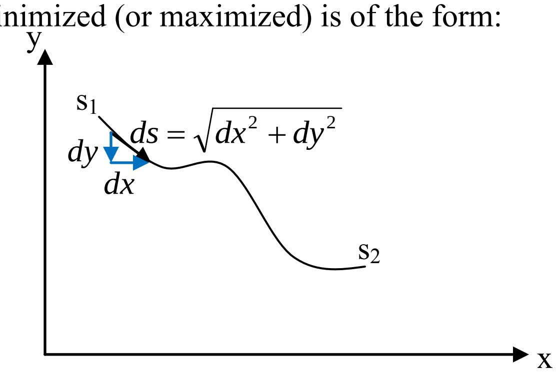
Mon.10/25 Tues 10/26 Wed.10/27 Fri.10/29	**6.3-.4** Applications and Complications **7.1** Lagrange's Equations with Unconstrained **7.2-.3** Lagrange's with Constrained	**HW6**
Mon., 11/1 Wed., 11/3 Thurs. 11/4 Fri., 11/5	**7.4-.5** Proof and Examples **7.6-.8** Generalized Variables & Classical Hamiltonian (Recommend 7.9 if you've had Phys 332) **8.1-.2** 2-Body Central Forces, Relative Coordinates.	**HW7**

Introduction.

Last time, we started down the path for developing a 'whole new' tool for mechanics – relating motion and interactions. Of course, in some sense it's just another restating of the observation contained in Newton's 2nd and the Work-Energy relation. The idea of this new tool is that a system will evolve along the path for which the "action", the momentum times displacement, is minimum. Unlike when we moved from force-momentum to work-energy, developing this tool isn't simple; a whole chapter is devoted to the first step – figuring out how to minimize *anything* to do with a path. We're going to finish that up today.

The motivational example we considered was, say we wanted to know the equation of the quickest path someone could take between two points across regions of differing speed limits. We can define $y' = dy/dx$, so the integral that must be minimized (or maximized) is of the form:



Let's think about the case of minimizing the path length.

First off, the path length between two points is

$$S = \int_{s_1}^{s_2} ds$$

That's the sum of the length of each infinitesimal step that takes you along the path from point 1 to point 2. On the infinitesimal scale, you can rephrase that step length as $ds = \sqrt{dx^2 + dy^2}$. For that matter, you could rephrase the vertical change in terms of the curve's local slope, $y'(x) \equiv \frac{dy}{dx}(x)$, as $dy = y'(x)dx$, then the differential bit of length along the path could be rephrased as $ds = \sqrt{dx^2 + \left(y'(x)dx\right)^2} = \sqrt{1 + \left(y'(x)\right)^2}\,dx$, so

The thing we want minimized, the transit time, is

$t = \int_{s_1}^{s_2} \frac{ds}{v(x,y)} = \int_{x_1}^{x_2} \left(\frac{\sqrt{1 + \left(y'(x)\right)^2}}{v(x,y)} \right) dx$ (the velocity is written to emphasize that it may vary from one location to the next.)

If we define the integrand as function f, then it has functional dependence on x, y, and dx/dy (which is itself dependent on x).

$$t = \int_{x_1}^{x_2} f(x, y, y'(x))dx$$

$$f(x, y, y'(x)) \equiv \left(\frac{\sqrt{1 + (y'(x))^2}}{v(x, y)} \right)$$

After an awful lot of math, we arrived at this condition: if we follow the quickest path, then the integrand must satisfy

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0 \qquad \text{where y is a function of x.}$$

More generally, if there's some property of a path that we want to minimize, then, expressing that path as an integral, the integrand must satisfy this differential equation (though, more generally, x and y are replaced by whatever independent and dependent variables are relevant to your integral.)

Lat time, we got a little practice *setting up* such integrals and identifying the relevant differential equation.

This Time

We'll now set about *solving* the differential equation.

First, the simple example that the book works

Example #1: Show that the shortest distance between two points in a plane is along a straight line.

