Mon. 10/1 Tues. 10/2 Wed. 10/3 Fri. 10/5	**4.7-.8** Curvilinear 1-D, Central Force **5.1-.3 (2.6)** Hooke's Law, Simple Harmonic (Complex Sol'ns) ***What (research) I Did Last Summer****: AHoN 116 @ 4pm* Review for Exam 1	 **HW4b**
Mon. 10/8 Wed. 10/10 Thurs 10/11 Fri., 10/12	**Study Day** **Exam 1**(Ch 1-4) **5.4-.5** Damped & Driven Oscillations	 **HW5a**

Equipment

- Solid as coupled harmonic oscillators movie
- Mass on spring with force probe
- 3-D simple harmonic oscillator python
- Physical ball-spring model
- LabPro, motion sensor, force probe and mass on spring hanging from rod

Reminder about this afternoon at 4pm

Introduction

With Chapter 5 we move on to yet one more familiar subject – simple harmonic oscillator. However, as usual, we'll be building up to a more detailed treatment than back in 231. We'll soon be considering the driven, dampened harmonic oscillator.

But first, we'll cover some old ground and start using some new tools for this job – complex exponentials.

Taylor Series Refresher

As you're familiar,

The Taylor Series of a function, $f(x)$ expanded around 0 is

$$f(x) = f(x_o) + \left.\frac{df(x)}{dx}\right|_{x_o}(x - x_o) + \tfrac{1}{2}\left.\frac{d^2 f(x)}{dx^2}\right|_{x_o}(x - x_o)^2 + \tfrac{1}{3!}\left.\frac{d^3 f(x)}{dx^3}\right|_{x_o}(x - x_o)^3 + \ldots$$

$$= f(x_o) + \sum_{n=1} \tfrac{1}{n!}\left.\frac{d^n f(x)}{dx^n}\right|_{x_o}(x - x_o)^n$$

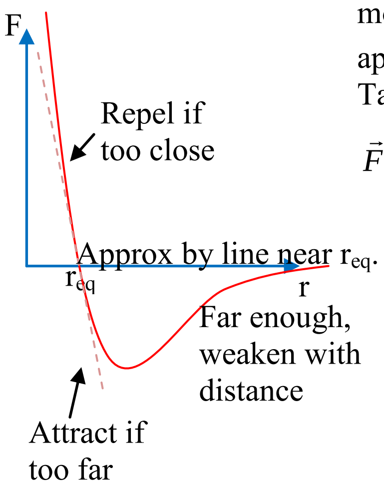
We're going to use this twice today.

Hooke's Law:

When you first met Hook's Law, it may have seemed like nice enough, but fairly limited in applicability. Even though it's not a "fundamental" force law, it's one of the *most* ubiquitous, and thus most important. Here's why.

Think of a Hydrogen atom.

- **Restoring Force.** Without getting into the quantum mechanical details, we can say that there is effectively a *restoring* force on the H atom – a force that tends to *restore* it back to its prefered, or *equilibrium* position. Furthermore, as experience may tell you with stretching and compressing things on the day-to-day scale, this restoring force grows the further out of equilibrium the atom is.
  - **Mathematical Representation / Linear Approximation.** So, there's some restoring force that grows with distortion: $\vec{F}(\vec{r}-\vec{r}_o)$ where $\vec{r}_o$ is the equilibrium position. For the gravitational interaction, we had a simple and exact mathematical form for the force. The similarly exact mathematical form of this restoring force depends upon the quantum mechanical details of the molecule; it's not quite so simple. However, we can do quite well without it by noting that the most of the time distortion, $\vec{r}-\vec{r}_o$, is relatively *small*. That allows us to approximate the exact mathematical form in terms of the first few terms in its Taylor series expansion in *r* about the equilibrium position, r_o.



$$\vec{F}=\vec{F}\Big|_{r_o}+\frac{d\vec{F}}{d\vec{r}}\Big|_{r_o}\cdot(\vec{r}-\vec{r}_o)+\ldots\approx\frac{d\vec{F}}{d\vec{r}}\Big|_{r_o}\cdot(\vec{r}-\vec{r}_o)=-k_s(\vec{r}-\vec{r}_o).$$

    - **If you've not met the Taylor Series before…**
    - The red curve shows how we'd expect the force between two bonded atoms to depend on their separation. If they get *too* close, they're strongly repulsive (a big positive force), if they get *too* far apart, they're pulled back toward each other. Of course, if they get *too too* far apart, they stop feeling each other all together – each atom is electrically neutral after all, so the force trails off toward 0.
    - Near in, there's some equilibrium position – not too far or too near, but just right. Notice that the curve is fairly smooth there, I can sketch a tangent line that, pretty well hugs that curve right near the equilibrium separation (and gets pretty far from the curve far from the equilibrium separation.) Being a "tangent curve" means it's got the same slop as the curve has right at that point
      - $slope=\frac{\Delta\vec{F}}{\Delta\vec{r}}\Big|_{r_o}$.
    - Like the function itself, the line hits F = 0 at r = r_{eq}. Putting these two together, gives the equation of the line as

- $\vec{F}_{line} = \left.\frac{d\vec{F}}{d\vec{r}}\right|_{r_o} \cdot (\vec{r} - \vec{r}_o)$.

