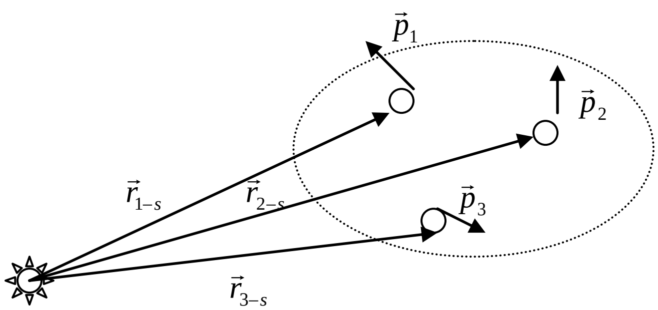
Mon. 9/24 Tues. 9/25 Wed. 9/26 Thurs. 9/27 Fri., 9/28	**3.5** Angular Momentum for multiple particles **4.1-.3, 4.9** Work & Energy, Force as a Gradient, 2 Particle Interaction ***Science Poster Session**: Hedco7~9pm* **4.4-.6** Curl of Conservative Force, Varying Potential, 1-D systems	 **HW3b** (3.D-.G), **Project Topic** **HW4a** (4.A, 4.B)
Mon. 10/1 Tues. 10/2 Wed. 10/3 Fri. 10/5	**4.7-.8** Curvilinear 1-D, Central Force **5.1-.3 (2.6)** Hooke's Law, Simple Harmonic (Complex Sol'ns) ***What (research) I Did Last Summer**: AHoN 116 @ 4pm* Review for Exam 1	 **HW4b** (4.C-.F)

Angular Momentum for Systems of Particles:

9.4 Angular momentum of multiparticle systems

- Now that we've got a real handle on angular momentum of a single object, we're going to branch out and consider systems of particles. When dealing with the motion of systems of particles in the past, we've found it convenient to separate out our description of the center of mass motion and the internal motion of the particles; this time will be no exception.
- Consider our old favorite – the interstellar gas cloud. While each individual particle is zipping this way and that, let's say that as a whole it's orbiting something, perhaps a star; heck, perhaps our cloud is a proto-planet itself in the process of collapsing to form a new planet. So, let's look at the total angular momentum of the cloud about the star.
- For a start, each individual particle has its own linear momentum (relative to the star's motion).

- Since we're interested in the angular momentum about the star, we need to consider each particle's position relative to the star.



- So, the total angular momentum of the cloud, relative to the star, is simply
$\vec{L}_{c-s} = \vec{L}_{1-s} + \vec{L}_{2-s} + \vec{L}_{3-s} + \ldots = (\vec{r}_{1-s} \times \vec{p}_1) + (\vec{r}_{2-s} \times \vec{p}_2) + (\vec{r}_{3-s} \times \vec{p}_3) + \ldots$
- From here, the book bee-lines it for the torque-anguler momentum relation for a multi-particle system.

(Angular Momentum Principle with) Multi-particle systems
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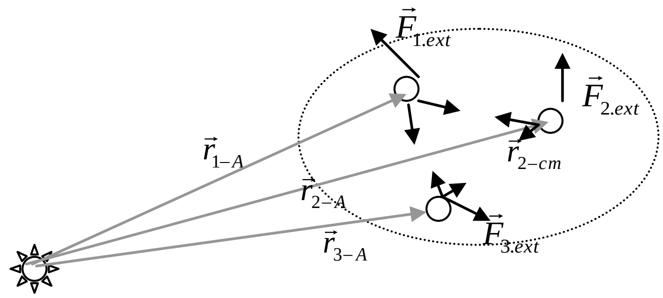
- **Change in angular Momentum for each particle.**
    - This is the particle's position vector crossed into the sum of forces on the particle.