The element of length on a plane is:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + (dy/dx)^2}\; dx = \sqrt{1 + y'^2}\; dx,$$

where we define $y' = dy/dx$. The total length between two points (x_1, y_1) and (x_2, y_2) along the path $y(x)$ is:

$$S = \int ds = \int_{x_1}^{x_2} \sqrt{1+y'^2}\, dx\,.$$

The function that goes into the Euler-Lagrange equation:

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0,$$

to minimize the length is:

$$f = \sqrt{1+y'^2}\,.$$

This gives:

$$0 - \frac{d}{dx}\left[\frac{\frac{1}{2}(2y')}{\sqrt{1+y'^2}}\right] = 0,$$

which means that:

$$\frac{y'}{\sqrt{1+y'^2}} = \text{constant} \equiv C\,.$$

Rearranging this gives:

$$y' = \sqrt{\frac{C^2}{1-C^2}} = \text{constant} \equiv a\,,$$

which means that $y = ax + b$, the equation for a straight line. If you are given endpoints, you still have to determine the constants. This gives you the form of the solution.

Variables:

In the derivation of the Euler-Lagrange equation:

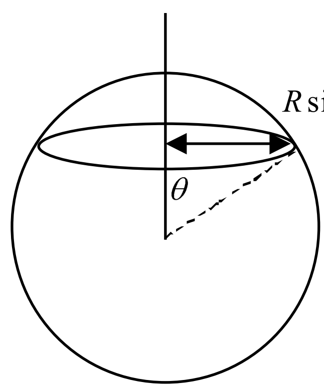
$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0,$$

we assumed that x is the independent variable and y is the dependent variable so $y=y(x)$ and $y'=dy/dx$. Sometimes we will use different names for the variables, but the form of the equation is the same.

Last time, we set this one up, I'll remind you of it, and then we'll find the minimized path length.

Example #2: (Probs. 6.1 & 6.16) (a) Find an integral for the length of a path joining two points on a sphere of radius R. (b) Show that the geodesic (shortest path) between two points on a sphere is a *great circle*, a curve defined by a plane that passes through the center of the sphere. An example is a longitude line, which has constant ϕ if we choose the z axis along the north pole.

(a) If θ increases by $d\theta$, the distance moved is $R\,d\theta$. If ϕ increases by $d\phi$, the path is along a circle of radius $R\sin\theta$ (see the diagram below) and the distance moved is $R\sin\theta\,d\phi$.



The distance moved for an arbitrary small displacement is:

$$ds=\sqrt{(R\,d\theta)^2+(R\sin\theta\,d\phi)^2}\,.$$

The path between two points on the surface of the sphere can be described by a function $\phi(\theta)$, so rewrite the equation above as:

$$ds=R\sqrt{1+\sin^2\theta(d\phi/d\theta)^2}\;d\theta=R\sqrt{1+\sin^2\theta\,\phi'(\theta)^2}\;d\theta,$$

where $\phi'(\theta)=d\phi/d\theta$. The total path length is:

$$S=R\int_{\theta_1}^{\theta_2}\sqrt{1+\sin^2\theta\,\phi'(\theta)^2}\;d\theta.$$

(b) The Euler-Lagrange equation is:

$$\frac{\partial f}{\partial \phi}-\frac{d}{d\theta}\frac{\partial f}{\partial \phi'}=0,$$

where $f\left(\phi(\theta),\phi'(\theta),\theta\right)=\sqrt{1+\sin^2\theta\,\phi'(\theta)^2}$.

In my notation I'm reminding you that, presumably, on the given curve ϕ and the slope can be expressed as functions of θ. Since $\partial f/\partial\phi=0$, the equation above reduces to:

$$\frac{\partial f}{\partial \phi'}=\text{constant (w.r.t}\theta),$$

Now, if it's a constant with respect to θ, then it's also a constant with respect to the ϕ an ϕ□ since they themselves depend on θ (or are, themselves constants).

$$\frac{\sin^2\theta\,\phi'}{\sqrt{1+\sin^2\theta\,\phi'^2}}=C.$$

Without loss of generality, we can choose the coordinates so that point 1 lies on the *z* axis and $\theta_1=0$. In this coordinate system, the constant must be $C=0$. But it's supposed to be a constant with respect to θ, so changing to another θ, the only way that it can still be 0 is if $\phi'=0$, so ϕ is constant as we move from one θ to another. What does that look like, tracing straight down a side of the sphere - path is a great circle.

