Mon. 10/15	**5.5-.6** Resonance	
Tues. 10/16		**HW5b** (5.26-.43)
Wed. 10/17	**5.7 -.8** Fourier Series	
Thurs 10/18		**HW5c** (5.46-.52), **Project Bibliography**
Fri. 10/19	**6.1-.2** Calculus of Variations – Euler-Lagrange	

Restructure for next year: Damped but not driven first day; Driven and Resonance next day, Resonance & Fourier the final day.

Equipment

- White boards and pens
- Harmonic Oscillator python
- Alan's resonance demo – three masses on wires of different lengths
- Mass with two springs, post and driver
- String set up for resonating
- ladder

Examples and Exercises:

Before plowing ahead, I want to take a moment to pause and work with what we've met so far. Here are the key relations:

Euler's Relations

$$\frac{e^{ix}+e^{-ix}}{2}=\cos x \qquad \frac{e^{ix}-e^{-ix}}{2i}=\sin x$$

Taylor Series

$$f(x)=f(x_o)+\left.\frac{df(x)}{dx}\right|_{x_o}(x-x_o)+\tfrac{1}{2}\left.\frac{d^2f(x)}{dx^2}\right|_{x_o}(x-x_o)^2+\tfrac{1}{3!}\left.\frac{d^3f(x)}{dx^3}\right|_{x_o}(x-x_o)^3+\ldots$$

Hook's law

$$F_x(x)=m\ddot{x}=-k(x-x_{eq}) \quad U(x)=\tfrac{1}{2}k(x-x_{eq})^2 \quad x(t)=A\cos(\omega t-\delta) \quad \omega_o=\sqrt{\frac{k}{m}}=\frac{2\pi}{T}$$

(Linear) Damped Oscillator

$$\ddot{x}+2\beta\dot{x}+\omega_o^2x=0 \qquad x_h(t)=\begin{cases} e^{-\beta t}A\cos(\omega_1 t-\delta) & \text{under damped} \\ e^{-\beta t}\left(C_1e^{\sqrt{\beta^2-\omega_o^2}\cdot t}+C_2e^{-\sqrt{\beta^2-\omega_o^2}\cdot t}\right) & \text{over damped} \\ e^{-\beta t}(A+Bt) & \text{critically damped}\end{cases} \quad \text{where } \omega_1=\sqrt{\beta^2-\omega_o^2}$$

(Linearly) Damped & Driven Oscillator

$\ddot{x}+2\beta\dot{x}+\omega_o^2 x = f(t)$ with $f(t) = f_o \sin(\omega_D t)$

Okay, where we left off last time was talking about the damped-driven oscillator. We'd made a reasonable guess and found it to be right!

$$x_{driven}(t) = A\cos(\omega_D t - \delta)$$

$$A = \frac{f_o}{\sqrt{(\omega_o^2 - \omega_D{}^2)^2 + (2\beta\omega_D)^2}} \qquad \delta = \arctan\left(\frac{2\beta\omega_D}{\omega_o^2 - \omega_D{}^2}\right)$$

However, a *general* solution to this second-order differential equation should have two free parameters (for inputting initial position and velocity), but our solution has *no* free parameters; even the amplitude is completely determined by the strength of the driving force, and the frequencies and damping constant.

Unfortunately, as good a solution as it is, it has *no* free parameters.

But that's not the completely general solution yet. Then again, recall that the solutions for the *un* driven equation, if plugged into the left hand side will give 0. Which means that we can add them to this solution and get another solution!