$$\frac{d\vec{L}_{1-A}}{dt} = \vec{r}_{1-A} \times \left(\vec{F}_{2\to1} + \vec{F}_{3\to1} + \vec{F}_{ext-1}\right)$$

$$\frac{d\vec{L}_{2-A}}{dt} = \vec{r}_{2-A} \times \left(\vec{F}_{1\to2} + \vec{F}_{3\to2} + \vec{F}_{ext-2}\right) = \vec{r}_{2-A} \times \left(-\vec{F}_{2\to1} + \vec{F}_{3\to2} + \vec{F}_{ext-2}\right)$$

$$\frac{d\vec{L}_{3-A}}{dt} = \vec{r}_{3-A} \times \left(\vec{F}_{1\to3} + \vec{F}_{2\to3} + \vec{F}_{ext-3}\right) = \vec{r}_{3-A} \times \left(-\vec{F}_{3\to1} - \vec{F}_{3\to2} + \vec{F}_{ext-3}\right)$$

Here, we've employed the Principle of Reciprocity for forces.

Now adding the three equations to each other gives the rate of change in total angular momentum of the system of particles.
Now, for central forces, such as we've thus far encountered, the torques due to interactions between the particles sum to 0, just as the forces themselves do. This leaves just the external forces to worry about.

$$\frac{d\left(\vec{L}_{1-A} + \vec{L}_{2-A} + \vec{L}_{3-A}\right)}{dt} = \vec{r}_{1-A} \times \vec{F}_{ext-1} + \vec{r}_{2-A} \times \vec{F}_{ext-2} + \vec{r}_{3-A} \times \vec{F}_{ext-3}$$

or

$$\frac{d\left(\vec{L}_{tot-A}\right)}{dt} = \vec{\Gamma}_{ext-1} + \vec{\Gamma}_{ext-2} + \vec{\Gamma}_{ext-3} = \vec{\Gamma}_{net.ext}$$

Just like for a point object.

Similarly, in the event that there *is no net torque,* the we have **Conservation of Angular Momentum** for the system of particles.

$$\frac{d\left(\vec{L}_{tot-A}\right)}{dt} = 0$$

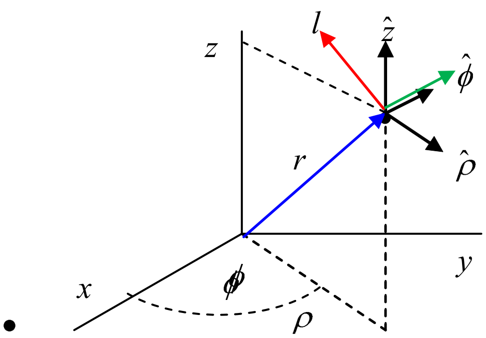
Note: the net torque is *not* simply the net force crossed by some representative position vector; it's the sum of individual torques which are the net forces on each individual part crossed by the position (relative to the axis)

Uniform rotation & moment of inertia
A common special case of angular motion for a system of particle is when they have the same angular speed. That makes it convenient to rephrase angular momentum in terms of that shared w rather than individual v's, and factor out the w. What you're left with is the object's moment of inertia.

Suppose I spin an object, say a ball; we'll call the axis of rotation the z-axis. So we'll describe the motion in polar coordinates. Let's focus on just one morsel of the ball, on particle. The position of a particle in the object is $\vec{r} = \rho\hat{\rho} + z\hat{z}$. If the object rotates with an angular speed $\omega = \dot{\phi}$ but maintains constant magnitude of ρ and z (going around and around, not in or out, up or down), then its velocity is simply $\vec{v} = \rho\dot{\phi}\hat{\phi} = \rho\omega\hat{\phi}$. The angular momentum of the particle is:

- $\vec{\ell} = \vec{r}\times\vec{p} = m\vec{r}\times\vec{v} = m(\rho\hat{\rho} + z\hat{z})\times(\rho\dot{\phi}\hat{\phi})$.

- The cross products of the unit vectors are $\hat{\rho}\times\hat{\phi} = \hat{z}$ and $\hat{z}\times\hat{\phi} = -\hat{\rho}$ (see the diagram below), so:
  - $\vec{\ell} = (m\rho^2\omega)\hat{z} - (mz\rho\omega)\hat{\rho}$.

