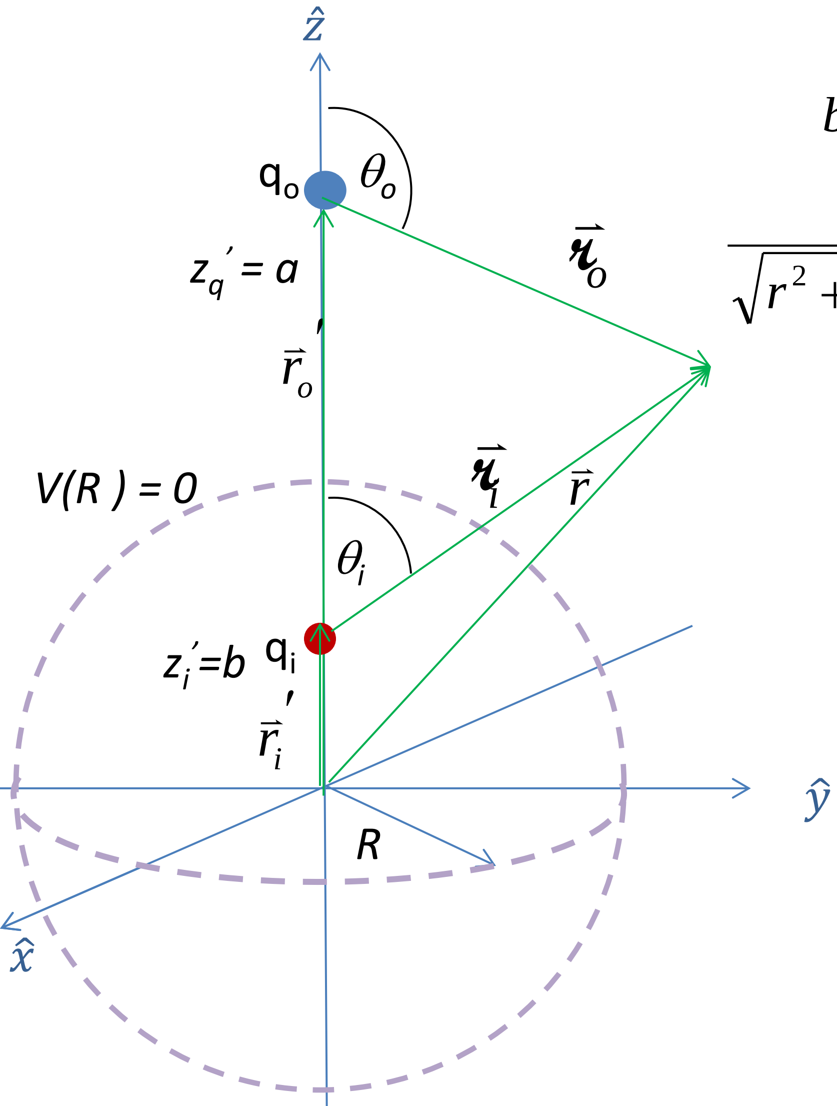
$$q_i = -q_o\frac{(R+R^2/a)}{(a+R)}$$

$$q_i = -q_o\frac{R}{a}\frac{(1+R)}{(1+R)}$$

$$q_i = -q_o\frac{R}{a}$$

Abbreviated Example: Charge outside grounded sphere

- If you've got a solution that works for the boundary and satisfies Poisson's equation, then you've got *the* solution.



$$b = R^2/a \qquad q_i = -q_o \frac{R}{a}$$

$$\frac{q_o}{\sqrt{r^2 + a^2 - 2ra\cos\theta_o}} + \frac{q_i}{\sqrt{r^2 + b^2 - 2rb\cos\theta_i}} = 4\pi\varepsilon_o V(\vec{r})$$