There is some subtlety to using the calculus of variation. You must be careful that the solutions found are what you're looking for (usually a minimum or maximum). In example 2, a great circle can also go around the sphere the long way, which is a *stationary* path (<u>not</u> min. or max.).

Problem 6B: (Helix) Last time I'd asked you to set up the integral for the length of a path between two points on a cylinder. You may not recall, but you got

$$L=\int_i^f\sqrt{1+R^2\phi'^2}\,dz$$

So, the integrand is

$f(\phi(z),\varphi'(z),z)=\sqrt{1+R^2\phi'^2}$ and then the equation that it must satisfy is

$$\frac{\partial}{\partial\phi}f(\phi')-\frac{d}{dz}\frac{\partial}{\partial\phi'}=0$$

So, in the homework, you'll plug this in, take the derivatives, and, ultimately find an expression for ϕ as a function of z.

$$0-\frac{d}{dz}\frac{\frac{1}{2}R^2 2\phi'}{\sqrt{1+R^2\phi'^2}}=0$$

$$\frac{d}{dz}\frac{\phi'}{\sqrt{1+R^2\phi'^2}}=0$$

Or $\frac{\phi'}{\sqrt{1+R^2\phi'^2}}$ is *completely* independent of z, but even ϕ is a function of z on our path, so this is a constant over the whole path.

$$\frac{\phi'}{\sqrt{1+R^2\phi'^2}}=C$$

$$\phi'^2\frac{1}{C^2}=1+R^2\phi'^2$$

$$\phi'^2=\frac{1}{\frac{1}{C^2}+R^2}$$

$$\phi'=\frac{1}{\sqrt{\frac{1}{C^2}+R^2}}$$

$$d\phi=\frac{dz}{\sqrt{\frac{1}{C^2}+R^2}}$$

$$\phi-\phi_i=\frac{1}{\sqrt{\frac{1}{C^2}+R^2}}(z-z_i)$$

This slope is a constant, so we can write it as simply

$$\frac{\phi'}{\sqrt{1+R^2\phi'^2}}=C$$

$$\phi'^2\frac{1}{C^2}=1+R^2\phi'^2$$

$$\phi'^2=\frac{1}{\frac{1}{C^2}+R^2}$$

$$\phi'=\frac{\Delta\phi}{\Delta z}=\frac{\phi-\phi_i}{z-z_i}$$

$$\phi-\phi_i=\frac{\Delta\phi}{\Delta z}(z-z_i)$$

$$\phi=\frac{\Delta\phi}{\Delta z}(z-z_i)+\phi_i$$

This defines a ‘straight line’ wrapping around the cylinder as it rises.

Example 3: minimum gravitational potential curve

$$dU = yg \cdot dm$$

$dm = \left(\frac{dm}{ds}\right)ds = \lambda ds$ where λ is short-hand for linear mass density.

$$U = \int_i^f g\lambda y ds = g\lambda \int_i^f y ds = g\lambda \int_i^f y\sqrt{\left(\frac{dx}{dy}\right)^2 + 1}dy$$

So, to minimize this gravitational potential energy, we need the integrand

$$f(y, x, x') = y\sqrt{x'^2 + 1}$$

to satisfy

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0$$

$$-\frac{d}{dy}\left(y\frac{x'}{\sqrt{x'^2 + 1}}\right) = 0$$

So,

$$y\frac{x'}{\sqrt{x'^2 + 1}} = C$$

$$y^2 x'^2 = C^2 \left(x'^2 + 1\right)$$

$$\left(\left(\frac{y}{C}\right)^2 - 1\right)x'^2 = 1$$

Equation (1)

$$\int dx = \int \frac{dy}{\left(\left(\frac{y}{C}\right)^2 - 1\right)^{1/2}}$$

$$x - x_i = C\left(\cosh^{-1}\left(y/C\right) - \cosh^{-1}\left(y_i/C\right)\right)$$