  - Now, as long as I'm only interested in r values *pretty* close to r_{eq}, the line is a *really* good approximation to the actual function. So,
    - $\vec{F} \approx \left.\frac{d\vec{F}}{d\vec{r}}\right|_{r_o} \cdot (\vec{r} - \vec{r}_o)$
  - The final touch is to make explicit the fact that the slope is negative
    - $\vec{F} \approx -\left|\left.\frac{d\vec{F}}{d\vec{r}}\right|_{r_o}\right| \cdot (\vec{r} - \vec{r}_o) = -k_{s,i}(\vec{r} - \vec{r}_o)$
  - Well, look at that! That's the force law for a spring! The "inter-atomic spring constant" is the slope, or derivative of the real force with respect to atomic separation.
- **Moral:** big things like springs and bars are "springy" because interatomic bonds are springy.

- **Ball & Spring model.** So a better model of molecules and solids is balls connected by flexible springs.

Similarly expand potential

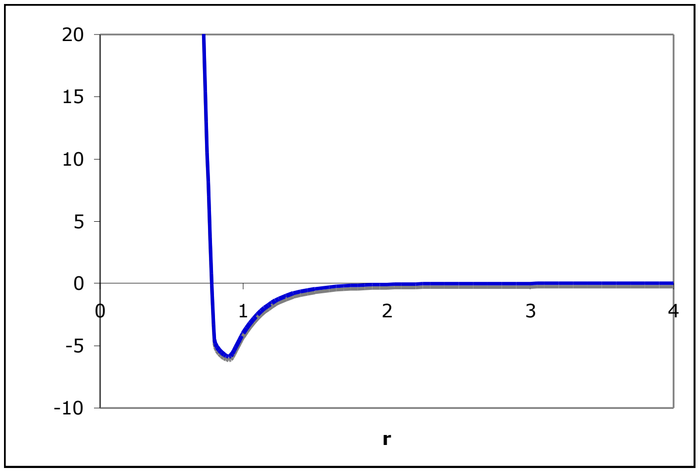
Then begin example problem

- Find equilibrium
- Find "spring constant"

Show Ball & Spring molecule and solid
Show ball – spring movie

<u>Example</u>/Exercise: Suppose one atom of a diatomic molecule is very heavy and remains fixed. The potential for the smaller atom of mass *m* can be approximated by the Leonard-Jones 6-12 potential:

$$U(x) = -\frac{a}{x^6} + \frac{b}{x^{12}},$$



where x is the distance between the atoms and a and b are both positive constants.

Let's make a little sense of this expression – make sure it does what we think an inter-atomic bond *should* do.

> **Qualitiatvely**: Two neutral atoms a great distance apart will not feel each other's pull at all, so the potential and force die off; however, as they come closer, they polarize each other more and more and make the interaction more and more attractive (the potential becoming negative.) Of course, they can't sit right on top of each other, so there's got to be some repulsion if they get too close (thus the potential starts climbing again for small enough r).
> **Qualitative math:** as r gets bigger, both $1/r^6$ and $1/r^{12}$ get smaller, but $1/r^{12}$ does so much more quickly, so the negative $1/r^6$ term dominates the expression – so the potential is negative (decaying to 0) for large r. Conversely, as r gets small, both terms get bigger, but $1/r^{12}$ does so more quickly, so eventually it comes to dominate – making the potential become positive for small r.

Okay. The Leonard-Jones expression is known as a "semi-empirical" because it's not derived from first principles (quantum), but it makes some qualitative sense. Still, not being derived from first principles means we don't have theoretical reasoning to tell us exactly what values a and b should have. We have to get those from experiment. And *that's* where our notion of approximating a complicated potential as a simple harmonic one near the equilibrium.

First, find an expression for the **equilibrium position** in terms of a and b.

Find the period of small oscillations around the equilibrium position.

$$0 = U'(x_e) = \left.\frac{dU}{dx}\right|_{x_e} = \frac{6a}{x_e^7} - \frac{12b}{x_e^{13}} = \frac{6}{x_e^7}\left(a - \frac{2b}{x_e^6}\right).$$

This gives:

$$x_e = \left(\frac{2b}{a}\right)^{1/6}.$$

So, if experimentalists can cook up a way to determine the bond length (say, from determining the mass and volume, thus density of a chunck of material and thus deducing the typical separation or dealing with the molecule in gas phase exciting rotational modes will lead you to beable to deduce the moment of inertia and thus atomic separation), then we have one equation that relates these two unknowns, *a* and *b*. We just need to generate a second equation for a second measurable in order to be able to solve for b and a.

