7.8 USING THE GENERAL METHOD

Some of the problems with the weakest-link rule were pointed out in Section 7.5. A better way to calculate uncertainties is to use the general approach outlined in section 7.3. With this method you calculate the variation in f when you vary each variable while leaving the other variables constant. For example, the variation in f with a variable a is $\delta f_a = (\partial f/\partial a)U[a]$, where $\partial f/\partial a$ is the partial derivative of f with respect to a. Equation 7.2 can be rewritten more explicitly as

$$U[f]=\sqrt{\left(\frac{\partial f}{\partial a}U[a]\right)^2+\left(\frac{\partial f}{\partial b}U[b]\right)^2+\ldots}\,. \tag{7.10}$$

This equation can be used to find the uncertainty for any function. A couple of common examples are given below.

If the function is a sum or difference, $f = a \pm b$, then the partial derivatives are $\partial f/\partial a = 1$ and $\partial f/\partial b = \pm 1$. Using equation 7.10, the uncertainty in f is

$$U[f]=\sqrt{(U[a])^2+(U[b])^2}\,. \tag{7.11}$$

The result is the same for both sums and differences.

If the function is a product, $f = a\cdot b$, then $\partial f/\partial a = b$ and $\partial f/\partial b = a$. The uncertainty in f is

$$U[f]=\sqrt{(bU[a])^2+(aU[b])^2}=\sqrt{(ab)^2\left\{\left(\frac{U[a]}{a}\right)^2+\left(\frac{U[b]}{b}\right)^2\right\}}$$

which can be written as

$$U[f]=|f|\sqrt{\left(\frac{U[a]}{a}\right)^2+\left(\frac{U[b]}{b}\right)^2}\,. \tag{7.12}$$

The result can be expressed simply in terms of fractional uncertainties as

$$Q[f]=\frac{U[f]}{|f|}=\sqrt{\left(\frac{U[a]}{a}\right)^2+\left(\frac{U[b]}{b}\right)^2}=\sqrt{(Q[a])^2+(Q[b])^2}\,. \tag{7.13}$$

Equations 7.12 and 7.13 also hold for division, $f = a/b$ (see exercise 7.4).

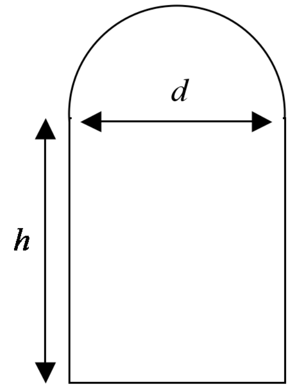
Example 7.6.2 (Revisited): Suppose that the number of bacteria in a certain colony at a certain time is $N = 305{,}000 \pm 15{,}000$. Find the uncertainty in $f = \ln N$ using the general approach. (You might need to know the uncertainty of the logarithm if you want to draw an uncertainty bar for this data point on a log-log graph.)

The partial derivative of f with respect to N is $\partial f/\partial N = 1/N$, so

$$U[f]=\sqrt{\left(\frac{\partial f}{\partial N}U[N]\right)^2}=\frac{1}{N}U[N]=\frac{15{,}000}{305{,}000}=0.049\approx 0.05, \tag{7.14}$$

where we have rounded to one significant digit. This result agrees with the other methods used in Section 7.6

Example 7.8.1: Suppose you measure the dimensions of the object below as $d = 4.24 \pm 0.03$ cm and $h = 6.07 \pm 0.03$ cm.



Are your measurements consistent with those of someone who reports $A = 31.3 \pm 0.6\ \text{cm}^2$?

