Fri. 10/22	**6.1-.2** Calculus of Variations – Euler-Lagrange	
Mon.10/25 Tues 10/26	**6.3-.4** Applications and Complications	**HW6**

Introduction.

Now for something completely different…

In studying mechanics, how interactions impact motion, first you meet the momentum – force relation. That's a great tool for describing how motion is affected by interactions. However, if you integrate both sides of Newton's 2^{nd} Law, you get the energy-work relation; in spite of having no real *new* information input to it, this relation proves handier for addressing certain problems (and more awkward for addressing others.) Now we're going to take some time to develop yet another way of relating motion and interactions, yet another tool for tackling mechanics problems. This tool will be a better fit for some problems, and worse for others. We're just adding one more tool to your tool box.

Where we're headed is that the path an object takes through space is that which minimizes the momentum*the path length, the "action." In and of itself, that may seem pretty mysterious, but that's where we're going. The first step is figuring out how to minimize *anything* about a path (its length, the time it takes to travel it,…) and what information we can read out of a minimized path.

That's where Calculus of Variations comes in. We're going to spend two days on this before we move on to apply it to energy.

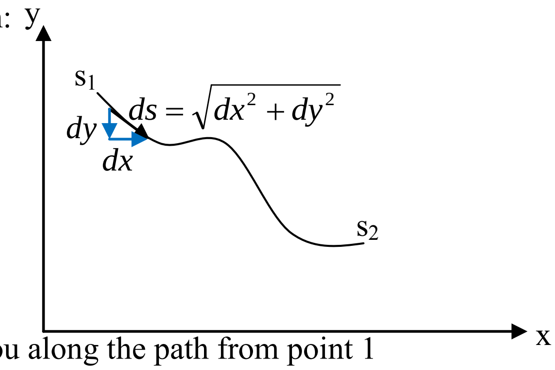
Calculus of Variations

Today will be an introduction to a subject called the *calculus of variations*. By way of motivation, the book poses two questions which can be answered by this technique (we will solve them next time.)

1. What is the equation of the shortest path between two points on a surface? In and of itself, the answer to this problem isn't too startling (the equation of a line), but it's a nice, simple test case.
2. What's the quickest path between two points if you're traveling through regions of different speed limits? (that's essentially the question Fermat answered to find the path a light ray follows through varying media).
3. Suppose you are given a starting point (x_1, y_1) and an ending point (x_2, y_2) in a plane where y is the vertical direction. If y_1 is higher than y_2, find the shape of the track between the points will give the shortest time for a particle sliding along it without friction. This is a little more complicated and contains some physics. It was also the problem that historically led to the development of this area of mathematics.

What do these problems have in common? In each case, the quantity that needs to be minimized is an *integral* related to the path, *not* the equation of the path itself (minimizing that would merely find its lowest point). For (1 & 2), the integral is over the length of small segments ds of the path. For (2), the integral is over the time intervals dt required to travel over small segments ds of the path. If the speed is *v*, which may depend on the position on the path, the time intervals are $dt = ds/v$.

It's easy to get lost in the book's derivation, and the result is important enough that it's worth understanding where it comes from. So, we'll retrace it's steps in deriving the *Euler-Lagrange equation*, which gives a differential equation for the desired path. We can define $y' = dy/dx$, so the integral that must be minimized (or maximized) is of the form:



Let's think about the case of minimizing the path length.

First off, the path length between two points is

$$S = \int_{s_1}^{s_2} ds$$

That's the sum of the length of each infintesmal step that takes you along the path from point 1 to point 2. On the infintesmal scale, you can rephrase that step length as $ds = \sqrt{dx^2 + dy^2}$. For that matter, you could rephrase the vertical change in terms of the curve's local slope, $y'(x) \equiv \frac{dy}{dx}(x)$, as $dy = y'(x)dx$, then the differential bit of length along the path could be rephrased as $ds = \sqrt{dx^2 + \left(y'(x)dx\right)^2} = \sqrt{1 + \left(y'(x)\right)^2}\,dx$, so

$$S = \int_{s_1}^{s_2} ds = \int_{x_1}^{x_2} \sqrt{1 + \left(y'(x)\right)^2}\,dx.$$

To consider an even more general problem, if we want to know the *time* it takes to travel between two points, through regions of varying speed, the integral would be

$$t = \int_{s_1}^{s_2} \frac{ds}{v(x,y)} = \int_{x_1}^{x_2} \left(\frac{\sqrt{1 + \left(y'(x)\right)^2}}{v(x,y)} \right) dx$$ (the velocity is written to emphasize that it may vary from one location to the next.)

