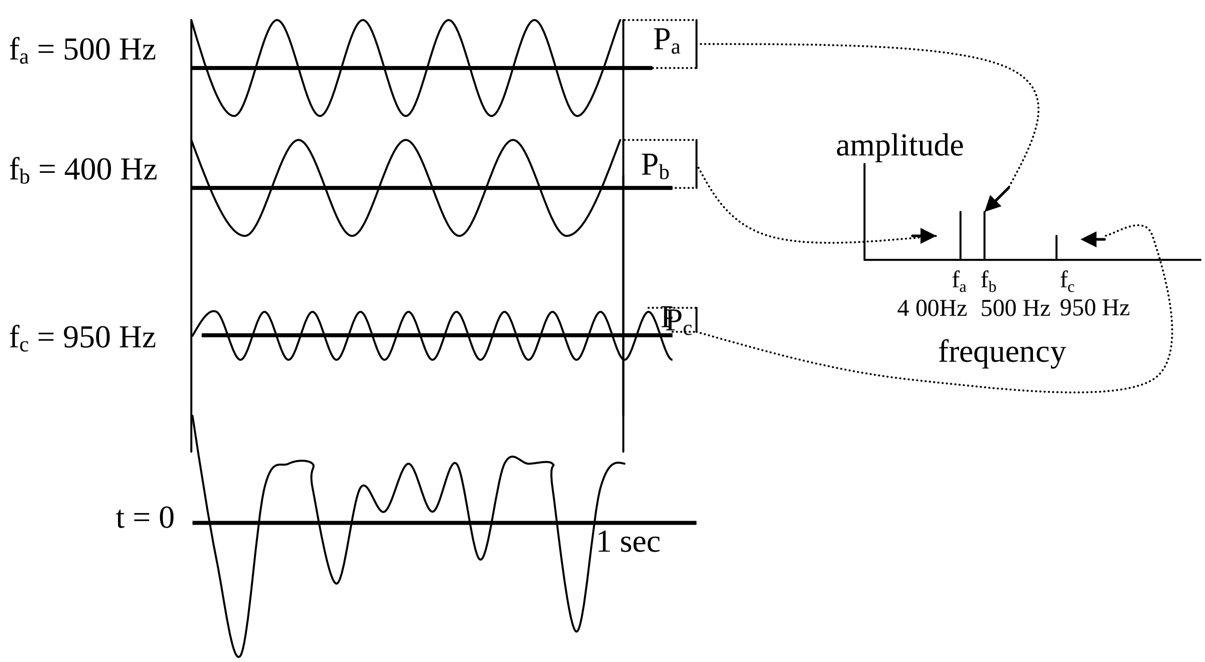
Quantum & Light: waves & photons

Chapter 6 is pretty abstract stuff. I'll endeavor to make it a *little* less so by relating it to things we already know. So bear with me while I first point out applicable stuff that you already know and then translate a fairly understandable scenario into the language of quantum mechanics.

Consider sound. From Fourier's theorem, no matter how complicated the actual state of sound in, say, a concert hall, we can think of all that complexity as the superposition of simple sinusoidal waves of different amplitudes, and frequencies (pitches). In Quantum Mechanics, we'll deal with waves and the same superposition principle applies. Say you're listening to a concert and over a given second, three distinct notes are played, to speak generically, call them notes *a*, *b*, and *c*. So the complicated pressure wave (graphically represented at the bottom left of the figure below) can be thought of as a superposition / simultaneous sounding of three distinct pure tones, (graphically represented in the figure just above the complicated composite pressure wave). Mathematically, we can write out

$$P_{sound}(t) = P_a \sin(2\pi f_a t + \phi_a) + P_b \sin(2\pi f_b t + \phi_b) + P_c \sin(2\pi f_c t + \phi_c)$$

(each phi is a measure of the initial phase of each wave.) With light or sound, one often speaks of the complicated state in terms of its simple ingredients by way of a "Fourier Spectrum." Since it's a given that each of these pure tones is sinusoidal, a sufficient and more compact representation of the same idea is the bar graph on the right. It just tells what the amplitude is for each frequency. That's known as a "Fourier Spectrum."