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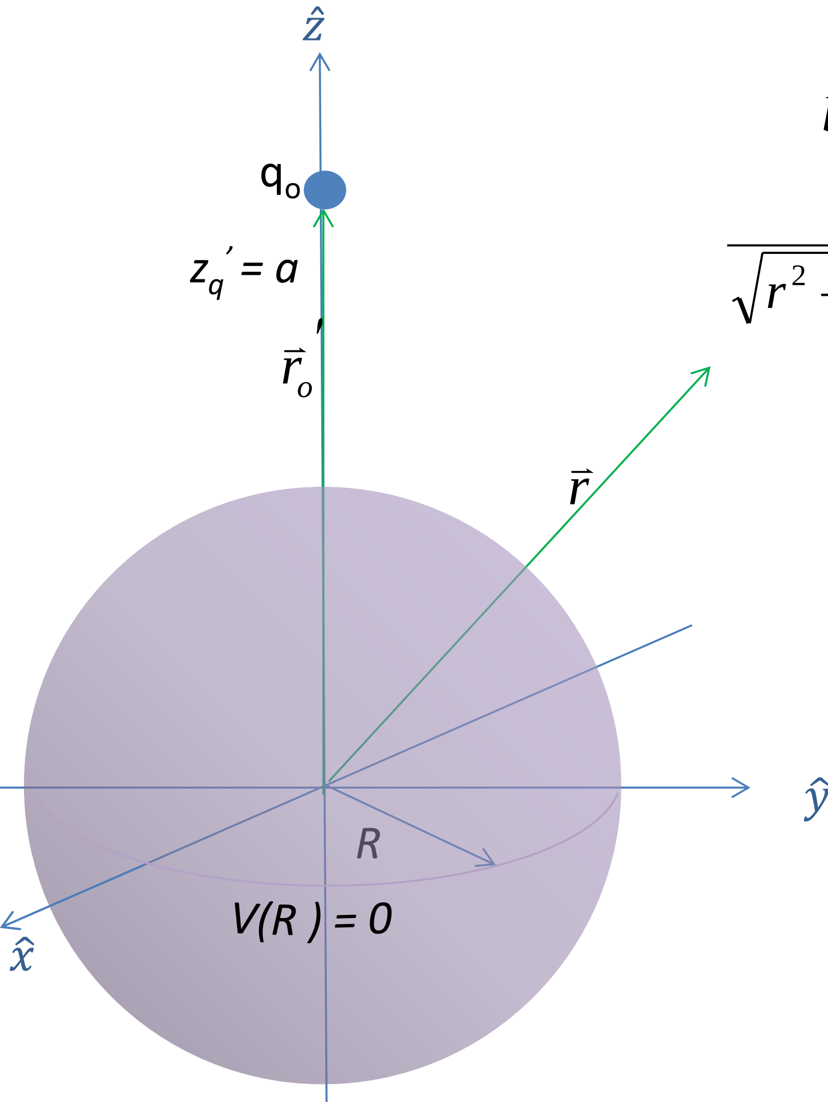
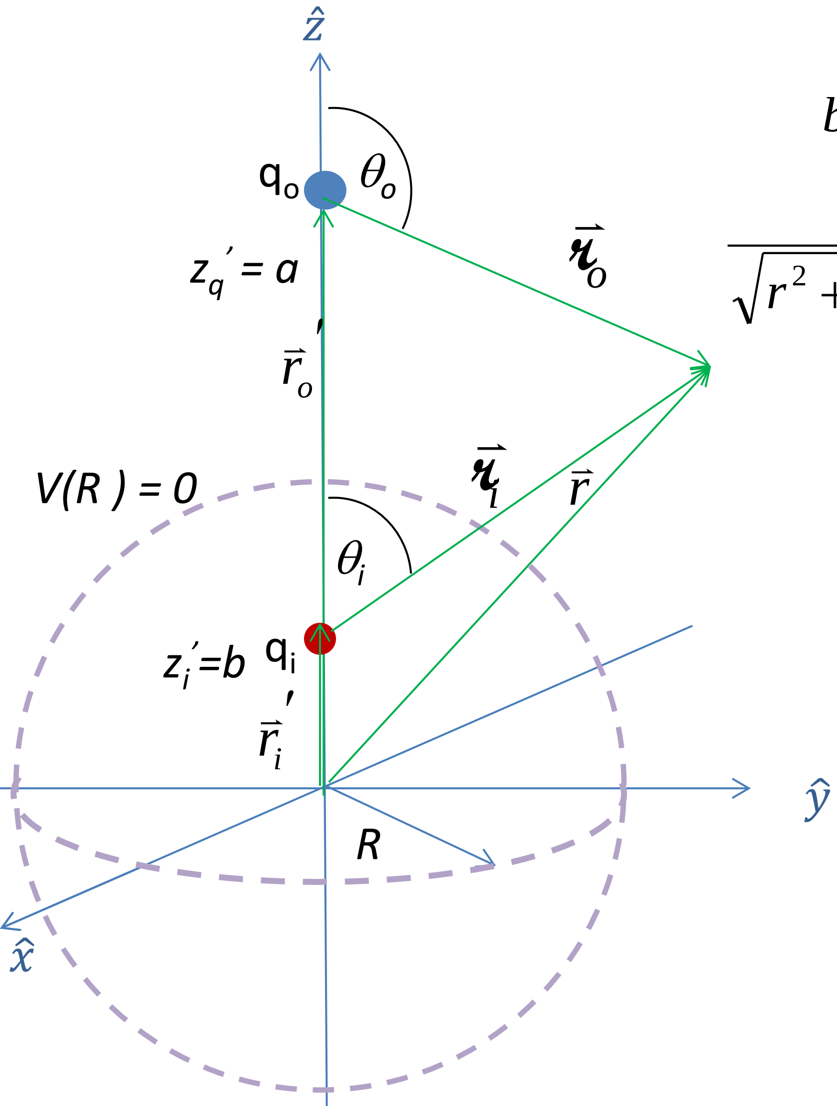


