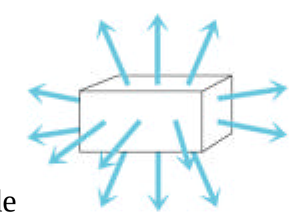
Fri., 3/20	**21.1-3** E Flux and Gauss's Law	**RE23**
Mon., 3/23 Wed., 3/25 Thurs., 3/26 Fri., 3/27	**21.4-6** Using Gauss's & Intro to Ampere's **21.7-9** Maxwell's, Gauss's, and Ampere's **Quiz** Ch 21, **Lab 9** Ampere's Law **(*write up*)** **22.1-2,10** Intro to Faraday's & Lenz's	**RE24 Lab notebook** **RE25** **RE26**

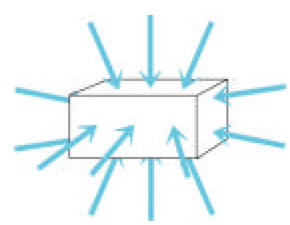
Check WebAssign

- **Intro**
    - Now that we've learned the basics of Electric and Magnetic interactions, and how we calculate things like the electric and magnetic fields generated by particular charge / current distributions, we're ready to take a second pass – develop more specialized and powerful tools. Gauss's Law and Ampere's Law are both geometric in nature. In fact, the general reasoning involved is broadly applicable – not just confined to electricity and magnetism.
    - The first tool we'll meet is Gauss's Law. It helps you to deduce charge distributions from the electric fields they generate.
    - Based on the reading..
    - **Julie's Slide 3**
- **Qualitative Motivational Examples.** You can determine what is inside a box by knowing the electric field at the surface. Sketch some examples:

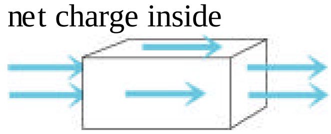
Vpython 21_Gauss_2.py



- E-field pointing out everywhere – positive (net) charge inside



- E-field point in everywhere – negative (net) charge inside
- E-field horizontal everywhere, so it is into one side of a box & outside the other – no net charge inside

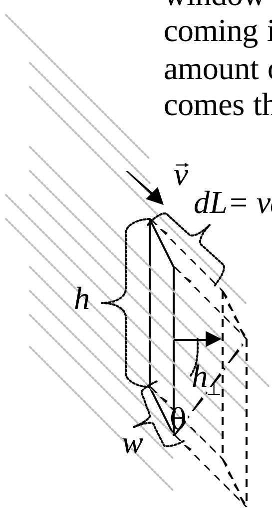


- Box could be between two capacitor plates
- E-field radiating linearly out of box – net positive charge inside – perhaps box straddles a positive plate that is sandwiched between two negative plates outside the box.

- We will get a quantitative relationship between the electric field on (through) a closed surface and the net charge (amount and sign) inside the closed surface.

- **Flux – the *general* idea / tool**
  - If the first time you're meeting a physics application of "flux" is here, with the electric field, it may not look it, but what we mean by "flux" is, generally, not that different from common usage: Usually, we mean "a flow", or more specifically, the rate at which some thing is transported across a boundary. In fact, we've already met a flux, only we didn't call it that – conventional current is a flux / flow of charge. But let's look at an even more tangible / everyday example:

- **Example: Rain**
  - **Flux through single, open area.** Say it's raining out and you have left your window open. Then, a reasonable question would be, at what rate is rain coming in your window. To make it concrete, let's say that we measure amount of rain in terms of the mass of water. What's the rate at which water comes through the window?



$$\Phi_{water\to window} = \frac{dm_{water}}{dt} = \frac{\Delta m_{water}}{\Delta Vol}\frac{dVol}{dt} = r_{water}\frac{(dL)wh_\perp}{dt} = r_{water}\frac{dL}{dt}wh_\perp = r_{water}vw(h\cos q)$$

$$\Phi_{water\to window} = r_{water}vA(\cos q)$$

$$\Phi_{water\to window} = r_{water}\vec{v}\cdot\vec{A}$$

- **Flux through whole closed area.** Now, what if we were interested in not just the flux through a particular window, but into the whole room. That's simply the sum of fluxes in through all windows, out through the door, down through the floor (boy, that's gonna be a mess to clean up!).

$$\Phi_{water \to room} = \Phi_{water \to window1} + \Phi_{water \to window2} + \Phi_{water \to floor} + \Phi_{water \to door} = \sum r_i \vec{v}_i \cdot \vec{A}_i$$

- **Area direction convention – "in."** In this case, it's convenient to have all the area vectors point *into* the room, so that you get positive contribution if the velocity points in and negative contribution if it points out (as with the door and floor.)