If we define the integrand as function f, then it has functional dependence on x, y, and dx/dy (which is itself dependent on x).

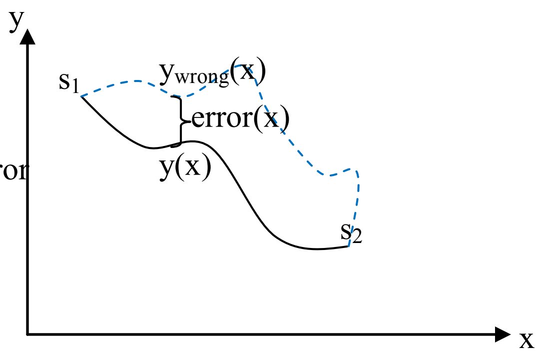
$$t = \int_{x_1}^{x_2} f(x, y, y'(x))\,dx$$

$$f(x, y, y'(x)) \equiv \left(\frac{\sqrt{1 + \left(y'(x)\right)^2}}{v(x,y)} \right)$$

The reason for choosing this example is that a) it's a reasonable thing to want to know and b) it has the kind of functional dependences characterize all the functions we'll be interested in.

Now, how do we go about *answering* the question? At this point, we'll start working formally / generally, and then we can return to specific questions. Imagine that, say *I* know the equation of the curve that minimizes this function, that's y(x), and *you* make a wild guess at the equation of the curve, $y_{wrong}(x)$. If I compare your guessed function and the correct function, I could define the difference between the two as

$$error(x) \equiv y_{wrong}(x) - y(x)$$



Now, since it will be useful later, I'll rewrite the error as $error(x) = \alpha\varepsilon(x)$, think of it as factoring out the error's amplitude if you want.

So, $y_{wrong}(x) = \alpha\varepsilon(x) + y(x)$

Now, if you went ahead and integrated the function to find, in this case the time to travel between the two points, you'd be integrating

$$t_{wrong} = \int_{x_1}^{x_2} f(x, y(x,\alpha)_{wrong}, y'_{wrong}(x,\alpha))dx$$

Where I've made explicit the *wrong* answer's dependence on α.

Now we're ready for the cute part. Imagine you've done the integral as I've laid it out (of course, if you *actually* knew the functional expression for the path, y(x), you wouldn't bother doing this *wrong* integral, but just bare with me.) Once you'd done it, you'd have the time expression as a function of alpha, and you want the *minimum* possible time, so you take the derivative and set it equal to 0. Then again, you *know* at what value of alpha the minimum occurs, when a=0 (since that would mean there'd be no difference between your guessed and the correct path, for which the time is minimized)

$$\left.\frac{dt_{wrong}(\alpha)}{d\alpha}\right|_{\alpha=0} = 0$$

$$= \int_{x_1}^{x_2} \left.\frac{df(x, y(x,\alpha)_{wrong}, y'_{wrong}(x,\alpha))}{d\alpha}\right|_{\alpha=0} dx$$

Now, the chain rule tells us that

$$\frac{d}{d\alpha} f(x, y(x,\alpha)_{wrong}, y'_{wrong}(x,\alpha)) = \frac{\partial f}{\partial x}\frac{dx}{d\alpha} + \frac{\partial f}{\partial y_{wrong}}\frac{dy_{wrong}}{d\alpha} + \frac{\partial f}{\partial y'_{wrong}}\frac{\partial y'_{wrong}}{d\alpha}$$