That is

$$\ddot{x}_{un} + 2\beta\dot{x}_{un} + \omega_o^2 x_{un} = 0$$

Where we'd found that $x_{un}(t) = e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_o^2}\cdot t} + C_2 e^{-\sqrt{\beta^2-\omega_o^2}\cdot t}\right)$

And now we've found that

$$\ddot{x}_{driv} + 2\beta\dot{x}_{driv} + \omega_o^2 x_{driv} = f_o\cos(\omega_d t)$$

Where we've now found that $x_{driv}(t) = A\cos(\omega t - \delta)$

But adding these two equations together, we'd also get that

$$(\ddot{x}_{driv} + \ddot{x}_{un}) + 2\beta(\dot{x}_{driv} + \dot{x}_{un}) + \omega_o^2(x_{driv} + x_{un}) = f_o\cos(\omega_d t) + 0$$

So, apparently, $(x_{driv} + x_{un})$ is *also* a solution to this driven case. *That* is our most general solution.

$$x(t) = x_{driv} + x_{un} = A\cos(\omega t - \delta) + e^{-\beta t}\left(C_1 e^{\sqrt{\beta^2-\omega_o^2}\cdot t} + C_2 e^{-\sqrt{\beta^2-\omega_o^2}\cdot t}\right)$$

The book couches this result in more general differential-equations language; for many of you, I hope that helps to make connections, and understand this better. I've avoided using that language (homogeneous, inhomogeneous, particular,…) in case it was more useful just focusing on the specific case at hand.

In the formal language of Differential Equations, what I just argued is that if you have a particular, linear differential equation,

$$\hat{D}x_p(t) = f(t) \quad \text{(using the Quantum notation of ‘hat’ for an operator)}$$

Then the general solution is the linear combination of a *particular* solution to this equation, $x_p(t)$ and the solutions to this simpler “homogenious” equation

$$\hat{D}x_h(t) = 0$$

Since

$$\hat{D}(x_p + x_h) = \hat{D}(x_p) + \hat{D}(x_h) = f(t) + 0$$

The homogeneous solution determined by the initial conditions must be added to this to get the general solution. Note that the homogeneous solution decays, so it can also be called the *transient solution*. Only the early motion of the oscillator depends on how it starts out. The particular solution is an oscillatory solution with the same frequency as the driving frequency, which can also be called the *steady-state solution*. The motion for large times only depend on the parameters of the system (including the driving force), not the initial conditions.

Example: (similar to Ex. 5.3) Suppose $\omega_o = 10\pi$ rad/s, $\beta = \omega_o/20 = \pi/2$ rad/s, $f_0 = 1000\text{ m/s}^2$, and $\omega = 4\pi$ rad/s (only difference from Ex. 5.3). If the oscillator starts at rest at the orign, find and plot the function for position as a function of time. Compare with the results for Ex. 5.3.

The frequency for the undriven oscillator (and the homogeneous solution) is:

$$\omega_1 = \sqrt{\omega_o^2 - \beta^2} = \sqrt{(10\pi)^2 - (\pi/2)^2} = 9.987\pi .$$

The amplitude of the particular solution is:

$$A = \frac{f_o}{\sqrt{(\omega_o^2 - \omega^2)^2 + 4\beta^2\omega^2}} = \frac{1000\text{ m/s}^2}{\pi^2\text{ rad/s}^2\sqrt{(10^2 - 4^2)^2 + 4(1/2)^2(4)^2}} = 1.177\text{ m},$$

and the phase angle is:

$$\delta = \tan^{-1}\left(\frac{2\beta\omega}{\omega_o^2 - \omega^2}\right) = \tan^{-1}\left(\frac{2(\pi/2)(4\pi)}{(10\pi)^2 - (4\pi)^2}\right) = 0.0465\text{ radians}.$$

The general solution for an underdamped, driven oscillator can be written as:

$$x(t) = A\cos(\omega t - \delta) + e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right],$$

where the coefficients B_1 and B_2 must be determined from the initial conditions $x_o = v_o = 0$. From the equation above:

$$x_o = A\cos(-\delta) + B_1,$$

$$B_1 = x_o - A\cos\delta = 0 - (1.177\text{ m})\cos(0.0465\text{ rad}) = -1.176\text{ m}.$$

Taking the derivative of $x(t)$ gives:

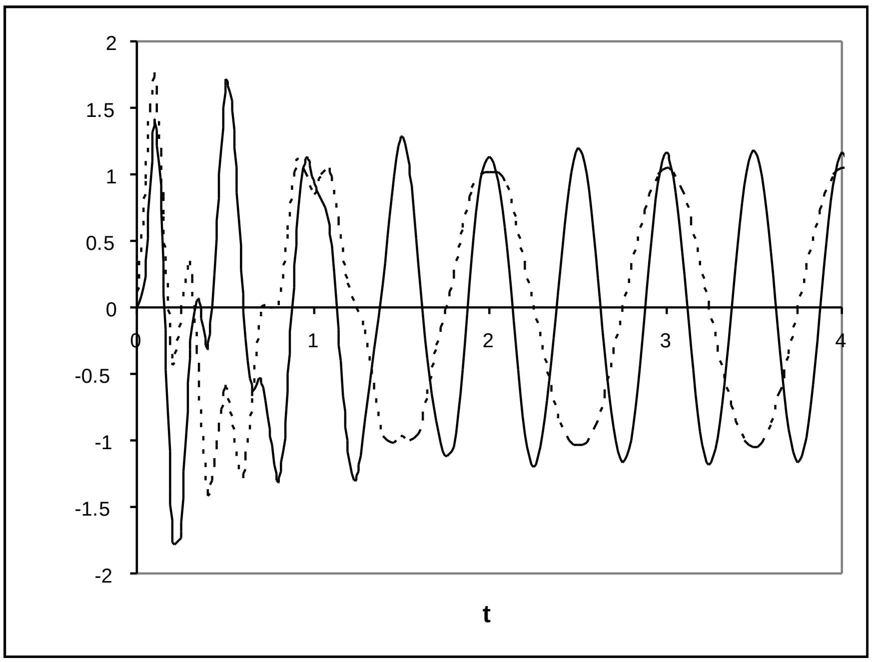
$$v(t) = -\omega A\sin(\omega t - \delta) - \beta e^{-\beta t}\left[B_1\cos(\omega_1 t) + B_2\sin(\omega_1 t)\right] + \omega_1 e^{-\beta t}\left[-B_1\sin(\omega_1 t) + B_2\cos(\omega_1 t)\right],$$

$$v_o = -\omega A\sin(-\delta) - \beta B_1 + \omega B_2,$$

$$B_2 = \frac{1}{\omega_1}\left(v_o - \omega A\sin\delta + \beta B_1\right) = \frac{1}{9.987}\left[0 - 4(1.177\text{ m})\sin(0.0465\text{ rad}) + (1/2)(-1.176\text{ m})\right],$$

$$B_2 = -0.807\text{ m}.$$

The graph of the solution is shown below (solid line) along with the solution of Ex. 5.3 (dashed line) where the driving frequency is $\omega = 2\pi\,\text{rad/s}$. The steady state solution for this example has a slightly larger amplitude because the driving frequency is closer to the natural frequency. It also lags a little farther behind the driving force.



Resonance:

As you're familiar, if an oscillating system is driven at the *right* frequency, its amplitude gets quite large – resonance.

Demo:

Alan's resonance demo (three masses on bars)

Mass on spring dirving

Qualitatively, we say that a system is in resonance when it has the most energetic response to a driving force – that is, it's oscillating with the greatest amplitude, it's moving with the greatest speed. Now we'll explore resonance in a damped-driven harmonic oscillator.

From last time, the steady state (large time) solution for a damped oscillator with driving force per unit mass of $f(t) = f_o \cos(\omega t)$ is:

$$x_p(t) = A\cos(\omega t - \delta),$$

where the phase shift of the steady state oscillation relative the driving force is:

$$\delta = \tan^{-1}\left(\frac{2\beta\omega_D}{\omega_o^2 - \omega_D{}^2}\right).$$

and the amplitude A is given by:

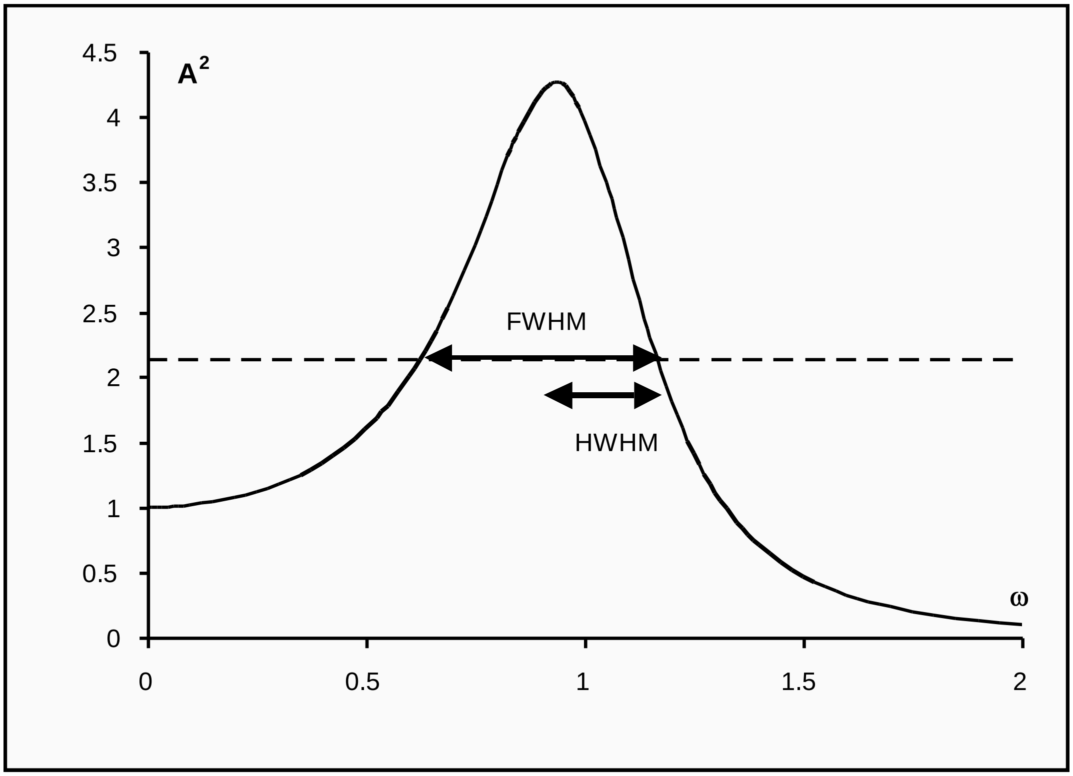
$$A = \frac{f_o}{\sqrt{\left(\omega_o^2 - \omega_D{}^2\right)^2 + 4\beta^2\omega_D{}^2}},$$

Recall that the natural frequency is $\omega_o = \sqrt{k/m}$ and the damping constant is $\beta = b/2m$.

Now, often one is interested in the energy associated with a resonance, and since energy goes like $\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$, the square of the amplitude is relevant to that question – maximize that, and you've maximized the energy in the system.

$$A^2 = \frac{f_o^2}{\left(\omega_o^2 - \omega_D{}^2\right)^2 + 4\beta^2\omega_D{}^2}$$

We'll first get familiar with how the amplitude depends upon the driving frequency. A plot would look something like this:



Generally, when you've got a plot that's peaked like this, whether it's a resonance plot, or a plot of car accidents as a function of age, there are three quantities that roughly characterize it –

- **where** the peak is
- how **high** the peak is
- how **wide** it is.

Where is the Peak?

The dramatically larger response when the system is driven at the right frequency is called *resonance*. It is easy to see that the amplitude gets largest when $\omega_D \approx \omega_o$ because the first term in the denominator gets small. For the location, maximize the function (take its derivative and set it equal to 0).