We won’t go all the way, but the constant C can be nailed down by observing that the length of the string is a constant, L

The length is, of course, simply

$$L = \int_i^f \sqrt{x'^2 + 1}\,dy$$

Looking back at Equation 1, that can be rearranged so,

$$\int_i^f \sqrt{x'^2 + 1}\,dy = L$$

$$\int_i^f \sqrt{\frac{1}{y^2/C^2 - 1} + 1}\,dy = L$$

$$\int_i^f \sqrt{\frac{y^2/C^2}{y^2/C^2 - 1}}\,dy = L$$

$$\tfrac{1}{2}C\int_i^f \sqrt{\frac{1}{u - 1}}\,du = L$$

$$C\left(\sqrt{(y_f/C)^2 - 1} - \sqrt{(y_i/C)^2 - 1}\right) = L$$

It's transcendental, but, for a given length, and given initial and final heights, C can be numerically calculated.

<u>Example #4</u>: (Brachistochrone) Suppose point 1 is higher than point 2. What shape should a frictionless track be so that an object will slide between the points in the shortest time?

The time to travel a short distance *ds* is ds/v, where *v* is the speed. If we take point 1 as the origin with the *y* axis <u>downward</u>, the speed by conservation of energy is

$$\tfrac{1}{2}m\left(v^2 - v_i^2\right) = mg\left(y - y_i\right)$$

$v = \sqrt{2gy}$. The length of a short segment of the path is (see Example 1):

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{(dx/dy)^2 + 1}\;dy = \sqrt{x'^2 + 1}\;dy,$$

where $x' = dx/dy$. We are treating the path as $x(y)$ because the speed depends on *y*. The total time is:

$$t = \frac{1}{\sqrt{2g}}\int_0^{y_2} \frac{\sqrt{x'^2 + 1}}{\sqrt{y}}\;dy.$$

The integrand is:

$$f(x, x', y) = \frac{\sqrt{x'^2 + 1}}{\sqrt{y}},$$

so the roles of *x* and *y* are switched and:

$$\frac{\partial f}{\partial x} - \frac{d}{dy}\frac{\partial f}{\partial x'} = 0.$$

The first term is $\partial f/\partial x = 0$, so $\partial f/\partial x'$ is a constant (w.r.t. y). The derivative is:

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{y}\sqrt{x'^2+1}},$$

so the square of this is also a constant:

$$\frac{x'^2}{y\left(x'^2+1\right)} = \text{constant} = \frac{1}{2a}.$$

The unusual constant is chosen for convenience. This can be rearranged to get:

$$2ax'^2 = y\left(x'^2+1\right),$$

$$(2a-y)x'^2 = y,$$

$$x' = \frac{dx}{dy} = \sqrt{\frac{y}{2a-y}}.$$

Integrate to get:

$$x = \int\sqrt{\frac{y}{2a-y}}\,dy.$$

Make the change of variables:

$$y = a(1-\cos\theta) \quad \text{and} \quad dy = a\sin\theta\,d\theta,$$

which gives:

$$x = a\int\sqrt{\frac{a(1-\cos\theta)}{2a-a(1-\cos\theta)}}\sin\theta\,d\theta = a\int\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\sin\theta\,d\theta.$$

Multiply and divide the integrand by $\sqrt{1-\cos\theta}$ and use $\sin^2\theta = 1-\cos^2\theta$ to get:

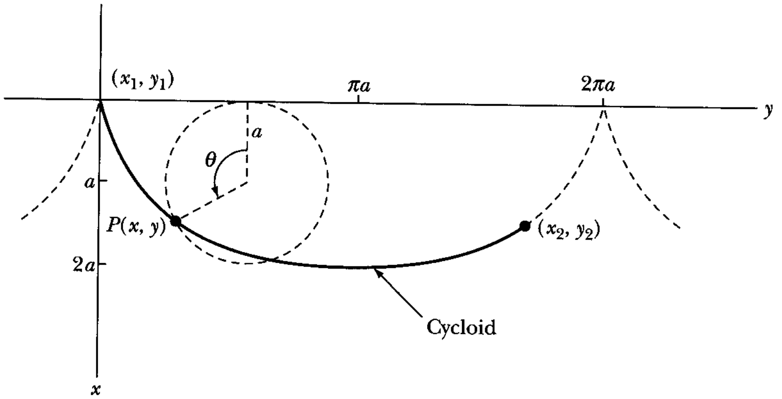
$$x = a\int\sqrt{\frac{(1-\cos\theta)^2}{(1+\cos\theta)(1-\cos\theta)}}\sin\theta\,d\theta = a\int\sqrt{\frac{(1-\cos\theta)^2}{1-\cos^2\theta}}\sin\theta\,d\theta = a\int(1-\cos\theta)\,d\theta$$

