Phys 331: Ch 9, .1-.3 Noninertial Frames: Acceleration, Ties

Fri., 11/16	**9.1-.2** Noninertial Frames: Acceleration, Tides, Angular Velocity	
Mon., 11/19	**9.4-.5** Noninertial Frames: Time derivatives, Newton's 2nd.	**HW9a** (9.2, 9.8)

Non-inertial Frames

Say you go down to the Rose Parade and see a Pool-hall's float go gliding by. On that float, of course, they have a pool table and two people playing pool, let's say I'm one of them.

At some moment, relative to you, the front of the float has position, velocity, and acceleration $\vec{R}_f, \vec{V}_f, \vec{A}_f$.

If *you* observe the cue ball, relative to *you*, *you'd* measure its position, velocity, and acceleration to be $\vec{r}_o, \dot{\vec{r}}_o$, and $\ddot{\vec{r}}_o$.

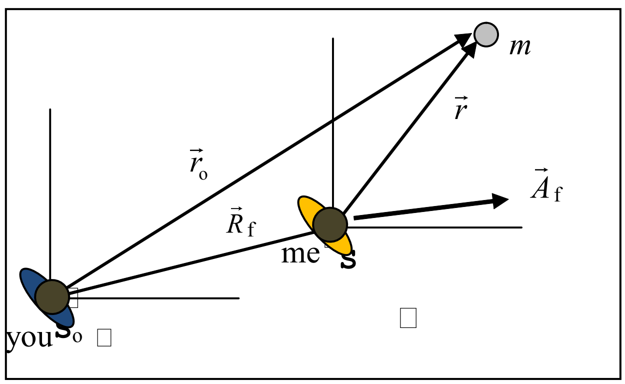
Of course, *I*, riding at the front of the float, would measure the ball's position, velocity, and acceleration relative to *myself* as

$$\vec{r} = \vec{r}_o - \vec{R}_f$$

$$\dot{\vec{r}} = \dot{\vec{r}}_o - \vec{V}_f$$

And

$$\ddot{\vec{r}} = \ddot{\vec{r}}_o - \vec{A}_f$$



Essentially what you measure, minus his own position, velocity, and acceleration.

(note: this is the classical approximation, as is appropriate for this class; taking into consideration that folks moving at different speeds measure time and distance differently is the subject of special and general relativity)

None of these are particularly remarkable kinematic observations; however, kinematics meets *dynamics* (interactions, pushes and pulls) in Newton's 2nd Law where force is related to acceleration, *but only as measured in a non-accelerating frame.*

$$\vec{F}_{net} = m\ddot{\vec{r}}_o$$

Or

$$\ddot{\vec{r}}_o = \frac{\vec{F}_{net}}{m}$$

So, for example, if you could measure all the forces on the cue ball, then you could use this to predict how it would accelerate *relative to you on the curbside.*

That means, it would accelerate away from the person riding the float as

$$\ddot{\vec{r}} = \frac{\vec{F}_{net}}{m} - \vec{A}_f$$

Tada! It's that simple. If so, then why on earth have we shied away from analyzing motion in non-inertial frames for so long? Over the course of this chapter we'll see that some situations are rather easy, while others are rather tricky.

Fictitious Inertial / Frame Force

If we multiply our acceleration relation by mass, of course we get

$$m\ddot{\vec{r}} = \vec{F}_{net} - m\vec{A}_f$$

In the same spirit with which we'd defined a "centrifugal potential" and an "effective" potential in the last chapter, we could define a "frame force" and an "effective net force" here (indeed, centrifugal force is a "frame force" for a rotating system.)

$$\vec{F}_{frame} \equiv -m\vec{A}_f$$

$$\vec{F}_{net.eff} \equiv \vec{F}_{net} + \vec{F}_{frame}$$

$$m\ddot{\vec{r}} = \vec{F}_{net.eff}$$

In what sense is the frame (or, as the book calls it "inertial", which seems a strange choice since it's an artifact being in a *non*-inertial frame) force like and not like a 'real' force?

