Mon. 9/13	**1.5,.7** Newton's 3rd & Polar Coordinates	
Wed. 9/15	**2.1-.2** Air Resistance - Linear	
Thurs 9/16		**HW1**
Fri. 9/17	**2.3** Trajectory and Range with Linear Resistance	

Homework Hint: 1.D from section 1.6.

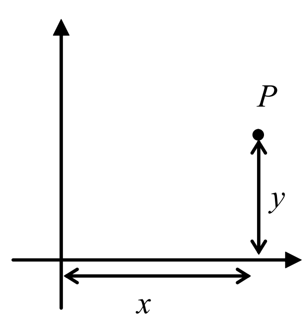
You want to know the maximum launch angle for a projectile to follow a path whose distance from the launch point / origin is ever growing (in contrast, shooting something almost straight up makes it start out getting further away, but then it starts coming closer again.) So, you'll want to develop an equation for the range of the projectile (actually, the square of the range is easier) in terms of launch angle and *t*. Since you want range to always grow with time, you want its derivative with respect to time to never change from + to -, as long as the projectile is in flight. Another way of saying that is if you set this relation equal to 0 and then solve for the *time* at which that happens (so the time at which the projectile *does* turn around and starts coming closer again), you'll find that your expression depends on the launch angle in such a way that if θ is too large, then your expression for *t* becomes nonphysical (like, a negative under a square root). So the angle at the threshold of that happening is the maximum launch angle for which there is no time at which the projectile would start coming closer to you. Enjoy

Section 1.7

Motivation. In general, a wise choice of coordinate systems makes it easier to do the math of a problem. Sometimes, that's as simple as rotating a Cartesian coordinate system to align with the slope of a plane (as we did last time) rather than aligning with gravity. Sometimes it's choosing polar rather than Cartesian coordinates. For example, when a ball is flying through the air, we usually choose a Cartesian coordinate system because then the graviatational force is always and purely in the –y direction, and never in the x. That makes the x-component equations particularly simple. Then again, if we're dealing with the Earth flying around the Sun, choosing polar coordinates would mean the gravitational force is always in the –r direction which makes the ϕ component equations particularly simple. It's all about making it easier to solve a problem.

Introduction: Cartesian vs. Spherical Coordinates.

- Expressing something's location in Cartesian coordinates is like saying "Take 3 paces due East, and 4 paces due North."

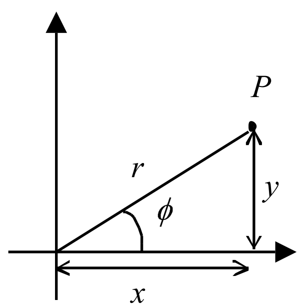


- **Last time: Equation of Motion in Cartesian coordinates under constant force**

- We brushed up on using the equation of motion,
  - $\vec{F} = m\vec{a} = m\dot{\vec{v}} = m\ddot{\vec{r}}$
- in Cartesian coordinates; that is, when it's convenient to speak of all vector quantities in terms of their x, y, and z components.
- Under a constant force, this lead to equations like
  - $F_x = m\ddot{x}$, $\dot{x} = \dot{x}_o + \ddot{x}\Delta t$, $x = x_o + \dot{x}_o\Delta t + \frac{1}{2}\ddot{x}(\Delta t)^2$ and their counterparts in the y and z direction. (Eq'ns 1)
  - One reason for the relative *simplicity* of these expressions is that the directions themselves, $\hat{x}, \hat{y}$, and $\hat{z}$, are constants.

This time: Equation of Motion in Polar

- Alternatively, when telling someone where something is, you could say "take 5 paces 53° North of East. That is, you can say how far away, and in what direction. That is speaking in terms of Polar (or spherical) coordinates.



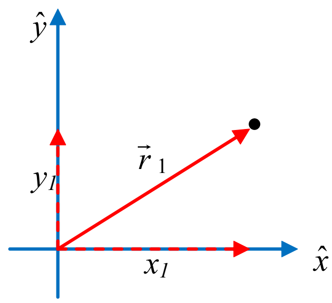
- You translate between these two expressions of position by using familiar trig functions and Pythagorean's Theorem:

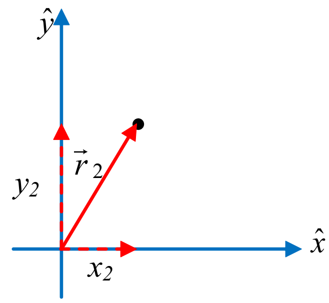
Cartesian Polar

- $$\left.\begin{array}{l} x = r\cos\phi \\ y = r\sin\phi \end{array}\right\} \longleftrightarrow \left\{\begin{array}{l} r = \sqrt{x^2 + y^2} \\ \phi = \tan^{-1}(y/x) \end{array}\right.$$

- Similarly, you could specify velocity not in terms of its x and y components, but in terms of its magnitude, i.e., speed, and direction. In the examples that we looked at last time, we translated between knowing the speed and direction of v_o to its x and y components so we could then tackle the problems in the convenient Cartesian coordinates. Now, what was convenient about those coordinates was that gravity pointed in a constant Cartesian direction – down, and the normal force pointed in a constant Cartesian direction – perpendicular to the plane.
- What if you've got a force that points toward a point rather than in a direction – for example, the tension in a string that tethers a ball? Then it might be convenient to work the problem in Polar coordinates. We're going to derive and use the equivalent of Equations 1 for *polar* coordinates. As you'll see, they're a bit messier.