- 

- If the axis we're considering is an axis of symmetry, then the inward component of the angular momentum for this morsel cancels with that for another; however, we still have the z-component to consider. The *z* component of the angular momentum for a particle is $\ell_z = m\rho^2\omega$, so the *z* component of the total angular momentum, summing over all particles that make up the object is:
  - $L_z = \sum_{\alpha=1}^{N} \ell_{\alpha z} = \sum_{\alpha=1}^{N} m_\alpha \rho_\alpha^2 \omega$.
- We can write $L_z = I\omega$ if we define the *moment of inertia*, *I*, by:
  - $I_z \equiv \sum_{\alpha=1}^{N} m_\alpha \rho_\alpha^2$,

  (subscript z because we could have considered roatation about a *different* axis)
- where ρ_α is the distance of mass m_α from the axis. The other pieces often cancel for an object (remember $\hat{\rho}$ is not in the same direction for all points). We will be concerned with the whole answer in Ch.10.
- so, then we have $L_z = I_z\omega_z$

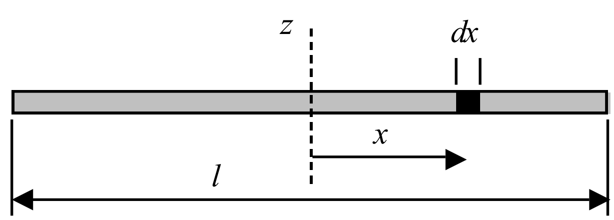
- **What *is* moment of inertia – conceptually?**
  - It play s an analogous role to mass for linear motion.
  - $L_z = I_z \omega_z \qquad\qquad p_z = mv_z$
  - $\Gamma_{net.z} = \dot{L}_z = I_z \dot{\omega}_z \qquad F_{net.z} = \dot{p}_z = m\dot{v}_z$

  "Inertia" is the idea that an object's going to keep on doing what it's doing. The bigger the "inertia", the harder it is to change its state of motion. For linear motion, mass represents inertia, but for angular motion, the moment of inertia does that job. The bigger the moment of inertia, the bigger the torque is required to change the angular velocity. You can see that in the equation, relating torque and angular velocity. Also look at the definition of moment of inertia, $I_z \equiv \sum_{\alpha=1}^{N} m_\alpha \rho_\alpha^2$. –the more mass and the further it is from the axis of rotation, the harder it is to change its rotation.

Before we use moment of inertia in describing motion, let's think about determining the moment of inertia.

<u>Example 1</u>: Find the moment of inertia of a thin rod of mass *m* and length *l* about its center.

Choose the *x* axis to be along the rod with the origin at the center. Divide the rod into small pieces of length *dx* (as shown below). The mass of each piece is $(dx/l)m$.



The moment of inertia is:

$$I_{center} = \sum m\rho^2 = \sum (dx)x^2 = \sum\left(\frac{m}{l}dx\right)x^2 \rightarrow \frac{m}{l}\int_{-l/2}^{l/2} x^2 dx$$

$$I_{center} = \frac{m}{l}\left[\frac{x^3}{3}\right]_{-l/2}^{l/2} = \frac{ml^2}{12}$$

<u>Exercise</u>: What is the moment of inertia about the end of the rod?

For this exercise, move the origin to the end of the rod.

$$I_{end} = \sum m\rho^2 = \sum\left(\frac{m\,dx}{l}\right)x^2 \rightarrow \frac{m}{l}\int_0^l x^2 dx = \frac{ml^2}{3}$$

Notice:

$$I_{end} = \frac{ml^2}{3} = \frac{4ml^2}{12} = \frac{3ml^2}{12} + \frac{ml^2}{12} = \frac{ml^2}{4} + \frac{ml^2}{12} = m\left(\frac{l}{2}\right)^2 + \frac{ml^2}{12} = I_{cm.about.axis} + I_{about.cm}$$

This is an example of the "parallel axis theorem."

This is larger than the result for Example 1 because some of the mass is farther from the axis.

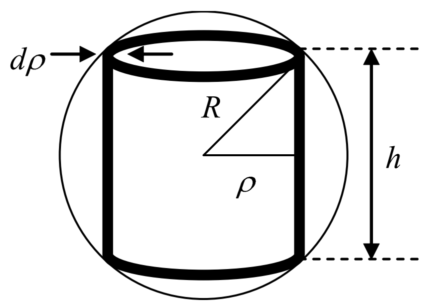
Example 2: (Prob. 3.32) Find the moment of inertia of a sphere of mass *M* and radius *R* an axis through its center.