Not a Force. In physics, we defined 'forces' in order to quantify interactions – pulls and pushes. Mind you, this isn't a 'real' force in the sense that two parties aren't pulling or pushing on each other. For example, when you hit the gas and your phone goes flying off the dash where you'd set it, it's not because there's an invisible hand pushing it back, rather, it's because the dash accelerated out from under the phone.

$$m\ddot{\vec{r}}_o = 0$$

That's quite obvious to someone standing on the curb watching you zip by.

Like a Force. On the other hand, as measured by you who are accelerating along with the car, if you didn't know any better (say you're in the back of a windowless van), the phone flew back as if someone turned on a 'tractor beam' that pulled it.

$$m\ddot{\vec{r}} = \vec{F}_{net.eff}$$

$$m\ddot{\vec{r}} = 0 - \vec{F}_{frame}$$

So it *moves* relative to the *accelerating frame* as if there were this 'frame force.'

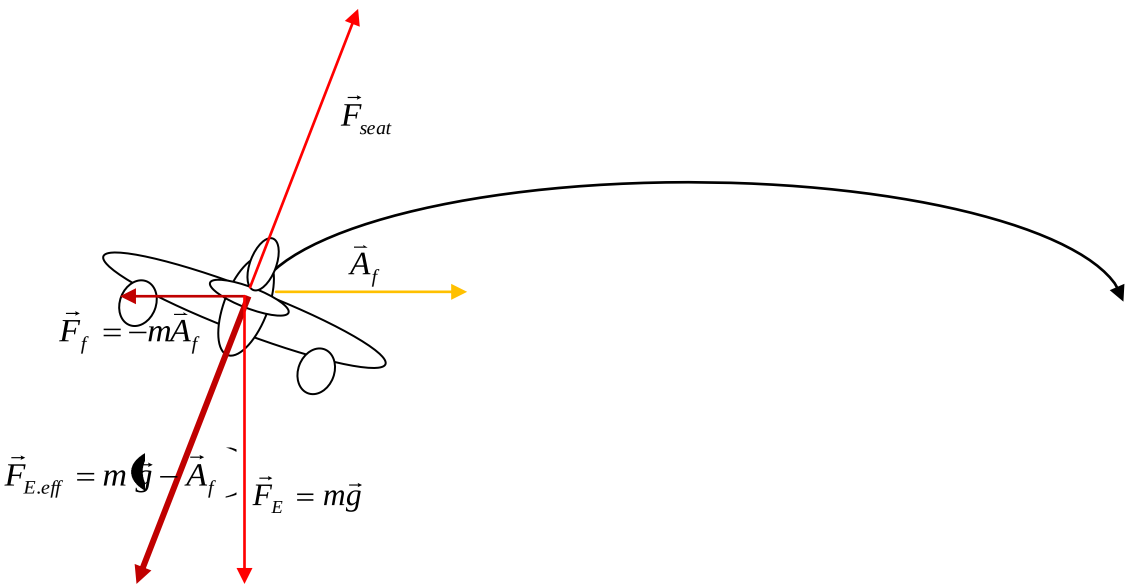
Note: of course, you too move in the car as if experiencing a frame force – everything measured against the accelerating frame does (thus calling it a 'frame' force). In your case, you and the seat press into each other as if the 'tractor beam' were pulling you back.

Frame Forces and Gravitation

Better yet, it's as if a gravitational field were pulling you back. Like the real gravitational force, the apparent frame force is proportional to mass (that, of course, is the only way that you could explain why all things, in spite of having different masses, would accelerate the same in response to the force)

$$\vec{F}_{frame} \equiv m\left(-\vec{A}_f\right)$$

has the same form as the force due to a uniform gravitational field. If you were in a plane in a cloud, the only sign you'd have that it was banking was that you felt 'heavier.'