To be continued…

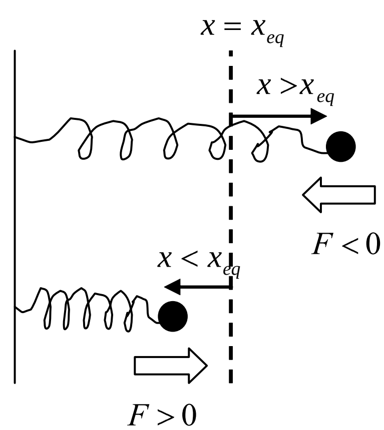
Simple Harmonic Motion:

Okay, so either we've got a real spring or we've got, as with the Leonard-Jones potential, something that behaves like a spring for small enough displacements from equilibrium. Let's think about how such a thing behaves.

For simplicity, let's consider a horizontal spring:

$$F_x(x) = -k(x - x_{eq}),$$

where *k* is the (positive) spring constant and x_{eq} is the equilibrium position of the end. Suppose the spring is attached to a wall on its left. Whether the spring is stretched or compressed, the force is a *restoring* force, which is directed back toward the equilibrium.



Parenthetically, if we integrate this, we get a potential energy associated with the force (taking $U(x_{eq}) = 0$):

$$U(x) = \tfrac{1}{2}k(x - x_{eq})^2.$$

Anyway, to make the math particularly clean, let's set the origin at the equilibrium point (it's easy enough to shift it elsewhere at the end of our arguments), $x_{eq}=0$:

$$m\ddot{x}(t) = -kx(t)\ ,$$

$$\ddot{x}(t) = -\frac{k}{m}x(t)\ ,$$

Pause and think like a mathematician:

- 2nd-order, linear Differential Equation means two independent solutions, or at least, two independent arbitrary constants.
- A linear combination of solutions is also a solution.
- One valid approach to finding a solution is simply guessing with some built in constants, plugging in, and if you're close enough, then you'll find out what your constants must be.

$\ddot{x}(t) = -\frac{k}{m}x(t)$ is of form $\ddot{x}(t) = Const \cdot x(t)$

There are several different ways to write the solution. But we're physicists, talking about physical systems, so maybe we should let observation suggest an guess.

Demo: mass bobs on spring with force probe and motion sensor watching.

I don't know about you, but to me, the curve of position as a function of time looks a lot like a cosine wave, except offset in time. So I'm going to guess

$$x(t) = A\cos(\omega(t - t_o))$$

$$x(t) = A\cos(\omega t - \omega t_o)$$

To translate this into the book's language, I'll identify $\delta = \omega t_o$

So

$$x(t) = A\cos(\omega t - \delta)$$

I plug this guess into our equation to see if it's a good guess, and I find that

$$\ddot{x}(t) = -\frac{k}{m}x(t)$$

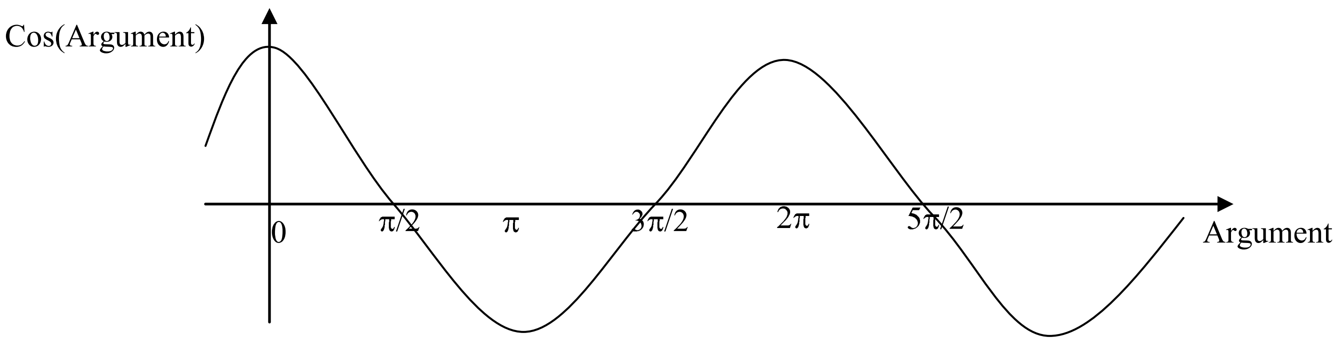
$$-\omega^2 A\cos(\omega t - \delta) = -\frac{k}{m}A\cos(\omega t - \delta)$$

