Wed., 11/28 Thurs., 11/29 Fri., 11/30	**22.8-9, .6** Energy, Diff. form, Superconductors **Quiz** Ch 22, **Lab 10** Faraday's Law **23.1,2,7** Ampere-Maxwell, E&M Pulse	**RE28**

Handouts:

- **Equipment:** maglev with superconductor demo (includes liquid nitrogen in a thermos)

Last Time

Faraday's law allows us to quantitatively speak about inductance (how a time varying current produces a "non-coulombic" electric field which is responsible for an *emf* that opposes the current's variation). Through this mechanism, if a current is varying in one wire, it can "induce" a current in a nearby wire. For that matter, it the current itself is subject to the non-columbic field – thus "self-induction."

Inductors are circuit components that have been optimized to exhibit this effect.

Energy in the Magnetic Field

- To the extent that Electric and Magnetic fields quantify interactions, it makes sense that we can phrase the energy invested in those interactions in terms of the fields, and thus, conceptually abstract from thinking of the energy as associated with the interactions but with the fields. We'd gone down that path for Electric fields in the very simple case of a capacitor. We found that we could phrase the energy invested in pilling charges on a capacitors plates could be phrased in terms of the associated electric field as
    - $\frac{Energy}{Volume} = \frac{1}{2}\varepsilon_o E^2$
- Now we're positioned to do the same thing for the energy invested in setting up a current in a solenoid. So, here we go.
- As always, when a charge is moved across a potential difference, work is done on the charge (just like a mass moving from one elevation to another has gravitational work done on it).
    - $W = -q\Delta V$
- The rate at which this work is done we call Power:
    - $P = \frac{W}{\Delta t} = -\frac{q\Delta V}{\Delta t}$
- If we take Dt to be the time for the charge to clear the voltage difference, then we can call q/Dt the current.
    - $P = -I\Delta V$
- Now let's specifically consider a voltage that's established by a time varying current:
    - $\Delta V = -L\frac{dI}{dt}$
- So, the *rate* at which energy is being transferred to these charged particles is

- $P = -I\Delta V = LI\frac{dI}{dt} = L\frac{1}{2}\frac{dI^2}{dt}$

- Then, the amount of energy transferred in the process of ramping up from 0 current to I is
  - $\int_i^f Pdt = Energy = \int_i^f L\frac{1}{2}\frac{dI^2}{dt}dt = \frac{1}{2}LI^2$
- Conversely, this much energy would have to be removed from the current in the solenoid if the current were turned off. So there is this much energy 'stored' in the current configuration.
- Then again, corresponding to ramping up *I* current is ramping up *B* magnetic field where, for our solenoid, $B = \frac{m_o NI}{d}$.
- So we could say that this much energy is invested in the *field* configuration, and we can even phrase the energy strictly in terms of the field:
- $B = \frac{m_o NI}{d} \Rightarrow I = \frac{Bd}{m_o N}$ and $L = \frac{m_o N^2}{d} pR^2$
- So our energy can be rephrased as
  - $Energy = \frac{1}{2}LI^2 = \frac{1}{2}\frac{m_o N^2}{d} pR^2\left(\frac{Bd}{m_o N}\right)^2 = \frac{1}{2}m_o B^2\left(pR^2 d\right)$
  - where the term in brackets is the volume of the solenoid, so we can rephrase this as
  - $\frac{Energy}{volume} = \frac{1}{2}m_o B^2$
- Though derived and justified in a very specific case, this is a very general result –the energy associated with setting up a magnetic field (by getting currents flowing) per volume through which the field exists is this.
- I should reiterate that you can equivalently think of this energy as a property of the current configuration. This would suggest that you don't absolutely *need* the concept of the magnetic field – it's just handy. The same was true about the energy associated with an Electric Field / a Charge Configuration. *However*, just as with the electric interaction, I don't see any way around treating the field as real when charges *accelerate*. Then they shed energy – radiate, and it's got to go *somewhere* – into the field.
- Exercises for Ch. 22

The Character of Physical Laws: (the name of a nice little book by Richard Feynman)