General Relativity aside: First Einstein had tackled the 'special' case of relating measurements in two coordinate systems that were moving relative to each other but not accelerating; that was 'special relativity.' Then he wanted to develop the more 'general' relations between measurements made in two reference frames that were accelerating relative to each other; that was 'general relativity.' Why then do we associate 'general relativity' with the gravitational force? Because Einstein made this observation – the effect of gravitation is 'equivalent' to that of an accelerating frame's 'frame force.'

Steady State / Statics.

Just like in Intro Physics, the easiest situations to which we can apply the force – acceleration relation is when there is no acceleration measured in the non-inertial frame.

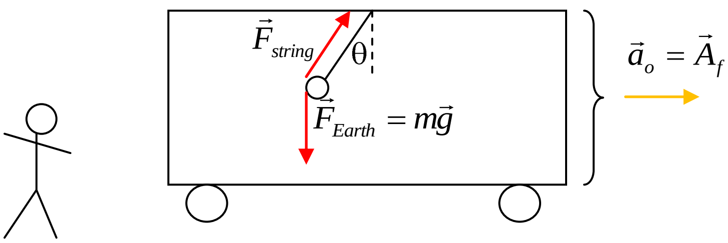
Example: boxcar pendulum

The book does this, but it's got a high enough pay-off/effort ratio that it's worth repeating.

A railroad car with a pendulum hanging from its ceiling (length L, mass m) accelerates forward at A_f. Let's say we know the mass of the ball and the acceleration of the car, then we want to solve for the force of the string and the angle at which the ball will hang.

I do

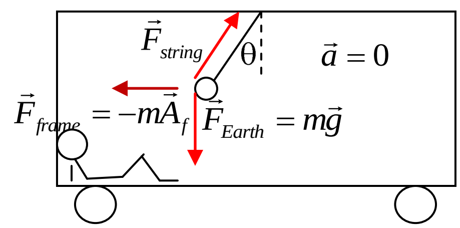
Inertial perspective



$$\vec{F}_{net} = m\vec{a}_o$$
$$\vec{F}_s + m\vec{g} = m\vec{A}_f$$
$$\hat{x} : F_s \sin\theta = mA_f$$
$$\hat{y} : F_s \cos\theta - mg = 0 \Rightarrow F_s \cos\theta = mg$$

I help draw, they do

Non-Inertial perspective



$$\vec{F}_{net.eff} = m\vec{a}$$
$$\vec{F}_s + m\vec{g} + \vec{F}_{frame} = 0$$
$$\vec{F}_s + m\vec{g} - m\vec{A}_f = 0$$
$$\hat{x} : F_s \sin\theta - mA_f = 0 \Rightarrow F_s \sin\theta = mA_f$$
$$\hat{y} : F_s \cos\theta - mg = 0 \Rightarrow F_s \cos\theta = mg$$

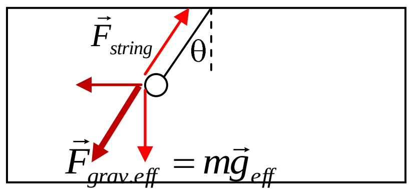
Angle: So, taking the ratio of the two equations tells us that

$$\frac{F_s \sin\theta}{F_s \cos\theta} = \frac{mA_f}{mg}$$
$$\tan\theta = \frac{A_f}{g}$$
$$\theta = \arctan\left(\frac{A_f}{g}\right)$$

Magnitude: Squaring and adding them gives

$$(F_s \sin\theta)^2 = (mA_f)^2$$
$$(F_s \cos\theta)^2 = (mg)^2$$
$$(F_s \sin\theta)^2 + (F_s \cos\theta)^2 = (mA_f)^2 + (mg)^2$$
$$F_s = m\sqrt{(A_f)^2 + (g)^2}$$

Effective gravitational force: this is the same result we'd get if there were a gravitational force $\vec{F}_{grav.eff} = m\vec{g}_{eff} = m(\vec{g} - \vec{A}_f)$ and $|F_{grav.eff}| = mg_{eff} = m\sqrt{(A_f)^2 + (g)^2}$



Period: While it isn't really that much harder to find the pendulum's frequency of oscillation about this angle than it is to find its period in an inertial (non-acclerating) frame,

Question: If the pendulum *weren't* in an accelerating box car, how would the frequency of its oscillation be related to g and L?