Or canceling off the like factors, I find that my guess works if

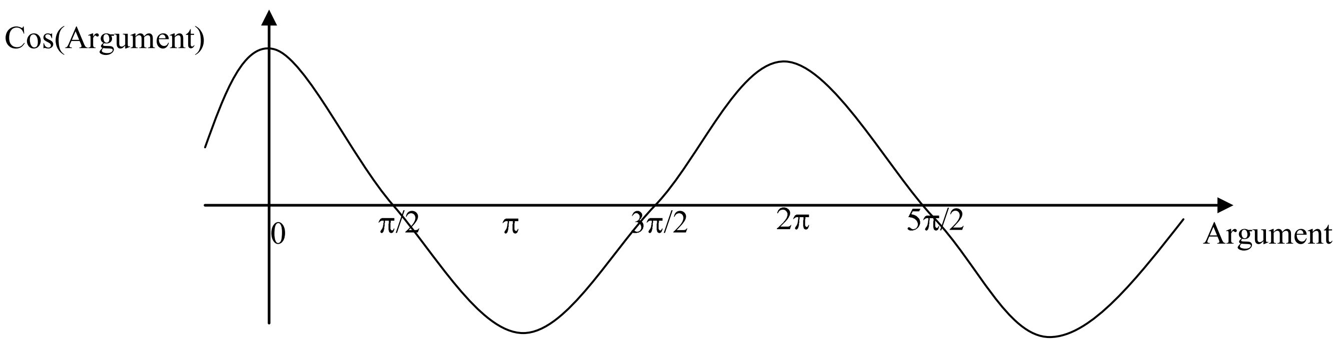
$$\omega \equiv \sqrt{\frac{k}{m}}\ .$$

What is the physical significance of this constant?

You know that if you plot out cosine, it repeats itself every time the argument goes through a multiple of 2π.



Now, in our case, the "Argument" = $\omega\left(t-\left(\frac{\delta}{\omega}\right)\right)$



$$0=\omega\left(t_i-\left(\frac{\delta}{\omega}\right)\right) \qquad 2\pi=\omega\left(t_f-\left(\frac{\delta}{\omega}\right)\right)$$

$$2\pi-0=\omega(t_f-t_i)$$

Taking the difference between these two, now, we define the interval of time over which a periodic function repeats itself as the "period", T, and obviously cosine repeats itself every time its argument changes by 2π. So $T=(t_f-t_i)$

$$2\pi-0=\omega T \Rightarrow \omega=\frac{2\pi}{T}$$

Tada! The constant we're calling w is the rate with which the 'angle' of the argument changes. In this context, we often refer to it as the "angular frequency."

Again, for our mass on a spring, this constant is $\omega \equiv \sqrt{\frac{k}{m}}$

So

$$\omega = 2\pi f = \frac{2\pi}{T} \equiv \sqrt{\frac{k}{m}}\,.$$

Example. In physics, sometimes it takes a little bit of work to translate from what you *measure* to your theoretical model. Here's an example of going from easily measured things to determining the values of the three constants in our expression for the position of the mass, $x(t) = A\cos(\omega t - \delta)$.

Say an 0.5kg mass on a spring of stiffness 5N/m is moving 2cm/s when 3cm from equilibrium at time $t_1 = 2$s. What are the constants in our expression – the amplitude, A, the angular frequency, ω, and the 'phase shift', δ?

First off, we know

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{5N/m}{0.5kg}} = \sqrt{10}/s \approx 3.16/s$$

Connecting the position and speed *values* with our corresponding expressions, we have

$$3cm = x(t_1) = A\cos(\omega t_1 - \delta)$$

And

$$2cm/s = v(t_1) = \dot{x}(t_1) = -\omega A\sin(\omega t_1 - \delta).$$

Comparing those two expressions, it occurs to me that

$\frac{v(t_1)}{\omega} = -A\sin(\omega t_1 - \delta)$ and so

$$\sqrt{\left(\frac{v(t_1)}{\omega}\right)^2 + (x(t_1))^2} = \sqrt{(A\sin(\omega t_1 - \delta))^2 + (A\cos(\omega t_1 - \delta))^2} = A\sqrt{(\sin(\omega t_1 - \delta))^2 + (\cos(\omega t_1 - \delta))^2} = A$$

Plugging in the numbers, I have

$$\sqrt{\left(\frac{2cm/s}{3.16/s}\right)^2 + (3cm)^2} = 3.066cm = A$$

Similarly,

$$\frac{v(t_1)/\omega}{x(t_1)} = -\frac{A\sin(\omega t_1 - \delta)}{A\cos(\omega t_1 - \delta)} = -\tan(\omega t_1 - \delta) \Rightarrow \omega t_1 - \delta = -\tan^{-1}\left(\frac{v(t_1)/\omega}{x(t_1)}\right)$$

$$\delta = \tan^{-1}\left(\frac{v(t_1)/\omega}{x(t_1)}\right) - \omega t_1 = \tan^{-1}\left(\frac{2cm/s/3.16/s}{3cm}\right) - (3.16/s)2s = -9.28rad$$