- I disagree with some of the details, but not the spirit of this section of the chapter. At the heart of the physics program is making observations about nature; from those, you try to deduce the most fundamental and concise truths, and the ramifications. These most fundamental truths are unexplained. We can state them and use them, but we don't know *why* they are the laws of nature. One of major program in physics is discovering the fundamental truths and that often leads to unification – realizing that something we *thought* was fundamental actually followed from something else. For example, all magnetic and electric phenomena (including Faraday's law) seem to follow from Coulomb's law and relativity. But why are Coulomb's law and relativity the way they are? We don't know – we might be able to rephrase things to give conceptual tools, and to subtly change the question (like, 'why is the speed of light constant?'), but underlying these two are two things we just haven't explained.
- At some point, even in physics, a dialog with a three-year-old (who incesintly asks "why") ends in "no one knows." That said, we keep pushing in hopes that some day we'll be able to move one more fact from the unexplained to the explained side of the ledger.

Differential Form of Faraday's Law:

- The amount that the electric field (the non-Coulomb part) curls around a point is related to the rate of change of the magnetic field there:
  - $\oint \vec{E} \cdot d\vec{\ell} = -\int \left(\frac{\partial}{\partial t} \vec{B} \right) \cdot \hat{n}\, dA$
- But recall that in the previous chapter we'd defined (cast in vague / general math terms)
- $\left[\mathrm{curl}\left(\vec{F}\right)\right]_z = \lim_{\Delta A_{\perp z} \to 0} \frac{\oint \vec{F} \cdot d\vec{\ell}}{\Delta A_{\perp z}}$ (here, F is some mathematical function that is a vector)
- At the time, the vector we were considering was, B, the magnetic field, but generally, this is what we mean by the curl of a function.
- Applying that to Faraday's relation gives
  - $\left[curl(\vec{E})\right]_z = \lim_{\Delta A_{\perp z} \to 0} \frac{\oint \vec{E} \cdot d\vec{\ell}}{\Delta A_{\perp z}} = \lim_{\Delta A_{\perp z} \to 0} \frac{-\int \left(\frac{\partial}{\partial t} \vec{B} \right) \cdot \hat{n}\, dA}{\Delta A_{\perp z}} = -\frac{\partial}{\partial t} B_z$
- Similarly for the x and y components, so
  - $curl(\vec{E}) = -\frac{\partial}{\partial t} \vec{B}$
- we also showed that $curl = \vec{\nabla} \times$
- So we now have

- $$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$
- This is perfectly equivalent to Faraday's Law.
- So, thus far we have these laws:

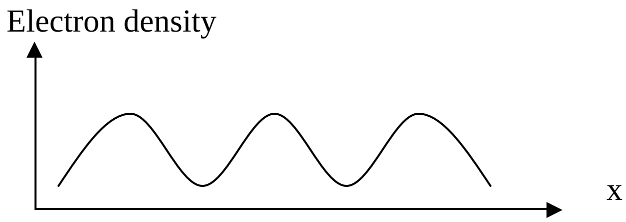
$\oint \vec{E} \cdot \hat{n}\, dA = \frac{\sum q_{\text{inside}}}{e_0}$	$\nabla \cdot \vec{E} = \frac{r}{e_0}$	Gauss's law for electricity
$\oint \vec{B} \cdot \hat{n}\, dA = 0$	$\nabla \cdot \vec{B} = 0$	Gauss's law for magnetism
$\oint \vec{E} \cdot d\vec{\ell} = -\int \left(\frac{\partial}{\partial t} \vec{B} \right) \cdot \hat{n}\, dA$	$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$	Faraday's law
$\oint \vec{B} \cdot d\vec{\ell} = m_0 \sum I_{\text{inside path}}$	$\vec{\nabla} \times \vec{B} = m_o \vec{J}$	Ampere's law (incomplete)

- In Ch. 23, we will fix up Ampere's law, then we'll have the complete set of Maxwell's equations!