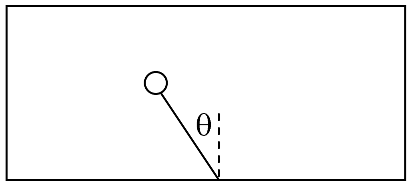
A: $\omega = \sqrt{\frac{g}{L}}$

Now, what's our new effective g_{eff}? And what's the frequency we'd expect in the accelerating box car?

We can skip to the result in this case, $\omega = \sqrt{\frac{g_{eff}}{L}} = \sqrt{\frac{\sqrt{(A_f)^2 + (g)^2}}{L}}$

Example: balloon.

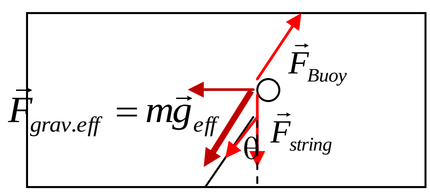
In problem 8.1, the book mentions a fun companion to this problem – a floating balloon. At first blush, you might expect it to hang like



But think for a minute about the forces that hold it in place. There's the gravitational force down, the 'frame force' back, the buoyant force…where *does* the buoyant force point?

Recall that in a non-accelerating situation, buoyancy results from the fact that density and pressure of the air increases as you decrease elevation (since part of that pressure is the result of all the air above bearing down on you thanks to the gravitational pull down on it), so just below a balloon the pressure is slightly greater than just above – the difference between these two pressures gives a net force up / the opposite direction of gravitation.

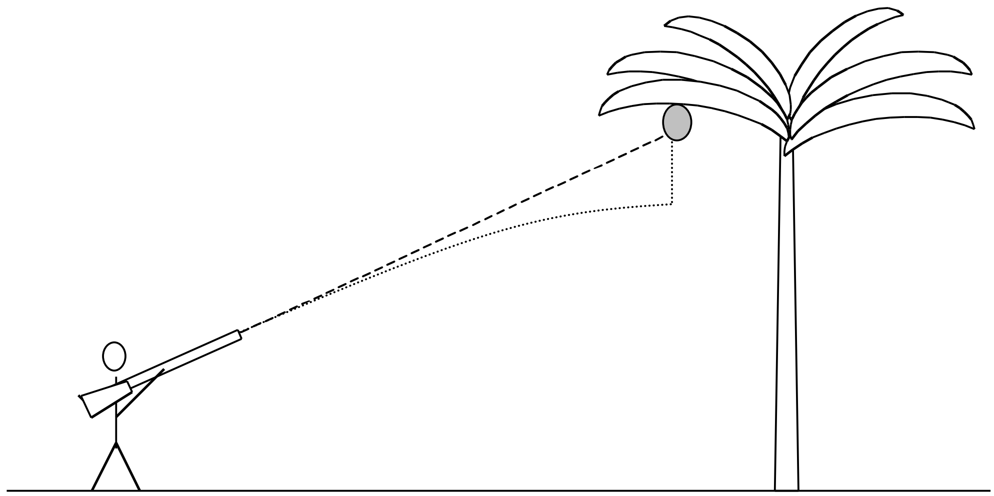
Now, in an accelerating box car there's also a pressure difference left to right since the trailing wall is pushing into the air in front of it while the leading wall is pulling away from the air behind it. In short, the air is *also* moving along with the accelerating reference frame, it *also* experiences an effective gravitational pull with $\vec{g}_{eff} = \left(\vec{g} - \vec{A}_f\right)$ and so the buoyant effect is that pressure decreases as you move up *and forward / the opposite direction of 'gravitation.'*



The balloon will lean *forward!*

One way to think of this is that the balloon is like a bubble, or a lack of mass. As all the mass in the air gets pushed down and left, that propagates the bubble up and right.