The exact frequency resulting in the maximum amplitude is found by locating the minimum of the denominator:

$$\left(\omega_o^2 - \omega_D{}^2\right)^2 + 4\beta^2\omega_D{}^2,$$

with respect to *ω*. This gives:

$$0=\frac{d}{d\omega_D}\left[\left(\omega_o^2-\omega_D{}^2\right)^2+4\beta^2\omega_D{}^2\right]=2\left(-2\omega_D\right)\left(\omega_o^2-\omega_D{}^2\right)+8\beta^2\omega_D,$$

$$\begin{aligned}0&=-\left(\omega_o^2-\omega_D{}^2\right)+2\beta^2\\ \omega_D{}^2&=\omega_o^2-2\beta^2\end{aligned},$$

so the frequency that gives the largest amplitude (maximum response) is:

$$\omega_{res}=\sqrt{\omega_o^2-2\beta^2}.$$

(interesting that , if $\sqrt{2}\beta>\omega_o>\beta$, then in a peculiar regime – underdamped, but resonance frequency is imaginary? This must be signaling that if b is big enough (but not too big), you can get oscillation, but no resonance peak; the curve monatomically decays toward 0)

How High is the Peak?

Plugging that back in gives the peak height.

$$A^2=\frac{f_o^2}{\left(\omega_o^2-\omega_{res}{}^2\right)^2+4\beta^2\omega_{res}{}^2}$$

$$A^2=\frac{f_o^2}{\left(\omega_o^2-\sqrt{\omega_o^2-2\beta^2}^{\,2}\right)^2+4\beta^2\sqrt{\omega_o^2-2\beta^2}^{\,2}}$$

$$A^2=\frac{f_o^2}{4\beta^2\left(\omega_o^2-\beta^2\right)}$$

(recall, we are considering the under-damped driven case, so $\omega_o>\beta$.)

If the damping constant β is small compared to the natural frequency ω_o, then $\omega_{res}\approx\omega_o$ and the maximum amplitude-squared is:

$$A_{max}{}^2\approx\left(\frac{f_o}{2\beta\omega_o}\right)^2.$$

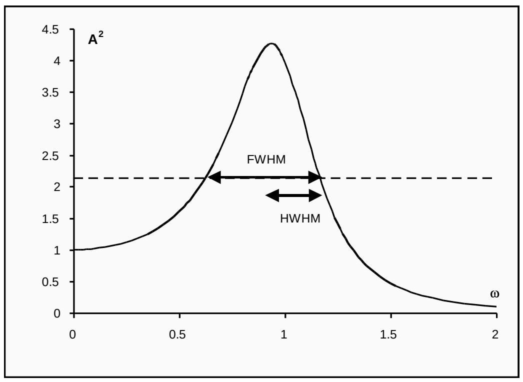
That tells us in a quantitative way what we qualitatively would expect: a smaller damping constant β leads to a larger amplitude and a frequency closer to the natural frequency.

So, we've identified the frequency and amplitude of the peak. What remains is characterizing how 'wide' it is.

How wide is it?

Of course, nothing but a rectangular peak has a *set* width, so it's sort of a matter of convention what mathematical representation we us for characterizing a peak's width. The common convention is the *full width of the peak half-way up towards is maximum*. That's a nice, meaningful convention since, regardless of the functional form of a peak, you can calculate this and it reflects whether the peak's aspect ratio is quite broad or sharp.

The *full width at half maximum* (FWHM) and the *half width at half maximum* (HWHM) of the amplitude <u>squared</u> are defined as shown below.



We can find these widths approximately (Prob. 5.41). The maximum of the amplitude squared is approximately:

$$A_{\max}^2 \approx \frac{f_o^2}{4\beta^2\omega_o^2},$$

so the amplitude squared is half as large when the denominator is twice as large:

$$\tfrac{1}{2}A_{\max}^2 \approx \frac{1}{2}\frac{f_0^2}{4\beta^2\omega_o^2} \approx \frac{f_0^2}{\left(\omega_o^2-\omega_{\frac{1}{2}}^2\right)^2+4\beta^2\omega_{\frac{1}{2}}^2},$$