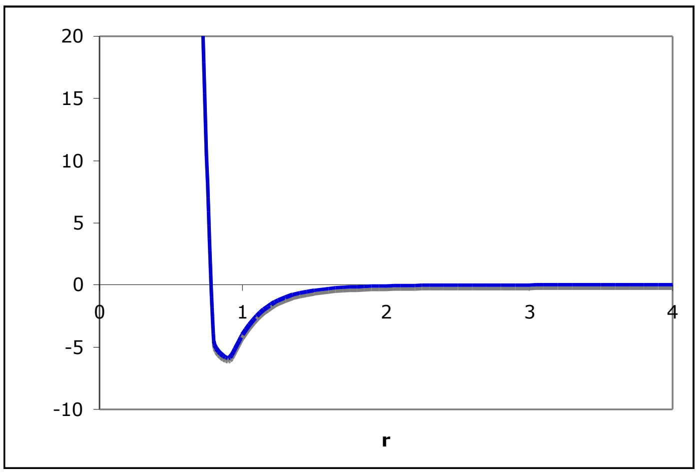
Which, given the periodic nature of these trig functions, is equivalent to -9.28rad + 4π = 3.287rad or about 1.04*π.

Okay, we've gotten familiar with the solution to the simple harmonic oscillator problem of a mass hanging from a spring. Again, the reason physicists are so interested in the simple harmonic oscillator is that almost *any* system that has an stable equilibrium can be, for small displacements from equilibrium, modeled in the same way. To pick up where we left off on this point, let's return to the Leonard-Jones potential.

Example: Suppose one atom of a diatomic molecule is very heavy and remains fixed. The potential for the smaller atom of mass *m* can be approximated by the Leonard-Jones 6-12 potential:

$$U(x) = -\frac{a}{x^6} + \frac{b}{x^{12}},$$

where *x* is the distance between the atoms and *a* and *b* are both positive constants. Find the **period of small oscillations around the equilibrium position**.



Solution: First, we found find the equilibrium position:

$$0 = U'(x_e) = \left.\frac{dU}{dx}\right|_{x_e} = \frac{6a}{x_e^7} - \frac{12b}{x_e^{13}} = \frac{6}{x_e^7}\left(a - \frac{2b}{x_e^6}\right).$$

That gave:

$$x_e = \left(\frac{2b}{a}\right)^{1/6}.$$

Now, we'd argued that, for small displacements, most any function could be approximated by

$$U(x) \approx U(x_{o\,eq}) + \tfrac{1}{2}\left.\frac{d^2U(x)}{dx^2}\right|_{x_{eq}}(x - x_{eq})^2$$

Which has the basic form of $U(x) \approx Const + \tfrac{1}{2}k(x - x_{eq})^2$. That is to say, the system will oscillate about its equilibrium point just as would a mass on a spring of stiffness

$$"k_{sp}"=\left.\frac{d^2U(x)}{dx^2}\right|_{x_{eq}}$$

So, taking what we learned from actually considering a mass on a spring,

$$\omega=\frac{2\pi}{T}\equiv\sqrt{\frac{k}{m}}\Rightarrow T=2\pi\sqrt{\frac{m}{k}}$$

Our mission is now clear. The curvature of the potential at the equilibrium is (take second derivative, <u>then</u> plug in x_e):

$$U''(x_e)=\left.\frac{d^2U}{dx^2}\right|_{x_e}=\frac{-42a}{x_e^8}+\frac{156b}{x_e^{14}}=\frac{6}{x_e^{14}}\left(-7ax_e^6+26b\right)$$

$$U''(x_e)=\frac{6}{(2b/a)^{14/6}}\left[-7a(2b/a)+26b\right]=72b\left(\frac{a}{2b}\right)^{7/3}$$

which is equal to the spring constant *k*. Since $\omega=2\pi f=\sqrt{k/m}$ and $\tau=1/f$, the period of oscillation is:

$$\boxed{\tau=2\pi\sqrt{\frac{m}{k}}=2\pi\sqrt{\frac{m}{72b}\left(\frac{2b}{a}\right)^{7/3}}=\pi\sqrt{\frac{m}{18b}\left(\frac{2b}{a}\right)^{7/6}}}.$$

Getting this relation is of practical use because experimentalists can determine the frequency at which the bond vibrates (IR spectroscopy) and thus get a value for this. So, between measuring the bond separation and the frequency of vibration you have two equations for the two unknowns *a* and *b*. That allows you to express the potential relation and make other predictions – like what it takes to break the bond.

Alternative representations of solutions.

While our little experiment with a mass on a spring suggested an obvious way to express our solution, thanks to trig identities and Euler's relation, there are a couple other ways to express it. Observe that the solution is some function that equals a constant times its own second derivative (to within a constant factor) We know a function that does that:

$$x(t)=Ce^{\alpha t}$$

(I know, you may be tempted to guess sine or cosine, but the math is actually easier and more easily generalized to tougher problems if we go with the exponential)

so $\dot{x}=\alpha Ce^{\alpha t}$ and $\ddot{x}=\alpha^2Ce^{\alpha t}$. Plugging in this guess gives:

$$\alpha^2e^{\alpha t}=-\frac{k}{m}e^{\alpha t},$$

So, we've got a good guess on our hands if

$$\alpha^2 = -\frac{k}{m}$$

or

$$\alpha = \pm i\omega$$

$$\omega \equiv \sqrt{\frac{k}{m}}$$

(the *auxiliary equation*).