Example #1: Monkey & Hunter (the easy way!) A hunter shoots a bullet at a monkey in a tree holding a coconut. The initial velocity of the bullet is toward the monkey and coconut. At the instant the shot is fired, the monkey releases the coconut. Show that the bullet will hit the coconut.



In the hunter's (inertial) frame, $\vec{F}_b = m_b\vec{g}$ for the bullet and $\vec{F}_c = m_c\vec{g}$ for the coconut. We could show that if the initial velocity of the bullet is towards the coconut, they will collide as the coconut falls. However, this requires a bit of effort.

Ignoring air resistance, the acceleration of the coconut is $\vec{A}_f = \vec{g}$. In the coconut's (noninertial) frame, we must include the inertial force $\vec{F}_\mathrm{f} = -m_b\vec{A}_f = -m_b\vec{g}$ in Newton's second law for the bullet:

$$m_b\ddot{\vec{r}} = \vec{F}_b - m_b\vec{A}_f = m_b\vec{g} - m_b\vec{g} = 0$$

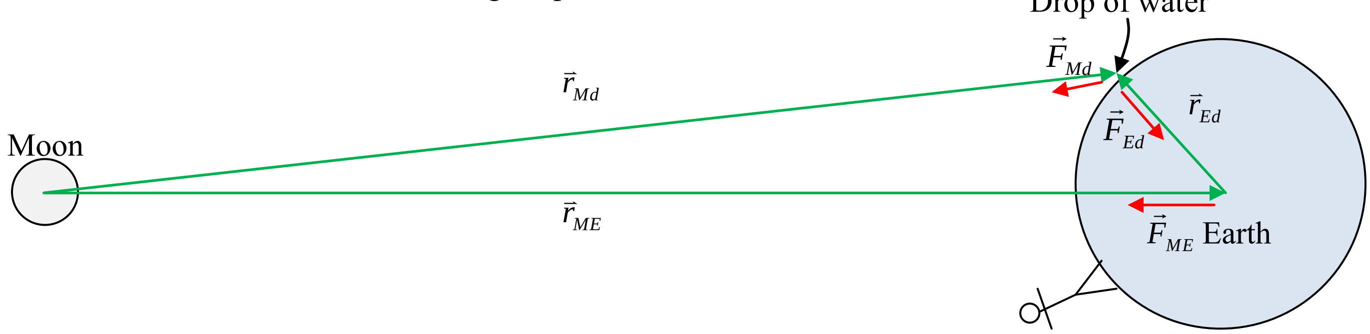
$$\dot{\vec{r}} = const$$

Therefore, if the bullet is initially going toward the origin of the coconut's frame, it will continue to go toward it.

9.2 Tides.

Now, there may be *some* cause for analyzing the motion of things in accelerating box cars, but by far the most commonly experienced accelerating reference frame is the Earth. It's spinning on its axis, it's orbiting the sun. It's ever-so-slightly wobbling in response to the moon's pull. We'll go after this last effect first.

Purely in terms of forces, the moon's gravitational pull decreases as you get further from it. That means that the near side of the Earth feels a greater gravitational pull than does the middle which feels a greater pull than does the far side. Since water is much free-er to deform than is solid rock, the result is that the water near the moon moves toward it a little, the solid earth beneath moves toward it a little less, and the water on the far side moves toward it still less – thus, relative to the solid earth, the water bulges up on both the near and far side.