- **Derivatives of Cartesian position vectors.** Suppose we want to write the position vector for a point. When using Cartesian coordinates, a vector is described by two constant unit vectors $\hat{x}$ and $\hat{y}$. These are in the same direction regardless of where the point is.





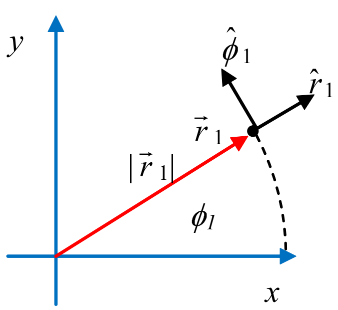
$$\vec{r} = x\hat{x} + y\hat{y}$$

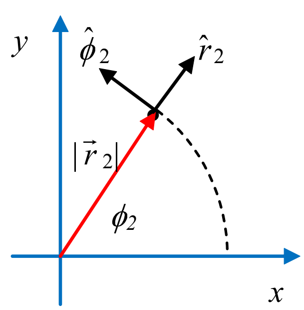
So, as an object moves from one point to another over time, the component directions remain constant:

$$\frac{d\vec{r}}{dt} = \frac{dx}{dt}\hat{x} + \frac{dy}{dt}\hat{y} \qquad \text{And} \qquad \frac{d^2\vec{r}}{dt^2} = \frac{d^2x}{dt^2}\hat{x} + \frac{d^2y}{dt^2}\hat{y}$$

In Polar coordinates, we're not so fortunate.

For plane polar coordinates, we will define the unit vectors $\hat{r}$, which is in the radial direction, and $\hat{\phi}$, which is perpendicular and points in the direction the point would move if ϕ were increased. These are shown in the diagram below.





Clearly, these unit vectors change as the point moves! The unit vector $\hat{r}$ points in the direction of $\vec{r}$ and has a length of one, so it can be expressed as:

$$\hat{r} = \frac{\vec{r}}{|\vec{r}|} = \frac{\vec{r}}{r}$$

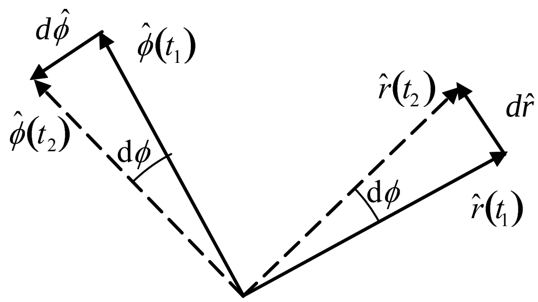
That means the position vector is:

$$\vec{r} = r\hat{r},$$

And $\hat{\phi}$ is the direction perpendicular to that.

Now, $$\frac{d\vec{r}}{dt} = \frac{d(r\hat{r})}{dt} = \frac{dr}{dt}\hat{r} + \frac{d\hat{r}}{dt}r = \dot{r}\hat{r} + \frac{d\hat{r}}{dt}r$$

What is $\frac{d\hat{r}}{dt}$? Well, what's just $d\hat{r}$? the little 'hat' over top of $\hat{r}$ may make it seem rather mysterious, but just think of it as a vector who happens to have length 1.



You may recall that an arc length is equal to the associated angle (measured in radians) times the radius, so the arc length that gets between the initial and final locations of the $\hat{r}$ tip would be $ds = |\hat{r}|d\phi = 1d\phi$ (last step since $|\hat{r}| = 1$.

To the extent that the angle change is infinitesimal, then the arc length is essentially equal to the straight path between the two points:

$$|d\hat{r}| = ds = 1d\phi$$

As for the *direction* of $d\hat{r}$, notice that it's virtually parallel to the $\hat{\phi}$ unit vectors; and, again, to the extent that the change in angle is infinitesimal, it *is* parallel to $\hat{\phi}$, so that's the direction. So, finally, we have reasoned out that

$$d\hat{r} = (d\phi)\hat{\phi}$$

Putting that into our expression for the velocity,

$$\dot{\vec{r}} = \dot{r}\hat{r} + \frac{d\phi}{dt}\hat{\phi}r$$

$$\boxed{\dot{\vec{r}} = \dot{r}\hat{r} + r\dot{\phi}\hat{\phi}}$$

Angular speed: you probably recognize $\frac{d\phi}{dt}$ as the "angular speed" or "angular frequency," often denoted $\omega \equiv \frac{d\phi}{dt}$, which tells us how frequently an object orbits or spins around.