Where

$$y_{wrong}(x) = \alpha\varepsilon(x) + y(x) \text{ so } \frac{\partial}{\partial\alpha} y_{wrong}(x) = \varepsilon(x)$$

Similarly,

$$y'_{wrong}(x) = \alpha\varepsilon'(x) + y'(x) \text{ so } \frac{\partial}{\partial\alpha} y'_{wrong}(x) = \varepsilon'(x)$$

Then

$$\frac{d}{d\alpha} f(x, y(x,\alpha)_{wrong}, y'_{wrong}(x,\alpha)) = \frac{\partial f}{\partial x}0 + \frac{\partial f}{\partial y_{wrong}}\varepsilon(x) + \frac{\partial f}{\partial y'_{wrong}}\varepsilon'(x)$$

Next, recall that we're going to be evaluating this derivative for the known minimizing value of $\alpha = 0$ and *f* has the same functional dependence on y_{wrong} as and y'_{wrong} as it does on y and y', so we might as well rewrite these derivatives as

$$\frac{d}{d\alpha} f(x, y(x,\alpha)_{wrong}, y'_{wrong}(x,\alpha)) = \frac{\partial f}{\partial y}\varepsilon(x) + \frac{\partial f}{\partial y'}\varepsilon'(x)$$

Now that we've simplified the integrand as much as we can, let's return to the integral.

$$0 = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\varepsilon(x) + \frac{\partial f}{\partial y'}\varepsilon'(x)dx = \int_{x_1}^{x_2} \frac{\partial f}{\partial y}\varepsilon(x)dx + \int_{x_1}^{x_2} \frac{\partial f}{\partial y'}\varepsilon'(x)dx$$

Applying the chain rule on the second integral,

$$\int_{x_1}^{x_2} \underbrace{\frac{\partial f}{\partial y'}}_{u} \underbrace{\frac{d\varepsilon(x)}{dx} dx}_{dv} = \underbrace{\frac{\partial f}{\partial y'}}_{u} \underbrace{\varepsilon(x)}_{v}\Bigg|_{x_1}^{x_2} - \int_{x_1}^{x_2} \underbrace{\varepsilon(x)}_{v} \underbrace{\frac{d}{dx}\frac{\partial f}{\partial y'} dx}_{du}$$

Now, looking back at our sketch that compares the right and wrong paths, it's kind of a given that they both start and end at the same end points. So, regardless of the value of alpha,

$y_{wrong}(x_{1,2}) - y(x_{1,2}) = \alpha\varepsilon(x_{1,2}) = 0$. In other words, ε(x) is 0 at these points, that kills the first term.

$$\int_{x_1}^{x_2} \frac{\partial f}{\partial y'} \frac{d\varepsilon(x)}{dx} dx = -\int_{x_1}^{x_2} \varepsilon(x) \frac{d}{dx}\frac{\partial f}{\partial y'} dx$$

Returning to the full expression,

$$0=\int_{x_1}^{x_2}\frac{\partial f}{\partial y}\varepsilon(x)+\frac{\partial f}{\partial y'}\varepsilon'(x)dx=\int_{x_1}^{x_2}\frac{\partial f}{\partial y}\varepsilon(x)dx-\int_{x_1}^{x_2}\varepsilon(x)\frac{d}{dx}\frac{\partial f}{\partial y'}dx$$

$$0=\int_{x_1}^{x_2}\varepsilon(x)\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}\right]dx$$

Now, the only way to *ensure* that this integrand comes to 0 *regardless of the functional form of ε(x)* is if it's multiplying by 0, i.e., if

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\frac{\partial f}{\partial y'}=0$$

That is our result.

Then, for example, quickest path between two points, across regions of different speed limits, is the path for which this differential equation is satisfied.

$$t=\int_{x_1}^{x_2}f(x,y,y'(x))dx$$

$$f(x,y,y'(x))\equiv\left(\frac{\sqrt{1+\left(y'(x)\right)^2}}{v(x,y)}\right)$$

Example (choose a problem not assigned)

Monday we'll do something with this.

Next two classes:

- Monday – Applications of Calculus of Variations
- Wednesday – start Ch. 7