(for now, consider ω a randomly selected symbol for the sake of not having to write the square root over and over again. In not too long, we'll identify it with the *other* use we've had for ω in this class – rate with which an angle changes.)

The two independent solutions are:

$$x(t) = Ce^{+i\sqrt{\frac{k}{m}}t} \quad \text{and} \quad x(t) = Ce^{-i\sqrt{\frac{k}{m}}t},$$

so the general solution is a linear combination of the two options:

$$x(t) = C_1 e^{i\omega t} + C_2 e^{-i\omega t}.$$

Note: it may be disturbing that we've got some complex terms; however, remember that we're *not done yet* . *Something* of this form is a solution, but we haven't yet imposed our boundary conditions / initial conditions, our knowledge about the position or velocity at some given time.

Given that we're talking about a *real* mass on a *real* spring in *real* space, we can safely set the very general condition that the positions be *real.* To see what it takes to do that, we can rewrite the exponential using Euler's formula:

$$e^{\pm i\omega t} = \cos(\omega t) \pm i\sin(\omega t),$$

as:

$$x(t) = C_1(\cos(\omega t) + i\sin(\omega t)) + C_2(\cos(\omega t) - i\sin(\omega t)) = (C_1 + C_2)\cos(\omega t) + i(C_1 - C_2)\sin(\omega t)$$

$$x(t) = B_1\cos(\omega t) + B_2\sin(\omega t).$$

The new coefficients:

$$B_1 = C_1 + C_2 \quad \text{and} \quad B_2 = i(C_1 - C_2),$$

must be <u>real</u>, so that the position is real. The inverse relations between the sets of coefficients are:

$$C_1 = \tfrac{1}{2}(B_1 - iB_2) \qquad \text{and} \qquad C_2 = \tfrac{1}{2}(B_1 + iB_2),$$

which are <u>complex conjugates</u>, $C_2 = C_1^*$ (switch sign of imaginary term).

This is all good and well, but when you actually plot out the motion of a mass bobbing on a spring, it doesn't so much look like a sine + cosine as it does a sine or a cosine with a simple time shift (so it's not exactly 1/0 at t=0). It takes a little work to show, but that's actually equivilant to this form of the solution. I'll go in the opposite direction from how the book did this. Let's *assume* that it works, and then see how our new expression is related to the old one.

$$x(t) = A\cos(\omega(t - t_o))$$

Or

$$x(t) = A\cos(\omega t - \omega t_o)$$

To translate this into the book's language, I'll identify $\delta = \omega t_o$

So

$$x(t) = A\cos(\omega t - \delta)$$

Of course, we can rewrite that as

$$x(t) = A\left[\cos(\delta)\cos(\omega t) + \sin(\delta)\sin(\omega t)\right]$$
$$x(t) = (A\cos(\delta))\cos(\omega t) + (A\sin(\delta))\sin(\omega t)$$

So it looks like this is consistent with

$$x(t) = B_1\cos(\omega t) + B_2\sin(\omega t)$$

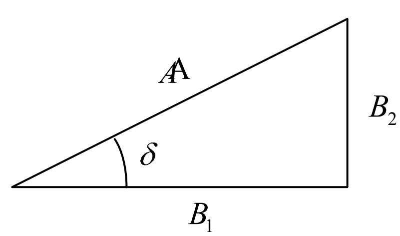
If

$$B_1 = (A\cos(\delta))$$
$$B_2 = (A\sin(\delta))$$

If it helps, one way of looking at that is

$$A = \sqrt{B_1^2 + B_2^2}\,,$$

which is the hypotenuse of a triangle with sides B_1 and B_2 as shown below.



This can also be written as the real part of a complex solution:

$$x(t) = \mathrm{Re}\, Ae^{i(\omega t - \delta)}.$$

Sometimes it is convenient to work with complex exponentials (they have simple derivatives and integrals) and take the real part at the end.

For all of the different forms, there are two coefficients to be determined by two initial conditions of the system (position and velocity).

Some morals are

- o You should get familiar with working with exponentials and fluidly translating between complex and real ones
- o The further your initial, general guess is from your particular solution, the more work it takes to get there.

Now, why do we call that constant ω?

- Recall, we'd defined this symbol as a short and for $\sqrt{\frac{k_s}{m}} = \omega = \frac{2\pi}{T}$

Two-Dimensional Oscillators: There are two main cases depending on whether or not the spring constant is the same in both dimensions.