Even for the *real* system

Abbreviated Example: Charge outside grounded sphere



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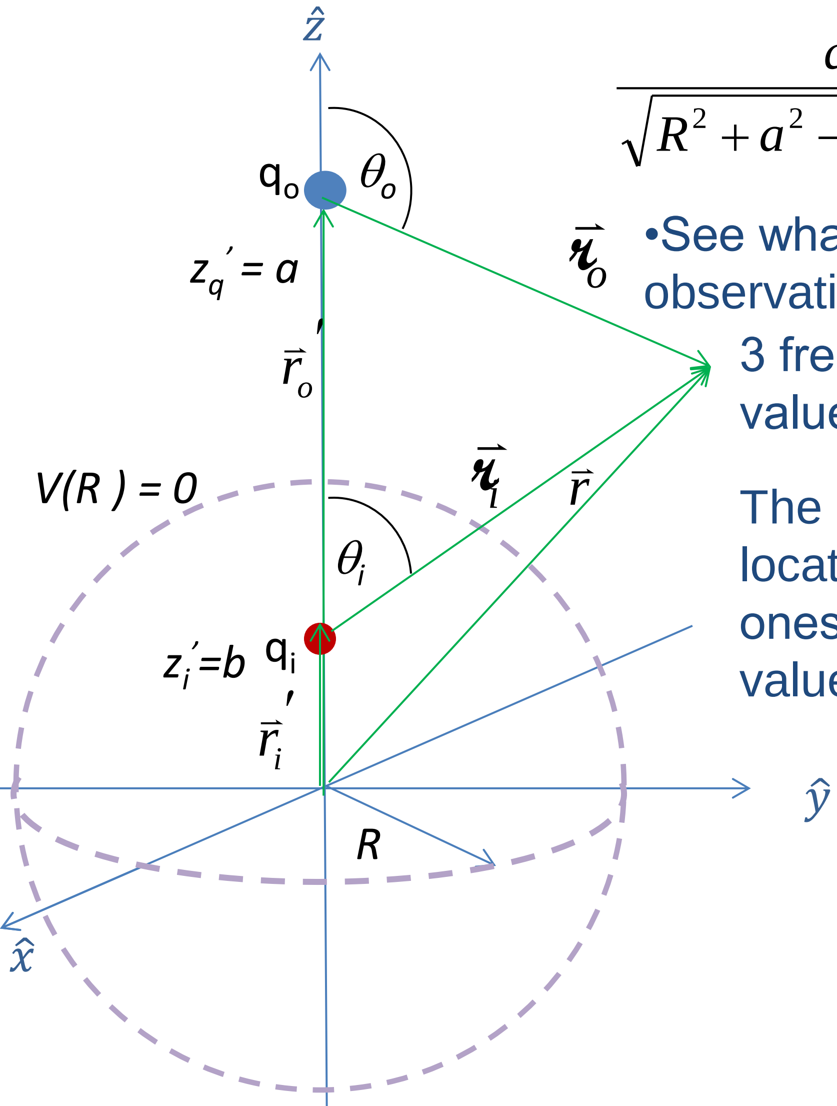
Abbreviated Example: Charge outside grounded sphere

- Apply the condition $\sum \frac{q_o}{\mathscr{r}_o} + \sum \frac{q_i}{\mathscr{r}_i} = const$ on conductor

$$\frac{q_o}{\sqrt{R^2 + a^2 - 2Ra\cos\theta_o}} + \frac{q_i}{\sqrt{R^2 + b^2 - 2Rb\cos\theta_i}} = 4\pi\varepsilon_o V(R) = 0$$

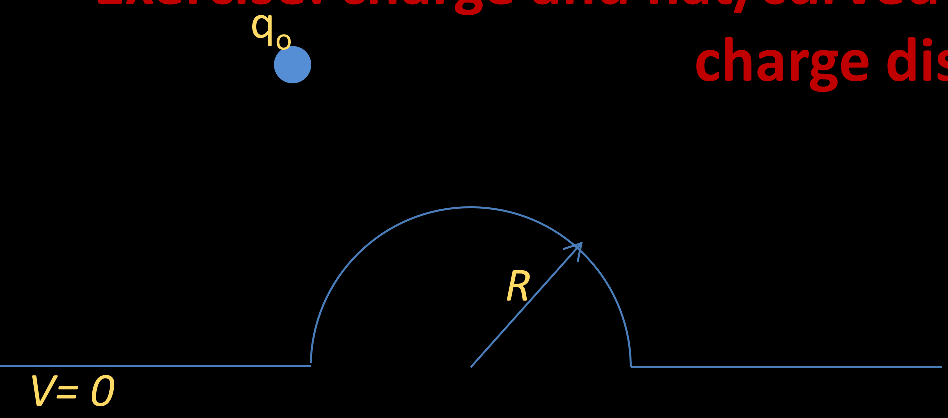


- See what you've got to do to remove dependence on the observation location on conductor.

3 free parameters: the constant, the image charge's value, and the image charge's location.

The relation should be true for *all* observation locations on the conducting surface, so choose easy ones to help you determine the three parameter's values.

Exercise: charge and flat/curved conductor, what's the equivalent charge distribution?



Fri.	3.2 Images T4 Relaxation Method	
Mon. Wed. Thurs	3.4.1-.4.2 Multipole Expansion 3.4.3-.4.4 Multipole Expansion	 HW4