Focusing on the denominator,

$$\left(\omega_o^2-\omega_{\frac{1}{2}}^2\right)^2 \approx 4\beta^2\omega_o^2.$$

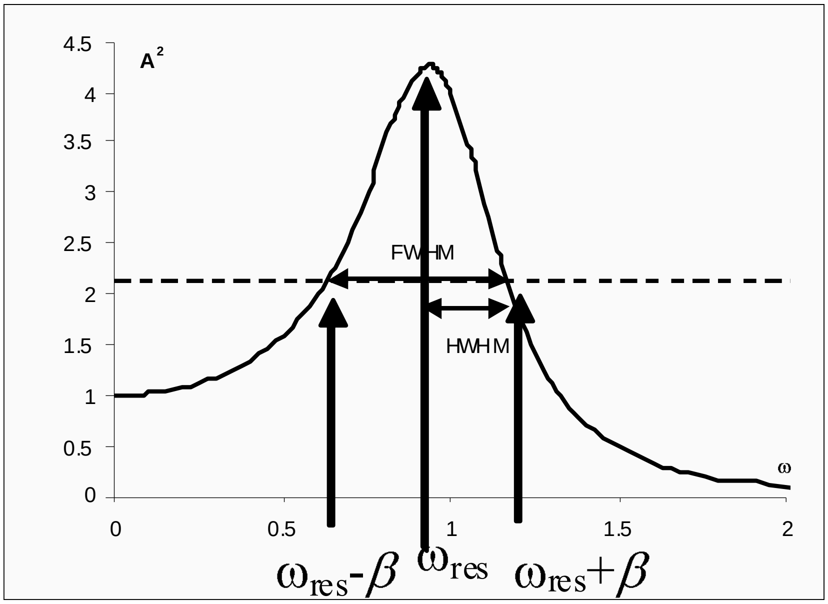
Take the square root and factor the left side:

$$\left(\omega_o+\omega_{\frac{1}{2}}\right)\left(\omega_o-\omega_{\frac{1}{2}}\right) \approx \pm 2\beta\omega_o.$$

The first term can be approximated as $\left(\omega_o+\omega_{\frac{1}{2}}\right) \approx 2\omega_o$, so:

$$\omega_o-\omega_{\frac{1}{2}} \approx \pm\beta,$$

$$\omega_{\frac{1}{2}} \approx \omega_o \mp \beta.$$



More precisely,

$$A^2{}_{1/2} = \frac{f_o^2}{\left(\omega_o^2 - \omega_{1/2}{}^2\right)^2 + 4\beta^2\omega_{1/2}{}^2} = \frac{1}{2}A^2{}_{res} = \frac{f_o^2}{8\beta^2\left(\omega_o^2 - \beta^2\right)}$$

$$\left(\omega_o^2 - \omega_{1/2}{}^2\right)^2 + 4\beta^2\omega_{1/2}{}^2 = 8\beta^2\left(\omega_o^2 - \beta^2\right)$$

$$\omega_o^4 - 2\omega_o^2\omega_{1/2}{}^2 + \omega_{1/2}{}^4 + 4\beta^2\omega_{1/2}{}^2 = 8\beta^2\left(\omega_o^2 - \beta^2\right)$$

$$\omega_{1/2}{}^4 - 2\left(\omega_o^2 - 2\beta^2\right)\omega_{1/2}{}^2 - 8\beta^2\left(\omega_o^2 - \beta^2\right) + \omega_o^4 = 0$$

$$\omega_{1/2}{}^2 = \left(\omega_o^2 - 2\beta^2\right) \pm \sqrt{\left(\omega_o^2 - 2\beta^2\right)^2 + \left(8\beta^2\left(\omega_o^2 - \beta^2\right) - \omega_o^4\right)}$$

$$\omega_{1/2}{}^2 = \omega_o^2 - 2\beta^2 \pm \sqrt{-4\omega_o^2\beta^2 + 4\beta^4 + 8\beta\omega_o^{2\,2} - 8\beta^4} =$$

$$\omega_{1/2}{}^2 = \omega_o^2 - 2\beta^2 \pm 2\beta\sqrt{\omega_o^2 - \beta^2}$$

$$\omega_{1/2}{}^2 = \omega_{res}^2 \pm 2\beta\sqrt{\omega_{res}^2 + \beta^2}$$

$$\omega_{1/2} = \sqrt{\omega_{res}^2 \pm 2\beta\sqrt{\omega_{res}^2 + \beta^2}}$$