So the rate with which the object is moving radially *in and out* from the origin is

$$v_r \equiv \dot{r}$$

And the rate with which the object moves angularly *about* the origin is

$v_\phi \equiv r\dot{\phi}$, or in possibly more familiar notation, $r\omega$.

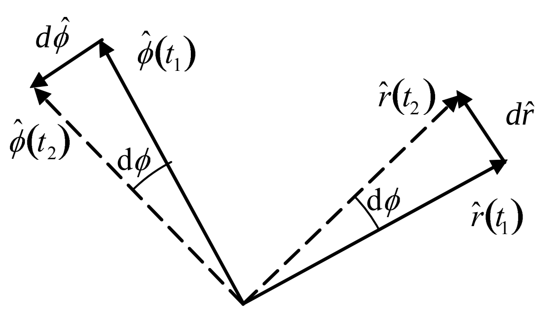
Now for the second derivative, acceleration,

$$\ddot{\vec{r}} = \frac{d}{dt}\left(\dot{r}\hat{r}\right) + \frac{d}{dt}\left(r\dot{\phi}\hat{\phi}\right)$$

$$\ddot{\vec{r}} = \frac{d\dot{r}}{dt}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\dot{\phi}\hat{\phi} + r\frac{d\dot{\phi}}{dt}\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt}$$

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\frac{d\hat{r}}{dt} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\frac{d\hat{\phi}}{dt}$$

Now, we've already figured out that $d\hat{r} = \left(d\phi\right)\hat{\phi}$, so we can substitute that in, but what is $d\hat{\phi}$? Looking back at our picture,



We can see that a very similar thing is happening with $d\hat{\phi}$ as with $d\hat{r}$. Thinking about the arc traced out as the $\hat{\phi}$ vector rotates from its initial to its final position,…

$$\left|d\hat{\phi}\right| = ds = \left|\hat{\phi}\right|d\phi = 1d\phi$$

Now, *this* change vector points virtually *anti*-parallel to the $\hat{r}$ vectors, so *its* direction is - $\hat{r}$. So we have

$$d\hat{\phi} = -\left|d\hat{\phi}\right|\hat{r} = -d\phi\hat{r}$$

Plugging this, and our expression $d\hat{r} = \left(d\phi\right)\hat{\phi}$, into the acceleration expression gives

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\frac{d\phi}{dt}\hat{\phi} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} + r\dot{\phi}\left(-\frac{d\phi}{dt}\hat{r}\right)$$

$$\ddot{\vec{r}} = \ddot{r}\hat{r} + \dot{r}\dot{\phi}\hat{\phi} + \dot{r}\dot{\phi}\hat{\phi} + r\ddot{\phi}\hat{\phi} - r\left(\dot{\phi}\right)^2\hat{r}$$

$$\ddot{\vec{r}} = \left(\ddot{r} - r\left(\dot{\phi}\right)^2\right)\hat{r} + \left(2\dot{r}\dot{\phi} + r\ddot{\phi}\right)\hat{\phi}$$

Or, recasting this in terms of angular speed, $\omega \equiv \dot{\phi}$, and angular acceleration, $\alpha \equiv \ddot{\phi}$,

$$\vec{a} = \left(\ddot{r} - r\omega^2\right)\hat{r} + \left(2\dot{r}\omega + r\alpha\right)\hat{\phi}$$

Speeding/slowing going in and out

Simply rotating

Going in & out *while* rotating

Speeding/slowing rotating

$$a_r = \ddot{r} - r\dot{\phi}^2 \qquad a_\phi = r\ddot{\phi} + 2\dot{r}\dot{\phi}$$

Now that we have expressions for position, velocity, and acceleration in terms of polar coordinates, we're ready to consider applying Newton's 2nd law.