Armed with these, we can tackle something pretty exotic:

Superconductors:

- **Qualitative Description.**
    - Superconductors are famous for having no resistivity. As the book also notes, it follows that they have peculiar magnetic properties. Here's a very qualitative explanation.
    - **Cooper Pairs.**
        - **Electron Density / Lattice Periodicity** Recall our picture of a conductor: an electron sea against the backdrop of periodic ion cores. Consider just one electron moving through a conductor. To zeroth order, we might imagine that it swims along against a uniformly positive backdrop. But, look a little closer, the positive charges are localized on the periodically located ions. The electron is attracted to the ions, so its path reflects the periodicity of the ion lattice. In particular, it would prefer to spend a little more time closer to an ion. Now imagine a whole stream of such electrons – they'll have a higher density near the ions than away from them. A stream headed in a particular direction might have a density like:



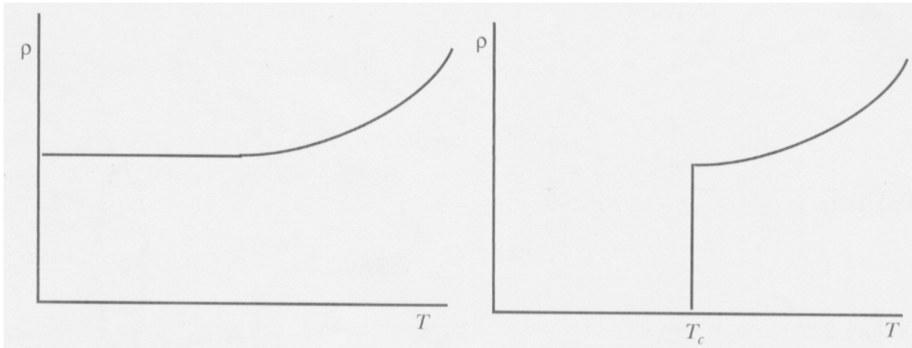
- Note that electrons flowing in the exact *opposite* way would have the exact same density profile as would our stream.

- **Electron – Phonon interaction: a wake.** Enough about how the ions impact the electron flow for now, now let's consider how the *electron flow* impacts *the ions*. Just as the ions attract the electrons, the electrons attract the ions. Imagine just one electron swimming along. Where ever it is, it draws the nearby ions slightly toward it, but since these ions are a bit massive and are bound elastically to each other, they're a little sluggish to respond and the scenario looks much like a boat leaving a wake across the surface of water – the moving electron leaves a "wake" of ion displacement. Should another electron come along soon after the first, it will encounter this "wake." If it 'rides in the wake' it's a bit easier going. This phenomenon is familiar in many mediums: birds fly in formation to take advantage of each other's wakes, and I often amuse myself during my son's bath time by 'pulling' his rubber ducky around with out actually touching it – the wake of my hand draws the duck along.

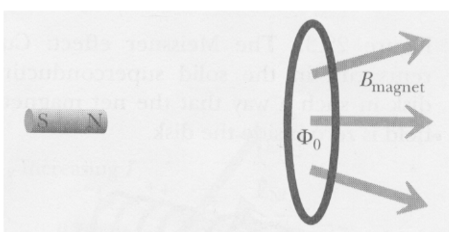
- **Phonon-mediate pairing.** Back to our picture of streams of electrons. Recall that an electron stream moving to the right has the same density pattern as one moving just as fast to the left – they can perfectly ride in each other's wakes. In this way, their ways are eased. They are referred to as a "Cooper Pair" (after the fellow who first recognized that this could happen.) Of course, the downside of pairing up like this is the familiar electrical repulsion two electrons feel for each other. Which effect dominates determines whether or not pairs are formed, and thus whether or not a material is superconducting (Abrikosov p334… (the discussion focuses on electrons near the Fermi level, presumably because only these have neighboring free states into which they can move, should a field be imposed, and thus only they ultimately contribute to currents, perhaps the pairing happens for lower energy electrons too, but they're electrically irrelevant).

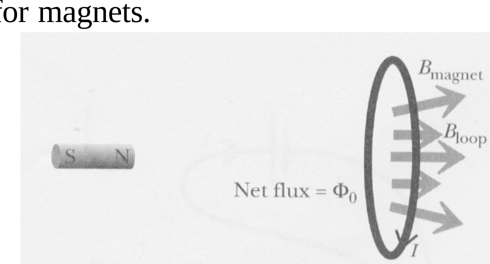
- One great feature of this pairing is that, since the electrons are swimming like a school of fish, all coordinated, they not only ease the way for each other, the continuous flow can build into the flight plan deflections around impurities and such. In this way, collisions with impurities that impede lone electrons are avoided – like a school of fish swimming around some kelp. Meanwhile, random thermal jiggling of the ions *can* interact with the electrons, but the effect is either that they are too small to impede them (adsorbed into the flow) or are large enough to break a cooper pair – rendering them 'normal' electrons again. Then again, 'normal' electrons will sometimes meet and pair up – at a given temperature, these two processes are in equilibrium and determine the population of paired electrons.