(1) Isotropic Osillator: The restoring force is $\vec{F} = -k\vec{r}$ or in component form $F_x = -kx$ and $F_y = -ky$. There is the same spring constant in both dimensions. The equations of motion are:

$$\ddot{x} = -\omega^2 x \qquad \text{and} \qquad \ddot{y} = -\omega^2 y,$$

where $\omega = \sqrt{k/m}$ so the solutions are:

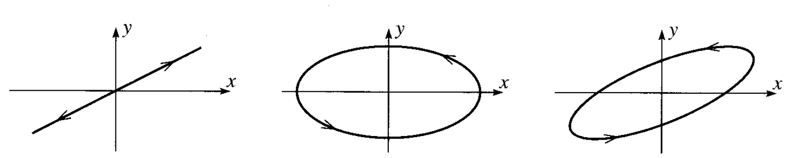
$$x(t) = A_x \cos(\omega t - \delta_x) \text{ and } \quad y(t) = A_y \cos(\omega t - \delta_y).$$

We can always redefine the time so that:

$$x(t) = A_x \cos(\omega t) \qquad \text{and} \qquad y(t) = A_y \cos(\omega t - \delta),$$

3-D simple harmonic oscillator.py (turn on just 2 degrees)

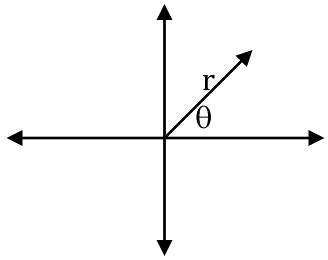
where δ is the *relative phase* of the motion in the *x* and *y* directions. Note that the amplitudes of the motions in the two dimensions, A_x and A_y, do not have to be equal. The general shape of motion is an ellipse (a circle is a special case). If the two motions are in phase, $\delta = 0$, the motion will be linear.



(a) $\delta = 0$ (b) $\delta = \pi/2$ (c) $\delta = \pi/4$

Here's a way to start with something we already know and think through to these results.

First, consider a point in the 2-D plane specified by either Cartesian or polar coordinates.

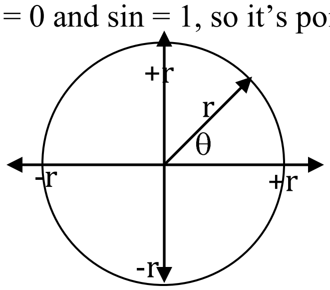


$X = r\cos(\theta)$
$Y = r\sin(\theta)$

Now let's say that the angle changes with time as $\theta(t) = \omega t$.
Then

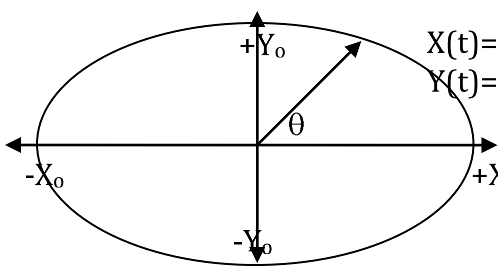
$X(t) = r\cos(\omega t)$
$Y(t) = r\sin(\omega t)$

At t= 0, sin = 0 and cos = 1, so we get the vector pointing along the x axis; a quarter period later cos = 0 and sin = 1, so it's pointing straight up… the vector sweeps out a circle.



$X(t) = r\cos(\omega t)$
$Y(t) = r\sin(\omega t) = r\cos(\omega t+\pi/2)$

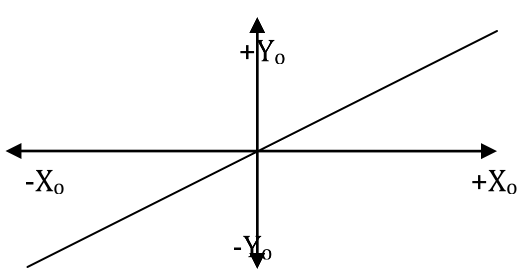
Now what if the amplitude of the oscillation in X were twice that of the oscillation in Y? Then we'd still get a nice closed shape, but it would be stretched along the x-axis.



$X(t) = X_o\cos(\omega t)$
$Y(t) = Y_o\sin(\omega t) = Y_o\cos(\omega t+\pi/2)$

I haven't proven, but that's an ellipse.

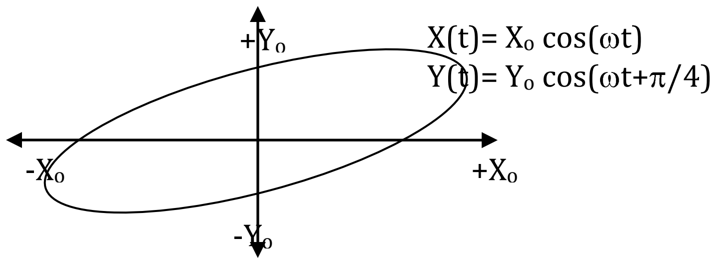
Now say that we phase shift the Y by $-\pi/2$. If you recall, $Y(t) = Y_o\sin(\omega t - \pi/2) = Y_o\cos(\omega t)$.
So we have



$X(t) = X_o\cos(\omega t)$
$Y(t) = Y_o\cos(\omega t)$

The x and y components maximize at the same time, hit 0 at the same time, minimize at the same time… we're skating up and down a straight line.