The net force can be resolved into $\hat{r}$ and $\hat{\phi}$ components:

$$\vec{F} = F_r\hat{r} + F_\phi\hat{\phi}$$
$$\vec{F} = ma_r\hat{r} + ma_\phi\hat{\phi}.$$

Therefore, in plane polar coordinates Newton's second law is:

$$F_r = m\left(\ddot{r} - r\dot{\phi}^2\right),$$
$$F_\phi = m\left(r\ddot{\phi} + 2\dot{r}\dot{\phi}\right).$$

So, depending on how an object's already moving, a given force can affect its motion differently. For example, consider a ball tethered to point by an elastic string; if the ball's initially just moving radially out, then the force of the string radially back just causes the ball to slow down and eventually start coming back.

$$F_r = m\left(\ddot{r}\right)$$

Then again, if the ball were initially moving tangentially and not radially at all, then the exact same force could be responsible for it to orbit at a constant radius

$$F_r = -m\left(r\dot{\phi}^2\right)$$

<u>Example</u>: A small coin is set on a turntable at a distance R from the center. At time $t = 0$, the turntable starts accelerating with a constant angular acceleration $\alpha = \ddot{\phi}$. If the coefficient of static friction between the coin and the turntable is μ_s, how much time will pass before the coin starts to slip?

Let's think before we dive in. It's through friction that the turn table drags the coin along – forces the coin to accelerate along with the turn table. However, static friction can only grow so large before breaking and letting the coin slip:

$$F_{fr} \le \mu N$$

Where N is the normal force with which the coin presses into the surface of the turn table; given that this is on the level, that should be balancing the coins weight, so

$$N = mg \qquad \text{so} \qquad F_{fr} \le \mu mg$$

We want to know at what time, *t,* the friction force hits its limit while forcing the coin to accelerate with the spinning turntable.

$$F_{fr} = \mu mg = m|a(t)|$$

It's going to be easiest to consider this in polar coordinates since, up to this moment of slipping, the distance from coin to origin (center of turn table) is constant.

$$|a| = \sqrt{a_r^2 + a_\phi^2}$$

$$a_r = \underbrace{\cancel{\ddot{r}}}_{0} - \underbrace{r}_{R}\dot{\phi}^2 \qquad a_\phi = \underbrace{r}_{R}\underbrace{\ddot{\phi}}_{\alpha} + 2\underbrace{\cancel{\dot{r}}}_{0}\dot{\phi}$$

So these simplify to

$$a_r = -R\dot{\phi}^2 \qquad a_\phi = R\alpha$$

So, how does the angular speed change with time? Well, the angular *acceleration* of the turntable is constant at:

$$\ddot{\phi} = \alpha\,,$$

so its angular speed grows linearly with time:

$$\dot{\phi}(t) = \alpha t\,.$$

(note: the relationship between ω and α is the same as that between *v* and *a*)

Putting all this together,

$$\mu g = |a(t)| = \sqrt{\left(R(\alpha t)^2\right)^2 + (R\alpha)^2}$$

Then solving for the time at which this equality holds,

$$\alpha^2 t^4 + 1 = (\mu g/R\alpha)^2$$

$$t = \frac{1}{\sqrt{\alpha}}\left[(\mu g/R\alpha)^2 - 1\right]^{1/4}$$

Reasonability Check. It's always good to pause and check that your final answer seems reasonable. Let's do that:

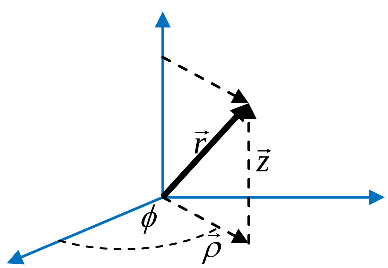
Units - The angular acceleration α is in radians/s^2, so the units work out (radians are "non-units").

Logic - The bigger α is, the shorter the time is. The larger the distance from the center, R, the shorter the time is. If the term in the square brackets is negative, $\mu g / R\alpha < 1$, then the coin will start to slip immediately. That's because the static friction is not as big as $F_\phi = mR\alpha$.

A note about Cylindrical Coordinates. In this chapter we pretty much stayed 2-D. Looking ahead, we'll certainly get into 3-D situations too. Depending on the symmetry of the problem, spherical or cylindrical coordinates might be best. Since the polar to cylindrical jump is pretty easy, I'll point it out now.

The z component of a point P is the same as in Cartesian coordinates. Also, the unit vector $\hat{z}$ is constant. If P' is the projection of the point in the xy plane, then the coordinates ρ and ϕ are defined like r and ϕ were above. The name of one of the coordinates changes because:

$$\vec{r} = \rho\hat{\rho} + z\hat{z}.$$



Then, we translate everything we'd said about r in 2-D to being about ρ and tack on the z term.

$$\vec{v} = \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{z}\hat{z}$$

$$\vec{a} = (\ddot{\rho} - \rho\omega^2)\hat{\rho} + (2\dot{\rho}\omega + \rho\alpha)\hat{\phi} + \ddot{z}\hat{z}$$

In general, problems in polar coordinates are fairly difficult to handle using Newton's Second Law. However, they are easier to handle with Lagrangians (Ch. 7).

Coming up next:

A common class of force problems – 'projectile motion' in a constant field (gravitational, electric, or magnetic)

Next two classes:

- Wednesday – Air Resistance
- Friday – Linear Air Resistance