$$x = a(\theta-\sin\theta)+C.$$

If we choose the first point to be the origin $x = y = 0$, then $C = 0$. The parametric solution (in terms of the parameter θ) is:

$$x = a(\theta-\sin\theta) \quad \text{and} \quad y = a(1-\cos\theta),$$

which is a *cycloid*. It is the path traced by the point on a circle of radius *a* that rolls along the *x* axis. The angle the circle has rotated through is θ.



More than Two Variables:

Suppose a problem involves one independent variable and more than one dependent variable. For example, the three Cartesian coordinates could depend on the parameter *u*, so $x = x(u)$, $y = y(u)$, and $z = z(u)$. We define the derivatives:

$$x' = \frac{dx}{du}, \qquad y' = \frac{dy}{du}, \qquad \text{and} \qquad z' = \frac{dz}{du}.$$

If we want to find paths for which the integral:

$$S = \int_{u_1}^{u_2} f\left[x(u), y(u), z(u), x'(u), y'(u), z'(u), u\right] du$$

is stationary, it must be stationary for variations of each of the variables. That will yield an Euler-Lagrange equation for each variable, so in this case:

$$\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0, \qquad \frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0, \qquad \text{and} \qquad \frac{\partial f}{\partial z} - \frac{d}{du}\frac{\partial f}{\partial z'} = 0.$$

Example #4 (Prob. 6.27) Show that the shortest distance between two points in 3-D is along a straight line.

The distance for a small displacement is:

$$ds = \sqrt{dx^2 + dy^2 + dz^2} = \sqrt{(dx/du)^2 + (dy/du)^2 + (dz/du)^2}\ du = \sqrt{x'^2 + y'^2 + z'^2}\ du.$$

The integral to minimize is:

$$S = \int_{u_1}^{u_2} \sqrt{x'^2 + y'^2 + z'^2}\ du,$$

where $x_1 = x(u_1)$, $x_2 = x(u_2)$, etc. The integrand is:

$$f = \sqrt{x'^2 + y'^2 + z'^2},$$

which does not depend explicitly on *x*, *y*, or *z*, so $\partial f/\partial x = \partial f/\partial y = \partial f/\partial z = 0$. The partial derivative of *f* with respect to x' is:

$$\frac{\partial f}{\partial x'} = \frac{\frac{1}{2}(2x')}{\sqrt{x'^2 + y'^2 + z'^2}} = \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}}.$$

The Euler-Lagrange equation associated with x is:

$$\frac{\partial f}{\partial x} - \frac{d}{du}\frac{\partial f}{\partial x'} = 0,$$

but in this case:

$$\frac{d}{du}\frac{\partial f}{\partial x'} = 0,$$

which implies that:

$$\frac{\partial f}{\partial x'} = \frac{x'}{\sqrt{x'^2 + y'^2 + z'^2}} = \text{constant} = a.$$

Similarly for the other components:

$$\frac{\partial f}{\partial y'} = \frac{y'}{\sqrt{x'^2 + y'^2 + z'^2}} = \text{constant} = b \qquad \text{and} \qquad \frac{\partial f}{\partial z'} = \frac{z'}{\sqrt{x'^2 + y'^2 + z'^2}} = \text{constant} = c$$

The three equations imply that the ratios $x' : y' : z'$ are constant, which means that x, y, and z change at constant ratios along the curve. In other words, the shortest path is a straight line.

Time for HW questions (due Wed. after the break)

Next class:

- Wednesday – start Ch. 7