So the full width would be the difference between these two options

$$\Delta\omega_{1/2} = \sqrt{\omega_{\text{res}}^2 + 2\beta\sqrt{\omega_{\text{res}}^2 + \beta^2}} - \sqrt{\omega_{\text{res}}^2 - 2\beta\sqrt{\omega_{\text{res}}^2 + \beta^2}}$$

$$\Delta\omega_{1/2} = \omega_{\text{res}}\left(\sqrt{1 + 2\frac{\beta}{\omega_{\text{res}}}\sqrt{1 + \frac{\beta^2}{\omega_{\text{res}}^2}}} - \sqrt{1 - 2\frac{\beta}{\omega_{\text{res}}}\sqrt{1 + \frac{\beta^2}{\omega_{\text{res}}^2}}}\right)$$

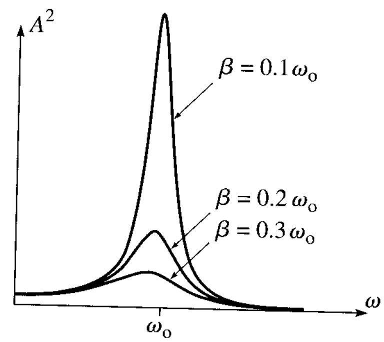
The HWHM of A^2 (which is related to energy) is:

$$HWHM \approx \beta,$$

and the FWHM is:

$$FWHM \approx 2\beta.$$

The smaller the damping constant, the sharper the resonance curve (see the figure below).



Quality Factor: Now, if our system had a resonance frequency of 10 Hz and a peak width of 5 Hz, we'd say the peak was pretty broad, but if our system had a resonance of 10 MHz and still a peak width of just 5 Hz, we'd say *that* was pretty sharp – drawn on an axis that stretches from 0 to say 12 MHz, that peak would look like a single spike of no width at all, a delta function. The point is that sometimes what matters isn't the absolute width, but the *relative* width of a peak.

Relative Width $= \dfrac{FWHM}{peak.value}$

Or phrased the other way around, we have what's known as the "quality factor", essentially a measure of the peak's relative sharpness

Quality Factor $= \dfrac{peak.value}{FWHM}$

Before I make this specific to our case, I'll point out that this idea of a "quality factor" comes up with any kind of peak – perhaps a histogram of how long it takes a ball to roll down an inclined plane (first couple of labs of Phys 233).

Okay, for the situation at hand, we're talking driving frequencies for a simple harmonic resonator; the peak is around $\approx \omega_o$ and the width is around $\approx 2\beta$:

$$Q \approx \frac{\omega_o}{2\beta}.$$

So, the less damping, the sharper the peak.

Conceptually, we can see this as the result of the interplay between oscillation and decay. The time for the oscillator to decrease in amplitude by a factor of 1/e, when not driven, is:

$$(\text{decay time}) = 1/\beta.$$

The period for one oscillation is:

$$\text{period} = 2\pi/\omega_1 \approx 2\pi/\omega_o.$$

Therefore, the quality factor can be written as:

$$Q = \frac{\omega_o}{2\beta} = \pi\frac{1/\beta}{2\pi/\omega_o} = \pi\frac{(\text{decay time})}{(\text{period})},$$