Now, qualitatively, think about the natural steps between these two extremes: the ellipse when the horizontal and vertical oscillations are in phase, and a line when they are 90° out of phase, then we'd expect something tilted and slightly squashed at some inbetween phase.



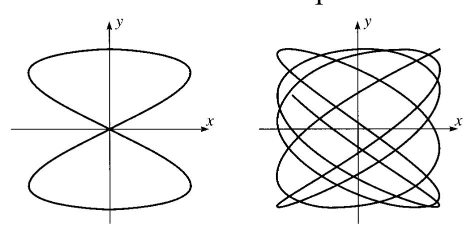
(2) Anisotropic Oscillator: There are different spring forces for each component so $F_x = -k_x x$ and $F_y = -k_y y$. In this case, there are different angular frequencies for each component, $\omega_x = \sqrt{k_x/m}$ and $\omega_y = \sqrt{k_y/m}$. The solutions are (choosing $t = 0$ appropriately):

$$x(t) = A_x \cos(\omega_x t) \qquad \text{and} \qquad y(t) = A_y \cos(\omega_y t - \delta).$$

There are two possible types of motion for an anisotropic oscillator:

(a) Periodic / Commensurable: If ω_x/ω_y is a rational fraction, then the motion will exactly repeat itself. Graphs of the motion are called *Lissajous figures*.

(b) Quasiperiodic / Incommensurable: If ω_x/ω_y is not a rational fraction, the motion never repeats. However, the motion in each dimension is periodic.



(a) $\omega_x = 2\omega_y$ (b) $\omega_x = \sqrt{2}\,\omega_y$

3-D simple harmonic oscillator .py

An okay model of N atoms in a solid is N 3-D harmonic oscillators, which, in terms of the math you'd do, is the same as 3N 1-D harmonic oscillators. If you impose that their oscillations are quantized, then you can make some fairly good predictions about the thermal properties of the material (a better model, as the movie suggests, is 3N *coupled* harmonic oscillators.)

Next time:

Friday we'll be reviewing for the exam which is when you get back next Wednesday. A resource for that review is the practice problems that are posted on the website. I encourage you to look over those, old homework, and the notes that are posted online (perhaps pay special attention to the examples worked out). Bring questions Friday.

Additional notes onComplex Exponentials

Now, before we set about solving this equation, I want to prepare you for aspects of that solution by returning to the Taylor Series. In particular, that for the exponential.

Again,

$$f(x) = f(x_o) + \sum_{n=1} \tfrac{1}{n!} \left. \frac{d^n f(x)}{dx^n} \right|_{x_o} (x - x_o)^n$$

For simplicity, let's expand around 0. Now,

$\left. \frac{d}{dx} e^{kx} \right|_0 = k e^{kx}\Big|_0 = k e^{k0} = k1 = k$ and so $\left. \frac{d^2}{dx^2} e^{kx} \right|_o = k \left. \frac{d}{dx} e^{kx} \right|_o = k^2 e^{kx}\Big|_o = k^2 e^{k0} = k^2$, or, for that matter, $\left. \frac{d^n}{dx^n} e^{kx} \right|_0 = k^n$

So, that means that the Taylor series

$$e^{kx} = 1 + \sum_{n=1} \tfrac{1}{n!} k^n x^n$$

Now, what if we were crazy enough to have $k = i$ as in $\sqrt{-1}$?

$$e^{ix} = 1 + \sum_{n=1} \tfrac{1}{n!} i^n x^n$$

Now, let's spell out the first few terms

$$e^{ix} = 1 + ix + \frac{1}{2!} i^2 x^2 + \frac{1}{3!} i^3 x^3 + \frac{1}{4!} i^4 x^4 + \frac{1}{5!} i^5 x^5 + \ldots$$

$$e^{ix} = 1 + ix + \frac{1}{2!}(-1)x^2 + \frac{1}{3!}(-1)ix^3 + \frac{1}{4!}1x^4 + \frac{1}{5!}ix^5 + \ldots$$

.

$$e^{ix} = \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots\right) + i\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots\right)$$

$$e^{ix} = \cos x + i \sin x$$

Well, that's remarkable!

It follows that $\frac{e^{ix}+e^{-ix}}{2}=\cos x$ and $\frac{e^{ix}-e^{-ix}}{2i}=\sin x$

This relation lends itself to a certain amount of unification in how we solve problems that have exponentially growing or shrinking behavior *and* how we handle ones that oscillate sinusoidally. You've probably already encountered this in Quantum.

Partly just to get used to this, we'll use exponentials to solve the simple harmonic oscillator. If that seems like making a simple problem harder than it needs to be and for insufficient reason, then, I'll hint that this will also prepare us for solving the damped simple harmonic oscillator system which both oscillates *and* exponentially decays at the same time.