Where

$$\vec{F}_{Md} = -G\frac{M_m m}{r_{Md}^2}\hat{r}_{Md} \qquad \vec{F}_{Ed} = -G\frac{M_E m}{r_{Ed}^2}\hat{r}_{Ed} \approx m\vec{g} \qquad \vec{F}_{ME} = -G\frac{M_E M_M}{r_{ME}^2}\hat{r}_{ME}$$

From an impartial, inertial observer's perspective, Newton's 2nd law applied to the water droplet is

$$\vec{F}_{net\to d} = m\vec{a}_o$$
$$\vec{F}_{Md} + \vec{F}_{Ed} + \vec{F}_{buoy} = m\vec{a}_o$$

Now you, riding on the Earth as you are, are *not* an impartial inertial observer. For the sake of this analysis, we'll just worry about how you and the Earth that you're riding accelerate toward the moon in response to its gravitational force, in particular,

$$\vec{A}_f = \vec{F}_{ME} = \left(-G\frac{M_E M_M}{r_{ME}^2}\hat{r}_{ME} \Big/ M_E \right) = -G\frac{M_M}{r_{ME}^2}\hat{r}_{ME}$$

So, looked at from your non-inertial perspective,

$$\vec{F}_{net\to d} = m\vec{a}_o = m\left(\vec{a} + \vec{A}_f\right)$$

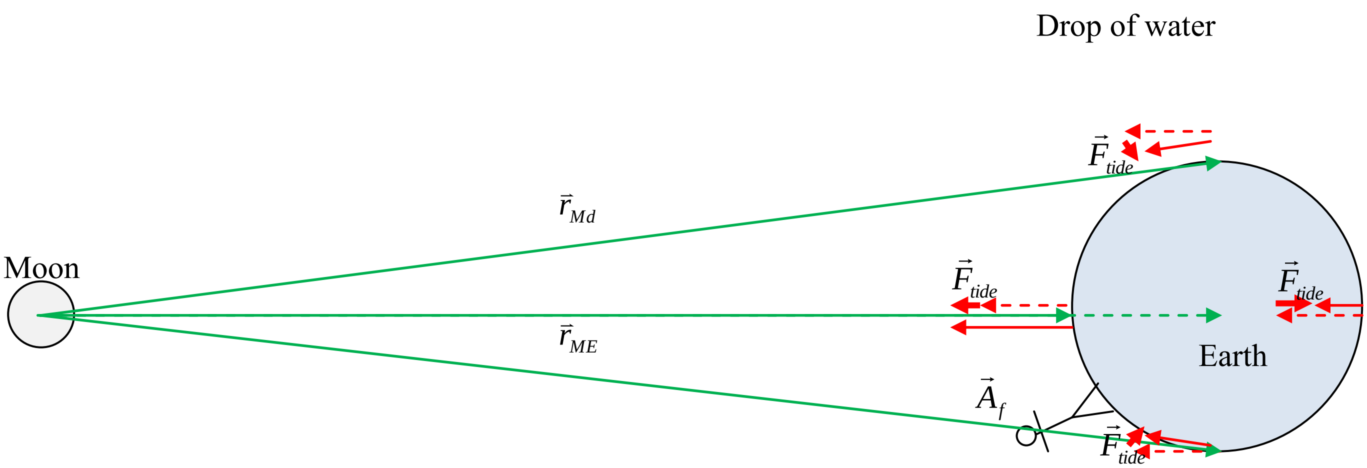
$$\vec{F}_{Md} + \vec{F}_{Ed} + \vec{F}_{buoy} - m\vec{A}_f = m\vec{a}$$

$$-G\frac{M_m m}{r_{Md}^2}\hat{r}_{Md} + m\vec{g} + \vec{F}_{buoy} - m\left(-G\frac{M_m}{r_{ME}^2}\hat{r}_{ME}\right) = m\vec{a}$$

$$\underbrace{-GmM_m\left(\frac{\hat{r}_{Md}}{r_{Md}^2} - \frac{\hat{r}_{ME}}{r_{ME}^2}\right)}_{\vec{F}_{eff.tidal}} + m\vec{g} + \vec{F}_{buoy} = m\vec{a}$$

$$\vec{F}_{eff.tidal} + m\vec{g} + \vec{F}_{buoy} = m\vec{a}$$

So the 'tidal' effect is due to that difference between what the moon's gravitational force on the water droplet *is* and what it *would be* if the droplet were at same location as the center of the Earth.