Now, we're going to venture a little into math land, the realm of 'yes, I guess you *could* do that mathematically, but it I don't know why you'd *want* to.' Of course, the reason for doing it is that we'll develop more general and more powerful tools that we'll need later. The main goal is to find ways to pars out the different pieces of information:

- The Amplitudes of the different waves
- The Sin waves themselves

The first step is to simply make two separate lists – one of the amplitudes for the different pure tones and one for the sine functions (can you say "eigen"?) of the different pure tones.

- List of amplitudes for each tone, essentially the info in the bar-graph plot.
  - $\langle P_{sound}| = \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix}$
- Individual simple sine waves (wavefunctions) for each tone.
  - $|a\rangle = \begin{bmatrix} \sin(2\pi f_a t + \phi_a) \\ 0 \\ 0 \end{bmatrix}, \; |b\rangle = \begin{bmatrix} 0 \\ \sin(2\pi f_b t + \phi_b) \\ 0 \end{bmatrix}, \; |c\rangle = \begin{bmatrix} 0 \\ 0 \\ \sin(2\pi f_c t + \phi_c) \end{bmatrix}$

In this funny notation, the way to recombine the two pieces of information is to dot product the amplitude list with one of the wavefunction lists. For example

- $\langle P_{sound} \| b\rangle = \begin{bmatrix} P_a \\ P_b \\ P_c \end{bmatrix} \bullet \begin{bmatrix} 0 \\ \sin(2\pi f_b t + \phi_b) \\ 0 \end{bmatrix} = P_a \cdot 0 + P_b \cdot \sin(2\pi f_b t + \phi_b) + P_c \cdot 0 = P_b \cdot \sin(2\pi f_b t + \phi_b)$

Thus one extracts back out, puts back together, the wave of pitch *b*. Similarly, to get the whole, complicated sound wave back, you can do this:

- $\langle P_{sound} | (|a\rangle + |b\rangle + |c\rangle) = \langle P_{sound} \| a\rangle + \langle P_{sound} \| b\rangle + \langle P_{sound} \| c\rangle =$
  $P_a \sin(2\pi f_a t + \phi_a) + P_b \sin(2\pi f_b t + \phi_b) + P_c \sin(2\pi f_c t + \phi_c) = P_{sound}(t)$

Now let's think about what we actually *hear*, what it is about this complicated sound wave that we perceive and then how we can get at *that* in our math. Practically speaking, when we hear sound, we don't perceive it as this complicated jumble; heck, we don't even perceive the phases, or the time variations beyond identifying a pitch (that is, when someone holds down middle C on a piano, you don't can't count out the pressure at your ear *undulating* every 3.8ms, you just perceive a *stead* pitch of middle C.) What you do perceive is three distinct notes of played with their different loudnesses. Our hearing system is pretty complicated but for the sake of this exploration it suffices to say that loudness relates to *Power* (the rate at which a sound delivers energy to the ear) which itself relates to the *square* of the sound pressure. So here's how you extract from the math (something akin to) what we actually *measure.*

- So, If we wanted to talk about how loud note *b* was, we'd want to know P_b^2. Now, recall from above that $\langle P_{sound} \| b\rangle = P_b \cdot \sin(2\pi f_b t + \phi_b)$ So, to mathematically pull out and square that magnitude, we could do this
  - $\frac{2}{T_b} \int_{T_b} |\langle P_{sound} \| b\rangle|^2 dt = \frac{2}{T_b} \int_{T_b} P_b^{\,2} \cdot \sin^2(2\pi f_b t + \phi_b) dt = P_b^{\,2} \frac{2}{T_b} \int_{T_b} \sin^2(2\pi f_b t + \phi_b) dt$
    $P_b^{\,2} \frac{2}{T_b} \left(\frac{1}{2} T_b \right) = P_b^{\,2}$

You may well be wondering 'if we *already* know P_{sound} as a function of P_b, why are we doing all this complicated math?' The answer is that 'even if we *don't* know P_{sound} as a function of P_b, if we know it as a completely different looking function, such as equation Q1.13 for a square wave, we could use this math to extract P_b.'

Recap. So, here's we've learned that's applicable to Q.M.

- **Eigenvector rule.** For a given measurable (pitch that you hear), there's a specific wavefunction, a.k.a eigenvector.
- **Statevector rule.** You can make an array, or vector, that lists the amplitudes of each simple wave (pure tone) that's represented in the complicated wave (the full three-tone combo.)
- **Superposition Rule.** You can pull back out the amplitudes of each simple wave by No matter how complicated a wave, you can represent it as a superposition / sum of the simple wavefunctions for different modes.