- The resistivity of ordinary materials decreases to a non-zero value as the temperature approaches absolute zero. For superconductors, the resistivity drops to zero below a critical temperature T_C. A current that is started in a superconductor can run for years! (no energy transfer)



- **No Resistance – No Drude Model: E field means *acceleration.*** Without reisistance, we no longer apply the Drude model: It's not the electron's drift velocity that's proportional to an imposed electric field, it's the electron's acceleration that is proportional.
- **Magnetic Effects.** Now think about Faraday's law: while the magnetic flux is changing, an *emf* is induced, and that *accelerates* (as opposed to 'maintains') a current. But this accelerating current produces a changing flux of its own, and in the opposite direction of the original one. The current will accelerate and accelerate until – *it's* changing flux counters that due to the withdrawl of the magnet – then there will be no net *emf* and the current will maintain its new value. The result is that there's a feedback loop so that there's no net change in magnetic flux – if there was flux through the material when it went superconducting, then the flux stays, even when you remove the source (the superconductor turns into an electromagnet), and if there was none before, then there never will be one, even if you bring a magnet near. A superconductor is like a perfect mirror for magnets.





- Type I superconductors "expel" the magnetic field when they become superconducting, which is known as the Meissner effect. This was an experimental surprise that can be explained with a quantum mechanical model (BCS theory). If a magnet is brought near a superconductor, there must be induced currents which produce a magnetic field in the opposite direction inside the superconductor. Since $\vec{B} = 0$ inside, Ampere's law tells us that the current must also be zero inside –they flow only on the surface! (otherwise, we'd have $B = \frac{m_o I}{2pr}$ *inside* the superconductor.)

- **Quantitative Description**
  - Now we can at least sketch an argument for their being neither current nor field inside a superconductor.

- What is that current? Imagine the process of turning on the current by turning on an external magnetic field. Turning on the external magnetic field means creating a curled electric field:

  - $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$ Eq. 1

- These fields will exist both inside and outside the superconductor (what counter fields the superconductor can generate is another matter).
- *En lieu* of any resistance, wherever there is electric field, there's an acceleration of charge:

  - $m\frac{d\vec{v}}{dt} = q\vec{E} \Rightarrow \frac{d}{dt}\left(\frac{m}{q^2 n}\vec{J}\right) = \vec{E}$ Eq. 2

- holds (note that now the rate of change rather than the current itself is proportional to E).
- Ampere's Law gives us another relationship between current density and magnetic field.

  - $\vec{\nabla}\times\vec{B} = m_o\vec{J}$ Eq. 3

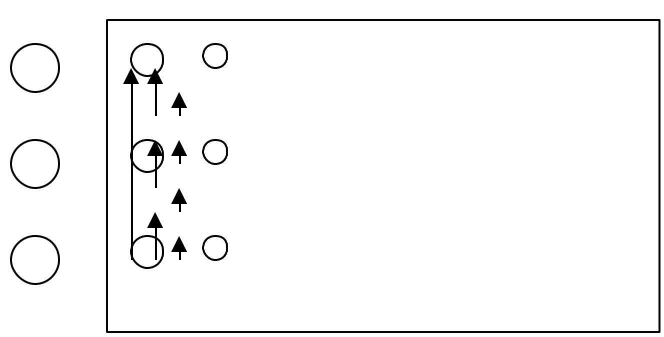
- Now we have three equations and three unknowns (E, B, and J); that should sound promising. They can be put together to solve for B or J (or E for that matter) independently.*

  - $$\nabla^2\left(\frac{m}{q^2 n}\vec{J}\right) = m_o\vec{J}$$
    $$\nabla^2\left(\frac{m}{q^2 n}\vec{B}\right) = m_o\vec{B}$$

- A function who's second derivative is itself times a constant…

  - $$B = B_o e^{-r/d}$$
    $$J = -\frac{B_o}{m_o d}e^{-r/d}$$

- where $d = \sqrt{\frac{m}{m_o q^2 n}}$ is the "London" penetration depth, which is on order of 10^{-7}-10^{-8}m.