So, we're going to need to integrate over the whole sphere, that means we start out by defining differentially small morsels of mass, and then summing over them; however, if we're clever about the geometry of those morsels, we can streamline the process.

$$I_{center} = \sum m\rho^2 = \sum\left(\frac{M}{Vol}dVol\right)\rho^2 = ...$$

Q: on what shape lie all points that contribute the same to the moment of inertia – that is, that have the same ρ?

Divide the sphere into thin hollow cylinders (thickness $d\rho$) because the each cylinder will have the same distance *r* from the axis (see below).



The height of each ring is $h = 2\sqrt{R^2 - \rho^2}$. The volume of a ring is $(2\pi\rho h\, d\rho)$, so its mass is:

$$\left(\frac{4\pi r\sqrt{R^2 - \rho^2}\; d\rho}{4\pi R^3/3}\right)M = \frac{3M}{R^3}\left(R^2 - \rho^2\right)^{1/2}\rho\, d\rho$$

The moment of inertia is:

$$I=\sum m\rho^2=\sum\left[\frac{3M}{R^3}\left(R^2-\rho^2\right)^{1/2}\rho\,d\rho\right]\rho^2\rightarrow\frac{3M}{R^3}\int_0^R\left(R^2-\rho^2\right)^{1/2}\rho^3 d\rho .$$

$$\frac{3M}{R^3}R^5\int_0^R\left(1-\left(\frac{\rho}{R}\right)^2\right)^{1/2}\left(\frac{\rho}{R}\right)^3 d\left(\frac{\rho}{R}\right)$$

Make the change of variables $q=\left(\frac{\rho}{R}\right)^2$ and $dq=2\left(\frac{\rho}{R}\right)d\left(\frac{\rho}{R}\right)$ to get:

$$I=\frac{3}{2}MR^2\int_0^1\left(1-q\right)^{1/2}q\,dq .$$

Add and subtract $\left(1-q\right)^{1/2}$ in the integrand:

$$I=\frac{3M}{2}R^2\int_0^1\left(1-q\right)^{1/2}\left[1-\left(1-q\right)\right]dq=\frac{3M}{2}R^2\int_0^1\left[\left(1-q\right)^{1/2}-\left(1-q\right)^{3/2}\right]dq$$

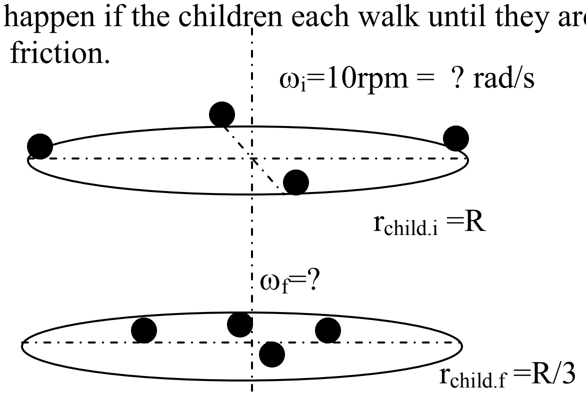
$$I=\frac{3M}{2}R^2\left[\left(\frac{-2}{3}\right)\left(1-q\right)^{3/2}-\left(\frac{-2}{5}\right)\left(1-q\right)^{5/2}\right]_0^1=\frac{3MR^2}{2}\left(\frac{2}{3}-\frac{2}{5}\right)=\frac{3MR^2}{2}\left(\frac{4}{15}\right)$$

$$I=\tfrac{2}{5}MR^2$$

Notice that spherical polar coordinates where <u>not</u> used.

Now for some simple use of moment of inertia in describing *motion.*

<u>Example 3</u>: Suppose four 60-kg children are standing at the edge of a merry-go-round that is spinning at 10 rpm. Assume the merry-go-round is a uniform 100-kg disk of radius 3 m. What will happen if the children each walk until they are 1 m from the center? Assume that there is no friction.