So, water near the moon gets pulled out, water far from the moon gets pushed out, and water half way between gets pushed inward.

Exercise: Find an approximate expression for the tidal force on a drop at the point nearest the moon (can use that $R_{dE} << R_{dM}$ and a binomial expansion)

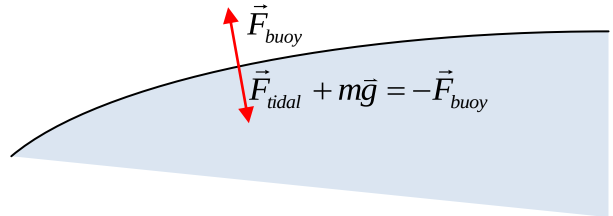
At this point, all forces are along the x-axis, and $R_{dM} = R_{ME} - R_E$.

$$\vec{F}_{tide} = -GmM_m\left(\frac{\hat{r}_{Md}}{r_{Md}^2} - \frac{\hat{r}_{ME}}{r_{ME}^2}\right)$$

$$\vec{F}_{tide} = -GmM_m\left(\frac{1}{r_{Md}^2} - \frac{1}{r_{ME}^2}\right)\hat{x} = -GmM_m\left(\frac{1}{\left(r_{ME} - r_E\right)^2} - \frac{1}{r_{ME}^2}\right)\hat{x} = -\frac{GmM_m}{r_{ME}^2}\left(\frac{1}{\left(1 - \frac{r_E}{r_{ME}}\right)^2} - 1\right)\hat{x}$$

$$\approx -\frac{GmM_m}{r_{ME}^2}\left(\left(1 + 2\frac{r_E}{r_{ME}}\right) - 1\right)\hat{x} = 2\frac{GmM_m}{r_{ME}^3}r_E\hat{x}$$

Magnitude of Tides



In equilibrium,

$$\vec{F}_{tidal} + m\vec{g} + \vec{F}_{buoy} = m\vec{a} = 0$$

$$\vec{F}_{tidal} + m\vec{g} = -\vec{F}_{buoy}$$

Now, the buoyant force of water on a droplet at the surface of the water is essentially a normal force – preventing the droplet from sinking. That means that the other two forces must, in equilibrium point normally *into* the surface of the water.

Now, if $\vec{F}_{tidal} + m\vec{g}$ is normal to the water surface, then if we imagine grabbing a droplet and dragging it *along* the surface, these two forces must do no work / constitute no change in their associated potential energy. So, if we consider the potential energies associated with then forces, then we can say the total potential at one location is equal to that at another. That will allow us to evaluate their elevations.

We can rewrite the forces as $\vec{F}_{tidal} + m\vec{g} = -\vec{\nabla}\left(U_{tide} + U_{Earth}\right)$ where

$$U_{tide} = -\int \vec{F}_{tide} \cdot d\vec{r} = \int \frac{GmM_m}{r_{Md}^2} dr_{Md} + \int \frac{GmM_m}{r_{ME}^2} dx = -\frac{GmM_m}{r_{Md}} - \frac{GmM_m}{r_{ME}^2} x$$