- **Integral Form.** More generally, this discrete sum can be written as a continuous integral.

$$\Phi_{water.into.room} = \oint r\vec{v} \cdot d\vec{A}_{in}$$

For the sake of moving future arguments, often one talks about the flux *out* through a closed area rather than *in* through it. Of course, that's just the opposite of *in* through the area.

$$\Phi_{water.out.room} = \oint r\vec{v} \cdot d\vec{A}_{out}$$

- **Generalizing: Any "vector field."** While our physics idea of "flux" connects best with the common usage when we're talking about something (in this case, water) being transported, the same mathematical and conceptual tool can be applied to any thing that's represented by a vector field, such as… the electric field.

- **Electric Flux**

  - Now let's apply the same idea to an Electric field.

    Through a patch of area,

$$\Phi_E \equiv \vec{E} \bullet \vec{A}$$

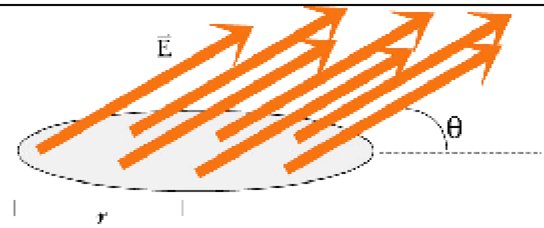
Clicker Question 21.2a

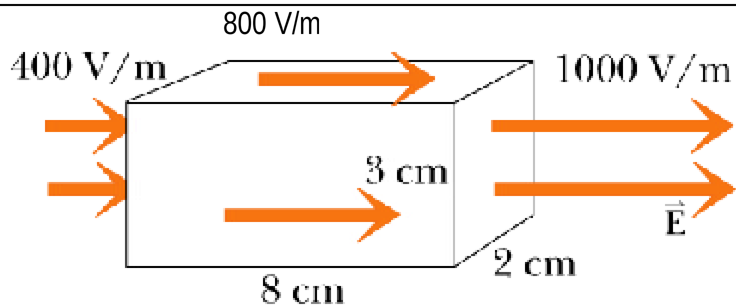
In or out of a closed surface

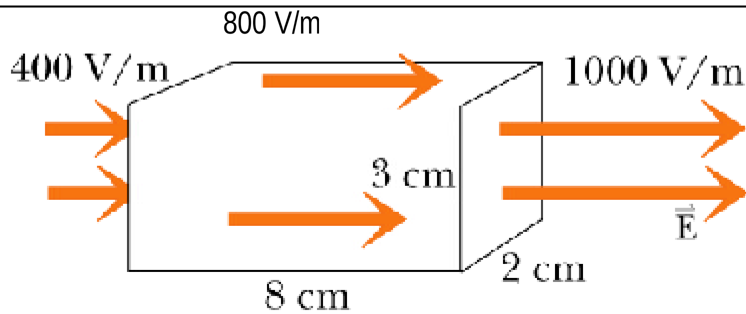
$$\Phi_E \equiv \oint \vec{E} \bullet d\vec{A}$$

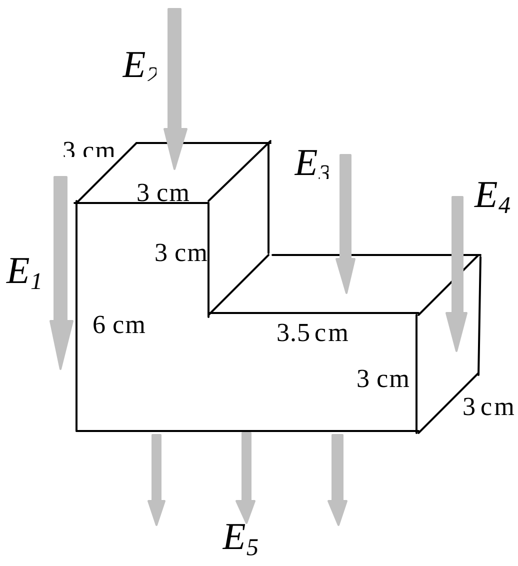
Clicker Queston 21.2b, 21.3a,b

E = 600 V/m, r=3 cm, θ = 25 degrees	What is the electric flux on this disk-shaped surface? 1. 600 V· m 2. 544 V· m

	3. 253 V· m 4. 1.70 V· m 5. 1.54 V· m 6. 0.717 V· m

	What is the net electric flux on the box? 1. 0 V· m 2. 0.36 V· m 3. 0.84 V· m 4. 8.04 V· m 5. 8.52 V· m

	What is the net charge inside the box? 1) 3.2 E –12 C 2) 7.6 E –12 C 3) –3.2 E –12 C 4) –7.6 E – 12 C 4) 1.3 E –11 C 6) –1.3 E –11 C