The moment of inertia of the merry-go-round is $I_m = MR^2/2$. The moment of inertia for each child (treating them as particles) is $I_c = mr^2$, where *r* is the distance from the center. Conservation of angular momentum (the direction if fixed, so just consider the magnitude) gives:

$$L_{initial} = L_{final}$$

$$(I_m + 4I_{co})\omega_o = (I_m + 4I_{cf})\omega_f$$

$$\left(MR^2/2 + 4mR^2\right)\omega_o = \left(MR^2/2 + 4m(R/3)^2\right)\omega_f .$$

Cancel out R^2 on both sides to get:

$$(M/2 + 4m)\omega_o = (M/2 + 4m/9)\omega_f .$$

Solve for the final angular speed:

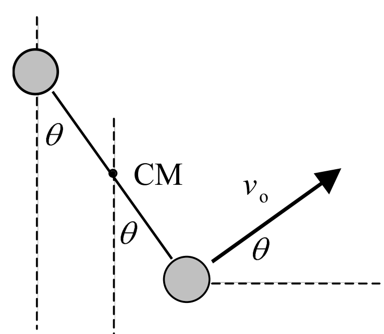
$$\omega_f = \frac{M/2 + 4m}{M/2 + 4m/9}\omega_o = \frac{(100\ \text{kg})/2 + 4(60\ \text{kg})}{(100\ \text{kg})/2 + 4(60\ \text{kg})/9}(10\ \text{rpm}) = 37.8\ \text{rpm} .$$

The merry-go-round will spin almost four times as fast!

<u>Angular Momentum about the CM</u>:

The result that $\dot{\vec{L}} = \vec{\Gamma}^{ext}$ was derived for an inertial reference frame (unaccelerated). However, the same result holds if $\vec{L}$ and $\vec{\Gamma}^{ext}$ are measured about the center of mass, even if the CM is accelerated! For example, the motion of a projectile can be described by the motion of its CM and rotation about the CM. Another way to put that is, you can break the problem up into the motion of the center of mass and the motion *about* the center of mass even if the CM is accelerating.

<u>Example 4</u>: Suppose two small spheres of mass *m* are attached by a light string of length *b*. Suppose someone holds one of the spheres and spins the other around at a speed v_o in a plane that passes through vertical. If the spheres are released when the string makes an angle θ with respect to <u>vertical</u>, describe the subsequent motion.



The diagram above shows the system at the instant of release. The CM will travel on a parabolic path as if it were a particle of mass 2*m* with initial velocity:

$$\vec{V} = \dot{\vec{R}} = \frac{m(0) + m\vec{v}_o}{2m} = \vec{v}_o/2 .$$

In other words, the initial velocity of the CM is $v_o/2$ at an angle θ above horizontal. At the instant of release, the angle of the string relative to vertical <u>at the CM</u> is related to the initial speed by:

$$v_o = (b/2)\dot{\theta},$$

so:

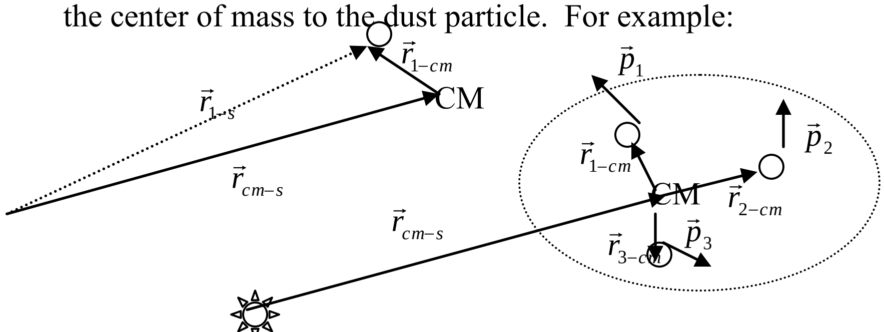
$$\dot{\theta} = 2v_o/b.$$

There is no net torque about the CM (the weights give opposite torques), so the angular momentum about the CM and $\dot{\theta}$ are constant.