In the intermediate step of the calculation, I'll keep an extra digit (marked with square brackets) and display the appropriate number of significant figures in the final step. To calculate the area from the measurements of the dimensions d and h, use the equation

$$A = hd + \frac{\pi r^2}{2} = hd + \frac{\pi d^2}{8}, \tag{7.15}$$

which gives

$$\begin{aligned} A &= (6.07\ \text{cm})(4.24\ \text{cm}) + \frac{\pi(4.24\ \text{cm})^2}{8} \\ &= 25.74\ \text{cm}^2 + 7.060\ \text{cm}^2 = 32.8\ \text{cm}^2. \end{aligned}$$

The uncertainties in the products hd and d^2 are

$$\begin{aligned} \sigma_{hd} &= hd\sqrt{\left(\frac{\sigma_h}{h}\right)^2 + \left(\frac{\sigma_d}{d}\right)^2} \\ &= (6.07\ \text{cm})(4.24\ \text{cm})\sqrt{\left(\frac{0.03\ \text{cm}}{6.07\ \text{cm}}\right)^2 + \left(\frac{0.03\ \text{cm}}{4.24\ \text{cm}}\right)^2} = 0.22\ \text{cm}^2, \end{aligned}$$

and

$$\sigma_{d^2} = d^2\sqrt{2\left(\frac{\sigma_d}{d}\right)^2} = d\sigma_d\sqrt{2} = (4.24\ \text{cm})(0.03\ \text{cm})\sqrt{2} = 0.18\ \text{cm}^2 \cdot$$

The uncertainty in the second term in the expression for the area, $\pi d^2/8$, is

$$\sigma_{\pi d^2/8} = \frac{\pi}{8}\sigma_{d^2} = \left(\frac{\pi}{8}\right)\left(0.18\ \text{cm}^2\right) = 0.071\ \text{cm}^2 \cdot$$

Using the rule for quantities that are added, the uncertainty in the area is

$$\sigma_A = \sqrt{\sigma_{hd}^2 + \sigma_{\pi d^2/8}^2} = \sqrt{\left(0.22\ \text{cm}^2\right)^2 + \left(0.071\ \text{cm}^2\right)^2} = 0.2\ \text{cm}^2 \cdot$$

The experimental value for the area is $32.8 \pm 0.2\ \text{cm}^2$, so the value is expected to be in the range $32.6 - 33.0\ \text{cm}^2$. The other person's measurement of $31.3 \pm 0.6\ \text{cm}^2$ means the value should be in the range $30.7 - 31.9\ \text{cm}^2$. Since the two ranges do not overlap, the measurements are not consistent.

Example 7.8.2: Suppose $f = \ln(a + b)$. What is the uncertainty of f in terms of a, b, and the uncertainties $U[a]$ and $U[b]$?

It is convenient to do this problem in steps. From equation 7.11, we know that the uncertainty in the sum $(a + b)$ is

$$U[a+b] = \sqrt{\left(U[a]\right)^2 + \left(U[b]\right)^2}\,.$$

From equation 7.14 in example 7.6.2, we have

$$U\left[\ln N\right] = \frac{1}{N}U\left[N\right].$$

Combining these two equations yields

$$U[f] = U\left[\ln(a+b)\right] = \frac{1}{a+b}U[a+b] = \frac{1}{a+b}\sqrt{\left(U[a]\right)^2 + \left(U[b]\right)^2}\,.$$

7.9 THE *REAL* BOTTOM LINE

You will be expected to state uncertainties of *all* quantities that you calculate based on your measurements in lab. You can use the weakest-link rule to estimate these uncertainties, which may be useful while you're performing an experiment. However, you must use the general method for your final uncertainty calculations. You can also use the *PropUnc* program to check your calculations.

ADDITIONAL EXERCISES

Exercise 7.5
Use the general approach to show that for division, $f = a/b$, the uncertainty is given by equation 7.12.

Exercise 7.6
Suppose that a sample of water with a mass of 1.1±0.1 kg starts at a temperature of 281±4 K and finishes at a temperature of 321±3 K. The specific heat of water is c = 4186 J/(kg·K) with negligible uncertainty. The amount of energy that must flow into the water to cause this temperature increase is given by $Q = mc\Delta T$, where $\Delta T = T_f - T_i$ is the change in temperature. Find the heat flow and its uncertainty. (Hint: First, find ΔT and it uncertainty. Treat ΔT as a single variable for the final part of the calculation.)