Light

Now that you've met the general tools that are applicable to *any* wave mechanics, we'll consider light and address a quintessentially quantum mechanical question: say you have white-light flashlight (so a spectrum of colors represented) shining on a wall. If you had it dim enough or a sensitive enough detector, you could detect one photon at a time hitting the wall. The question is what's the probability that a specific color (corresponding to frequency or wavelength) hits the wall next? To tackle this, rather than using a basis set of cosines and sines for my eigen functions I will, equivalently, use one complex exponentials. So, the field at a given point and time is

$$E_{total}(x,t) = |E_a| e^{i(k_a x-\omega_a t+\phi_A)} + |E_b| e^{i(k_b x-\omega_b t+\phi_b)} + |E_c| e^{i(k_c x-\omega_c t+\phi_c)} + \ldots$$

After the discussion of sound above, we could represent this in terms of an array containing all the amplitudes of the component simple electric fields and then eigen vectors for each simple component.

- $\langle E_{total}| = \begin{bmatrix} |E_a| \\ |E_b| \\ |E_c| \\ \ldots \end{bmatrix}$ and $|a(x,t)\rangle = \begin{bmatrix} e^{i(k_a x-\omega_a t+\phi_A)} \\ 0 \\ 0 \\ \ldots \end{bmatrix}$, $|b(x,t)\rangle = \begin{bmatrix} 0 \\ e^{i(k_b x-\omega_b t+\phi_b)} \\ 0 \\ \ldots \end{bmatrix}$, etc.

Eigenvector Rule: $|a(x,t)\rangle$, $|b(x,t)\rangle$, $|c(x,t)\rangle$, etc. are going to be the 'color' Eigenvectors for this problem. Each one corresponds to a specific wavelenfth / frequency / color. We won't yet dub the other array the "state vector"; we've got some mathematical massaging to do first.

Time-Evolution rule: if you recall that a general mathematical rule about dealing with exponents is $e^{x+y}=e^x e^y$, then you'll notice that you can write one of these eigenvectors as

- $|a(x,t)\rangle = \begin{bmatrix} e^{i(k_a x-\omega_a t+\phi_A)} \\ 0 \\ 0 \\ \ldots \end{bmatrix} = \begin{bmatrix} e^{i(k_a x+\phi_A)} \\ 0 \\ 0 \\ \ldots \end{bmatrix} e^{-i(\omega_a t)} = |a(x,0)\rangle e^{-i(\omega_a t)}$

- If you further recall that for a photon (indeed, we'll find it's true for all quantons) $\varepsilon = hf = \hbar\omega$, then you'll see that we can rewrite that time-dependent factor as $e^{-i\varepsilon t/\hbar}$, just as the book states.
- $|a(x,t)\rangle = |a(x,0)\rangle e^{-i(\varepsilon_a t/\hbar)}$: the eigenvector a time t later is equal to what it was at t=0 times this exponential factor.

Just as with the sound, the 'inner product" of the array of amplitudes and one of the eigenvectors returns the contribution to the total field of that particular mode / color.

- $$\langle E_{total} \| b(x,t)\rangle = \begin{bmatrix} |E_a| \\ |E_b| \\ |E_c| \\ \ldots \end{bmatrix} \bullet \begin{bmatrix} 0 \\ e^{i(k_b x - \omega_b t + \phi_b)} \\ 0 \\ \ldots \end{bmatrix} = |E_a| \cdot 0 + |E_b| e^{i(k_b x - \omega_b t + \phi_b)} + |E_c| \cdot 0$$
  $$\langle E_{total} \| b(x,t)\rangle = |E_b| e^{i(k_b x - \omega_b t + \phi_b)}$$

Another bit of math that will come in handy is taking the absolute square to get the magnitude square back (and losing all that $e^{i\ldots}$ stuff)

- $$|\langle E_{total} \| b(x,t)\rangle|^2 = \left(|E_b| e^{i(k_b x - \omega_b t + \phi_b)}\right)\left(|E_b| e^{-i(k_b x - \omega_b t + \phi_b)}\right) = |E_b|^2 e^{i(k_b x - \omega_b t + \phi_b) - i(k_b x - \omega_b t + \phi_b)}$$
  $$|\langle E_{total} \| b(x,t)\rangle|^2 = |E_b|^2 e^0 = |E_b|^2 1$$
  $$|\langle E_{total} \| b(x,t)\rangle|^2 = |E_b|^2$$