What follows is some additional discussion of angular momentum taken from my Phys 231 notes. So this is an extended refresher.

Angular momentum about center of mass

- This much is familiar, but now let's note that each position vector could be re-expressed as the sum of two vectors, one from the star to the cloud's center of mass, the other from the center of mass to the dust particle. For example:



- So, we can rewrite our expression for the cloud's angular momentum about the star and separate out the contribution due to orbiting the star and the contribution due to rotating about its own center of mass.

$$\vec{L}_{c-s} = \left((\vec{r}_{cm-s} + \vec{r}_{1-cm}) \times \vec{p}_1\right) + \left((\vec{r}_{cm-s} + \vec{r}_{2-cm}) \times \vec{p}_2\right) + \left((\vec{r}_{cm-s} + \vec{r}_{3-cm}) \times \vec{p}_3\right) + \dots$$

$$\vec{L}_{c-s} = (\vec{r}_{cm-s} \times \vec{p}_1) + (\vec{r}_{1-cm} \times \vec{p}_1) + (\vec{r}_{cm-s} \times \vec{p}_2) + (\vec{r}_{2-cm} \times \vec{p}_2) + (\vec{r}_{cm-s} \times \vec{p}_3) + (\vec{r}_{3-cm} \times \vec{p}_3) + \dots$$

- $$\vec{L}_{c-s} = \left(\vec{r}_{cm-s} \times (\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \dots)\right) + (\vec{r}_{1-cm} \times \vec{p}_1) + (\vec{r}_{2-cm} \times \vec{p}_2) + (\vec{r}_{3-cm} \times \vec{p}_3) + \dots$$

$$\vec{L}_{c-s} = (\vec{r}_{cm-s} \times \vec{p}_{tot}) + (\vec{L}_{1-cm} + \vec{L}_{2-cm} + \vec{L}_{3-cm} + \dots)$$

$$\vec{L}_{c-s} = \vec{L}_{cm-s} + \vec{L}_{c-cm} = \vec{L}_{translation-s} + \vec{L}_{rotation}$$

- **Translational /orbital.** The first term now looks just like the angular momentum for a point object, located at the center of mass and with the net linear momentum of the cloud. This term is what we'd call the "translational" or "orbital" angular momentum, for it describes the whole body up and translating or orbiting about the star.
- **Rotational.** The second term describes how the cloud rotates about its center of mass, thus it's called the "rotational" angular momentum.

- **Condensation of cloud to planet: Conservation of angular momentum.** Now say billions upon billions of years have passed and the dust cloud has indeed collapsed into a solid planet.
  - **Q:** How should the rotational speed of the newly formed, small, dense planet compare with that of the large, diffuse cloud it used to be?
    - **A:** Much larger.
    - **Q:** Why?
      - **A:** Just as with the spinning figure skater, negligible external torque, so the angular momentum is conserved. As I decreases, ω increases to maintain $L_{rot} = I\omega$.
- **Rigid body special case.** Now that we've got a fairly rigid body, the planet, we can take our expression for the rotational angular momentum a step further. Each of the particles of the planet now has the same angular velocity about the center of mass, we'll call that the rotational angular velocity since it corresponds to the planet rotating on its axis, $\vec{\omega}_{rot}$.

  $\vec{L}_{rot} = (\vec{r}_{1-cm} \times \vec{p}_1) + (\vec{r}_{2-cm} \times \vec{p}_2) + (\vec{r}_{3-cm} \times \vec{p}_3) + \ldots$

  - $\vec{L}_{rot} = (m_1 r^2_{1-cm}\vec{\omega}_{rot}) + (m_2 r^2_{2-cm}\vec{\omega}_{rot}) + \ldots$