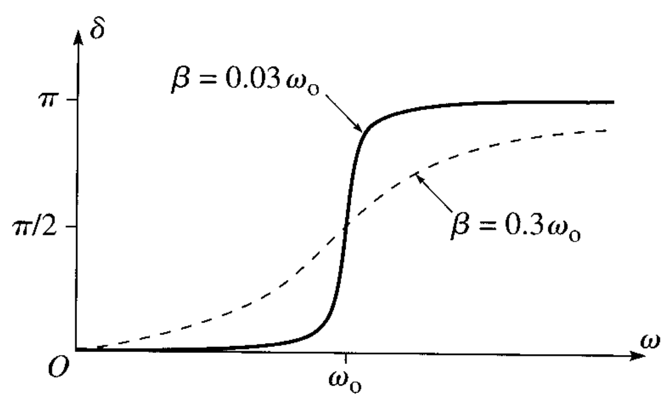
which is π times the number of oscillations in one decay time.

<u>Phase</u>:

Now, the amplitude isn't the only thing that varies according to the driving frequency, so does the phase. The phase shift of the steady state oscillation relative the driving force is:

$$\delta = \tan^{-1}\left(\frac{2\beta\omega_D}{\omega_o^2 - \omega_D^{\ 2}}\right).$$

This is a measure of how much the oscillator lags behind the driving force. If $\omega_D << \omega_o$, then δ is very small. That's when the driving frequency is much slower than the natural frequency – then the object just follows along. At $\omega = \omega_o$, the argument is infinite and $\delta = \pi/2$. The smaller β (or larger Q), the sharper the transition in the phase from 0 to $\pi/2$.



At the exact resonance, $\omega_{res} = \sqrt{\omega_o^2 - 2\beta^2}$

$$\delta = \tan^{-1}\left(\frac{2\beta\sqrt{\omega_o^2 - 2\beta^2}}{\omega_o^2 - \sqrt{\omega_o^2 - 2\beta^2}^{\,2}}\right) = \tan^{-1}\left(\frac{2\beta\sqrt{\omega_o^2 - 2\beta^2}}{2\beta^2}\right) = \tan^{-1}\left(\sqrt{\left(\frac{\omega_o}{\beta}\right)^2 - 2}\right)$$

So the phase isn't exactly π/2 (unless $\sqrt{2}\beta = \omega_o$) Again, we get a mathematical red flag that there is no resonance if β is big enough, but not too big, $\sqrt{2}\beta > \omega_o > \beta$; the phase is imaginary 'at resonance.')

> <u>Example</u>: (Prob. 5.42) Suppose a Foucault pendulum swings for about 8 hours before decreasing in amplitude by 1/e. If the length of the pendulum is 30 meter, what is the quality factor Q?
>
> Since oscillates many times, the pendulum is underdamped so the decay parameter is β and the amplitude decreases as $e^{-\beta t}$. If it takes 8 hours to decrease by 1/e, then:
>
> $$\beta(8\text{ hr}) = 1,$$
>
> $$\beta = (1/8\text{ hr})(1\text{ hr}/3600\text{ s}) = 3.47\times10^{-5}\text{ Hz}.$$
>
> The natural frequency of a pendulum is:
>
> $$\omega_o = \sqrt{g/L} = \sqrt{(9.8\text{ m/s}^2)/30\text{ m}} = 0.572\text{ Hz}.$$
>
> The quality factor is:
>
> $$Q = \frac{\omega_o}{2\beta} = \frac{0.572\text{ Hz}}{2(3.47\times10^{-5}\text{ Hz})} = 8.2\times10^3 \approx 8000.$$

Finally, I want to point out that, while we've been explicitly considering the case of a simple harmonic oscillator, the ideas and results largely hold for a *system* of *coupled* harmonic oscillators – imagine two masses joined by a spring, or three masses joined by springs, or… That's the subject of Chapter 11, which we'll not get to in this class, but you may find interesting reading. The main difference is that, for each mass you add to the system, you add a new resonance frequency. Of course, this relates to the system Francis and Michael studies over the summer. If you imagine adding more and more masses until eventually you can imagine a virtual continuum of masses joined by springs – i.e., atoms forming a string.

Demo. Drive oscillations on a string