Now, to get at the chance of a particular color (a, b, c,…) photon being detected, we need to translate this classical representation of the electric field to a photon picture. A good bridge between classical and quantum mechanical formulation is intensity since we can phrase that both in terms of amplitude of field and number of photons. The intensity of a beam of light is $I = c\varepsilon_o |E|^2$ from classical E&M. Meanwhile, we now understand that this energy is delivered in packets of 'photons.' So we could rephrase the intensity as follows. We can apply that for each color individually. Imagine light of color *a* , frequency f_a, shining a beam across the room to make a spot on a wall. The whole beam is a cylinder reaching from the light to the wall; it has volume *Vol* and a number n_a photons in it. Since each photon's moving toward the wall at speed *c*, the rate at which these photons hit a unit area of the wall is

- $\frac{n_a}{Vol} c$

Since each photon bears energy hf_a, then the rate at which *energy* is delivered to patch of wall, i.e., *Intensity* is

$$I_a = \frac{c(hf_a)n_a}{Vol}$$

Equating that with the expression we already have for intensity, in terms of the amplitude of the electric field,

$$I_a = \frac{c(hf_a)n_a}{Vol} = c\varepsilon_o |E_a|^2$$

$$\frac{(hf_a)n_a}{Vol} = \varepsilon_o |E_a|^2 = \varepsilon_o \left|\langle E_{total} \| a(x,t)\rangle\right|^2$$

$$\frac{n_a}{Vol} = \frac{\varepsilon_o}{hf_a}\left|\langle E_{total} \| b(x,t)\rangle\right|^2$$

Of course, the *number* of each color of photon in the cavity is

$$n_a = Vol\frac{\varepsilon_o}{hf_a}\left|\langle E_{total} \| a(x,t)\rangle\right|^2, n_b = Vol\frac{\varepsilon_o}{hf_b}\left|\langle E_{total} \| b(x,t)\rangle\right|^2, n_c = Vol\frac{\varepsilon_o}{hf_c}\left|\langle E_{total} \| c(x,t)\rangle\right|^2, ...$$

Q: Now, we're ready for answering our question, what's the probability that a photon of a particular color, say color *b*, is the next one we detect?

A: $\Pr(b) = \frac{n_b}{n_a + n_b + n_c + ...} = \frac{Vol}{N}\frac{\varepsilon_o}{hf_3}\left|\langle E_{total} \| b(x,t)\rangle\right|^2$

Alright, last step, I'm going to define an amplitude for each of these colors that sucks in all those factors outside the inner product.

$A_b = \sqrt{\frac{Vol\varepsilon_o}{Nhf_b}}|E_b|$ and ditto for the others. With that, I can write an array that I'll call the **State Vector** for determining color**.**

$$\langle\psi_{color}| = \begin{bmatrix} A_a \\ A_b \\ A_c \\ \cdots \end{bmatrix}$$

Finally, in terms of this state vector and the color eigenvectors, the probability of the next photon being color *b*, is

$\Pr(b) = \left|\langle\psi \| b(x,t)\rangle\right|^2$**.**

Recap of the Rules

- **State Vector Rule.** As we saw in this example, we can define a "statevector" which holds the basic information about how strongly each simple mode or "eigenvector" contributes to complex state of the system, in this case, the beam of multi-colored light.
- **Eigenvector Rule.** Each of these simple modes and their corresponding properties (colors) is represented by an "eigenvector."
- **Collapse Rule.** In this example, it kind of goes without saying that, you don't know which color it's going to be until you actually measure it.

- **Outcome Probability Rule.** The probability of finding any one particular color is calculated as above, the square magnitude of the inner product of the 'state' vector and the 'eigen' vector for the color you're interested in.
- **Time Evolution Rule.** This was the observation that the eigenvector at time t is a product of the eigenvector at time 0 and the time factor: $\left|a(x,t)\right\rangle = \left|a(x,0)\right\rangle e^{-i(\varepsilon_a t/\hbar)}$

Hopefully, seeing these rules applied to a couple of more tangible systems – sound's pitch, light's color – helps you to get a little more familiar with what they mean and where they come from. You noticed that most of these rules could be applied to a completely classical wave system – the sound. What's specifically 'quantum mechanical' about the light example is just that the light comes in units, photons, with energy related to frequency: $\varepsilon = hf = \hbar\omega$.