    $\vec{L}_{rot} = I\vec{\omega}_{rot}$
  - So, $\vec{L}_{planet-star} = \vec{r}_{pl.cm-s} \times \vec{p}_{pl.cm} + I_{pl.cm}\vec{\omega}_{rot}$
  - Now, we've been talking about dust clouds, planets, and stars, but those specifics were just for the sake of concreteness- the relations we've reasoned out apply more broadly. Say you spin a baton, then the angular momentum of that baton about your head can be determined using the above relation.
  - **Demo: Spin baton – orbit head and rotate about center of axis.**
      -
  - **Demo: Spinning Baton**
    - Rotate baton around me so that it always points at them
    - **Q**: What kind(s) of angular momentum does it have?
      - **A:** Just orbital: $\vec{L}_{trans.baton-head} = \vec{r}_{b.cm-h} \times \vec{p}_{b.cm}$
      - **09_barbell.py**
    - Rotate it so that it keeps synched up with me
    - **Q**: What kind(s) of angular momentum does it have?
      - **A:** orbital and rotational: $\vec{L}_{b-h} = \vec{r}_{b.cm-h} \times \vec{p}_{b.cm} + I_b\vec{\omega}_{rot}$

        Both positive
      - **09_barbell.py,** click once
    - Rotate and twirl with opposite direction of spin.
    - **Q**: Does it have more or less angular momentum now?
      - **A**: Less. What is the direction of the orbital angular momentum? What is the direction of the rotational angular momentum? $\vec{L}_{b-h} = \vec{r}_{b.cm-h} \times \vec{p}_{b.cm} + I_b\vec{\omega}_{rot}$

        One positive, one negative

- **09_barbell.py,** click a few times to get angular momentum vectors pointing in opposite directions**.**
- **Vectors:** Remember, angular momentum, like linear momentum, is a vector quantity, and it adds vector-wise. So, for example, you could have two pucks spinning with equal and opposite angular momentum, then the combined system would have 0 angular momentum. That's similar to two carts running at each other with equal and opposite linear momentum.

- **In terms of Center of Mass Translation and Rotation.**
  - Last time, we found that it was convenient to rephrase the total angular momentum of a system in terms of its rotation, or spin, and its translation, or orbit. So we can break the left-hand side up that way. Similarly, we can break the right-hand-side up in terms of torques about the center of mass and torques about the reference point, A.

$$\frac{d\vec{L}_{tot-A}}{dt} = \vec{r}_{1-A} \times \vec{F}_{ext-1} + \vec{r}_{2-A} \times \vec{F}_{ext-2} + \vec{r}_{3-A} \times \vec{F}_{ext-3}$$

$$\frac{d\vec{L}_{cm-A}}{dt} + \frac{d\vec{L}_{rot-cm}}{dt} = \left(\vec{r}_{cm-A} + \vec{r}_{1-cm}\right) \times \vec{F}_{ext-1} + \left(\vec{r}_{cm-A} + \vec{r}_{2-cm}\right) \times \vec{F}_{ext-2} + \left(\vec{r}_{cm-A} + \vec{r}_{3-cm}\right) \times \vec{F}_{ext-3}$$

$$\frac{d\vec{L}_{cm-A}}{dt} + \frac{d\vec{L}_{rot-cm}}{dt} = \vec{r}_{cm-A} \times \left(\vec{F}_{ext-1} + \vec{F}_{ext-2} + \vec{F}_{ext-3}\right) + \vec{r}_{1-cm} \times \vec{F}_{ext-1} + \vec{r}_{2-cm} \times \vec{F}_{ext-2} + \vec{r}_{3-cm} \times \vec{F}_{ext-3}$$

$$\frac{d\vec{L}_{cm-A}}{dt} + \frac{d\vec{L}_{rot-cm}}{dt} = \vec{r}_{cm-A} \times \vec{F}_{net.ext} + \vec{r}_{1-cm} \times \vec{F}_{ext-1} + \vec{r}_{2-cm} \times \vec{F}_{ext-2} + \vec{r}_{3-cm} \times \vec{F}_{ext-3}$$

$$\frac{d\vec{L}_{cm-A}}{dt} + \frac{d\vec{L}_{rot-cm}}{dt} = \vec{\tau}_{cm-A} + \vec{\tau}_{1-cm} + \vec{\tau}_{2-cm} + \vec{\tau}_{3-cm}$$

Time for HW questions

Next two classes:

- Wednesday – Work, Kinetic & Potential Energies
- Friday – Conservative Forces & 1-D Systems