*For instructor's eyes only: (more detailed derivation of London length)

If we Curl both sides of equation 2, we can bring in Faraday's Law to relate the current density to the *magnetic* field:

$$\vec{\nabla}\times\frac{d}{dt}\left(\frac{m}{q^2 n}\vec{J}\right)=\vec{\nabla}\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}$$

Strictly speaking, $\frac{d}{dt}=\frac{\partial}{\partial t}+\vec{v}\vec{\nabla}$, but Abrikosov argues that the second term is negligible in this case (p. 330)

$$\vec{\nabla}\times\frac{\partial}{\partial t}\left(\frac{m}{q^2 n}\vec{J}\right)=-\frac{\partial\vec{B}}{\partial t}$$

$$\frac{\partial}{\partial t}\left(\vec{\nabla}\times\left(\frac{m}{q^2 n}\vec{J}\right)+\vec{B}\right)=0$$

So, to within a time-independent constant, k, the term in the brackets is zero:

$$\vec{\nabla}\times\left(\frac{m}{q^2 n}\vec{J}\right)+\vec{B}+\mathrm{k}=0$$

But we know that initially J = B = 0 which implies that at that time (and so, for all time) k =0.

$$\vec{\nabla}\times\left(\frac{m}{q^2 n}\vec{J}\right)=-\vec{B}\quad\text{(Eq. 1)}$$

Now, Ampere's Law provides a second relationship between the magnetic field and the current: $\vec{\nabla}\times\vec{B}=m_o\vec{J}$.

Substituting this in gives

$$\vec{\nabla}\times\vec{\nabla}\times\left(\frac{m}{q^2 n}\vec{J}\right)=-m_o\vec{J}$$

$$\vec{\nabla}\times\vec{\nabla}\times\left(\frac{m}{q^2 n}\vec{B}\right)=-m_o\vec{B}$$

In general $\vec{\nabla}\times\vec{\nabla}\times\vec{B} = \vec{\nabla}\left(\vec{\nabla}\cdot\vec{B}\right) - \nabla^2\vec{B}$, but Gauss's Law for B tells us that $\vec{\nabla}\cdot\vec{B} = 0$ so $\vec{\nabla}\times\vec{\nabla}\times\vec{B} = -\nabla^2\vec{B}$. Similarly, one can argue that the current is continuously flowing with no source, thus no divergence: $\vec{\nabla}\times\vec{\nabla}\times\vec{J} = -\nabla^2\vec{J}$. So,

$$\nabla^2\left(\frac{m}{q^2 n}\vec{J}\right) = m_o\vec{J} \qquad B = B_o e^{-r/d}$$

$$\text{so} \qquad J = -\frac{B_o}{m_o d} e^{-r/d}$$

$$\nabla^2\left(\frac{m}{q^2 n}\vec{B}\right) = m_o\vec{B}$$

(the negative sign in the exponent is required to satisfy Eq. 1)

where $d = \sqrt{\dfrac{m}{m_o q^2 n}}$ is the "London" penetration depth, which is on order of 10^{-7}-10^{-8}m.

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Demo: float a magnet above a piece of superconductor

- The superconductor acts like a magnet with poles facing the opposite way, so it repels the bar magnet. There is "feedback" because the superconductor's current changes as the distance of the magnet changes.
- **Destroying Superconductivity**
- **Magnetic Field.** If a superconductor is exposed to too large an external field, H_c, the superconductivity is destroyed. This can be simply understood in terms of the effects of a magnetic field on the members of the cooper pairs – say the field points up, one half of the pair is moving right and the other half is moving left. In the presence of this field, the right bound one will want to arc counterclockwise up while the left bound will want to arc counter clockwise down – the field tries to split the pair. If the field is strong enough, this can overcome the interaction that binds the pair together.
- **Temperature.** If the superconductor is too hot, superconductivity can be destroyed. At any temperature, there is a rate at which cooper pairs are formed and there's a rate at which they are broken by random thermal jiggling of the ions. The latter rate is temperature dependent – the hotter it is, the more powerful jiggles there are, and thus the more likely that a pair would get knocked apart. The balance of these two processes determines the equilibrium population of cooper pairs. Only at low temperatures is that equilibrium population substantial.