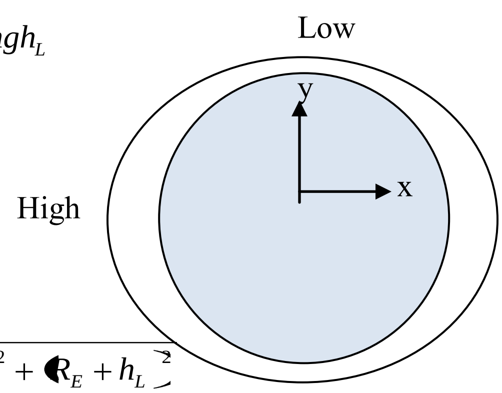
Varies as droplet moves

Constant, points in x direction

$$U_{tide} + U_{Earth} \equiv -GmM_m \left(\frac{1}{r_{Md}} + \frac{x}{r_{ME}^2} \right) + mgh$$

so, this must be constant at any two points on the water's surface, let's compare the high and low tide spots:

$$-GmM_m \left(\frac{1}{r_{Md.H}} + \frac{x_H}{r_{ME}^2} \right) + mgh_H = -GmM_m \left(\frac{1}{r_{Md.L}} + \frac{x_L}{r_{ME}^2} \right) + mgh_L$$



Or

$$-GmM_m \left(\left(\frac{1}{r_{Md.H}} - \frac{1}{r_{Md.L}} \right) + \left(\frac{\Delta x}{r_{ME}^2} \right) \right) = -mg\Delta h$$

Where

$$\Delta x = -\left(R_E + h_H\right) \quad r_{Md.H} = r_{ME} - \left(R_E + h_H\right) \quad r_{Md.L} = \sqrt{r_{ME}^2 + \left(R_E + h_L\right)^2}$$

Or to a pretty good approximation,

$$\Delta x = -\left(R_E\right) \quad r_{Md.H} = r_{ME} - \left(R_E\right) \quad r_{Md.L} = \sqrt{r_{ME}^2 + \left(R_E\right)^2}$$

$$\frac{GmM_m}{mg}\left(\left(\frac{1}{r_{ME}-R_E}-\frac{1}{\sqrt{r_{ME}^{\;2}+(R_E)^2}}\right)-\left(\frac{R_E}{r_{ME}^2}\right)\right)=\Delta h$$

Now, we could calculate at this point, but we could also simplify making use of the fact that r_{ME} $>>R_E$, and then do a binomial expansion on the two fractions.

Giving that,

$$\frac{GM_m}{g}\left(\left(\frac{1}{r_{ME}}\left(1-\frac{R_E}{r_{ME}}\right)^{-1}-\frac{1}{r_{ME}}\left(1+\left(\frac{R_E}{r_{ME}}\right)^2\right)^{-1/2}\right)-\left(\frac{R_E}{r_{ME}^2}\right)\right)=\Delta h$$

$$\frac{GM_m}{GM_E/R_E^{\;2}}\left(\left(\frac{1}{r_{ME}}\left(1+\frac{R_E}{r_{ME}}+\left(\frac{R_E}{r_{ME}}\right)^2\right)-\frac{1}{r_{ME}}\left(1-\tfrac{1}{2}\left(\frac{R_E}{r_{ME}}\right)^2\right)\right)-\left(\frac{R_E}{r_{ME}^2}\right)\right)\approx\Delta h$$

$$\frac{M_m}{M_E}\frac{R_E^{\;2}}{r_{ME}}\left(\left(\left(\left(\frac{R_E}{r_{ME}}\right)+\left(\frac{R_E}{r_{ME}}\right)^2+\tfrac{1}{2}\left(\frac{R_E}{r_{ME}}\right)^2\right)-\left(\frac{R_E}{r_{ME}}\right)\right)\right)\approx\Delta h$$

$$\tfrac{3}{2}R_E\frac{M_m}{M_E}\left(\frac{R_E}{r_{ME}}\right)^3\approx\Delta h$$

The book calculates this to be about 54cm.

Performing the same analysis for the sun's effect would go exactly the same; given that the sun is much further away, but also much more massive than the moon, you get a contribution of another 25cm.