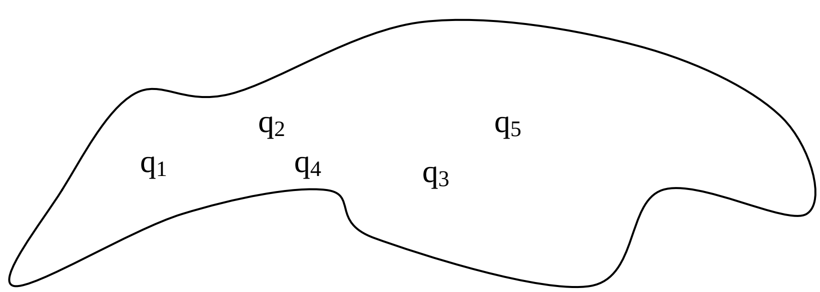
If $E_1 = E_2 = E_3 = E_4 = 826$ N/C and $E_5 = 600$ N/C, what is the magnitude and sign of the charge enclosed in th

1) -1.42e-11 C
2) 1.05e-11 C
3) -3.9e-12 C
4) 1.42e-11 C
5) 2.56e-12 C

- **Motivation – What's it good for?** That's a swell definition and all, but you may be wondering why we bother – what practical use is such a definition. It turns out that there's a very simple relation between electric flux and charge distribution. That means that if you know one, it's easy to find the other, and in some cases it's easier to solve for E through a flux argument than simply summing over the sources as we've done in the past.
- If that seems a worthy goal, then let's figure out what that but what's important is that, regardless of how messy the integral is, it has very simple solution. The book argued it out piece by piece, so now let's put all the pieces together in a coherent story.

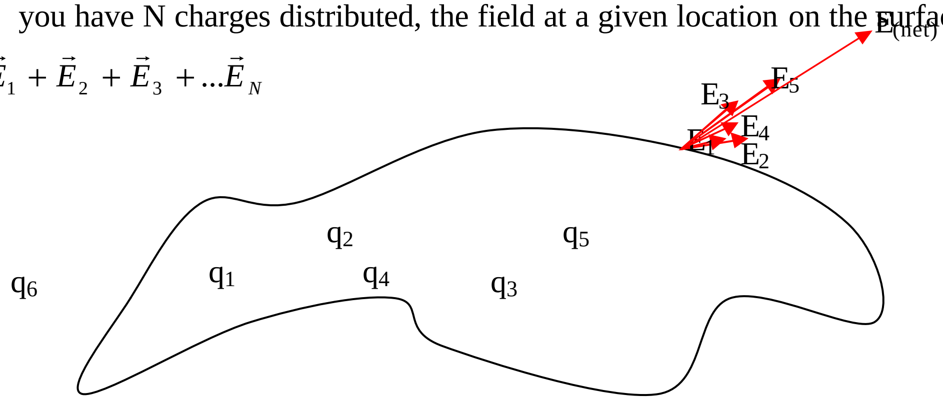
Relate Flux and Charge

- This is a very general tool, so I'll give you some vague visuals to help think about it, but don't take them too literally. Say we have some charge distribution like want to know the Flux of the electric field out a surface like this



- **First, Supperposition**: recall that the field of a charge distribution is simply the sum of the fields due to each point charge that makes up the distribution. That is, no matter how you have N charges distributed, the field at a given location on the surface is simply

$$\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots \vec{E}_N$$



That means that

$$\Phi_E \equiv \oint \vec{E} \bullet d\vec{A} = \oint \left(\vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \ldots \vec{E}_N\right) \bullet d\vec{A} = \oint \vec{E}_1 \bullet d\vec{A} + \oint \vec{E}_2 \bullet d\vec{A} + \oint \vec{E}_3 \bullet d\vec{A} + \ldots \oint \vec{E}_N \bullet d\vec{A}$$

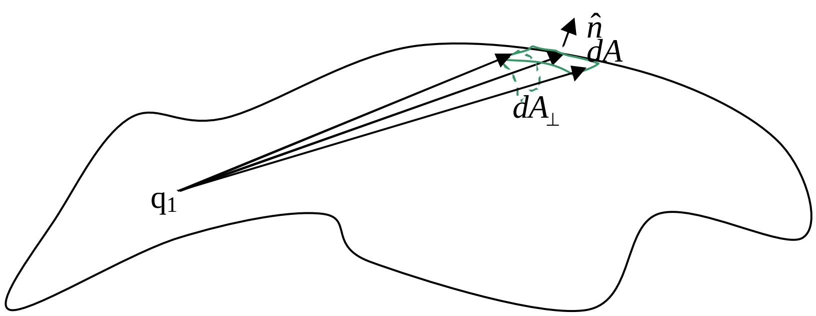
$$\Phi_E = \Phi_{E1} + \Phi_{E2} + \Phi_{E3} + \ldots \Phi_{EN}$$

Looking just at Charge 1 (which is *inside* the area). So, we can divide and conquer – find the flux due to one point charge, and then put it back together. Of course, the field at our observation location due to a point charge is simply $\vec{E}_1 = \frac{1}{4\pi\epsilon_o}\frac{q_1}{r_{0-1}^2}\hat{r}_{0-1}$.

So,

$$\Phi_{E1} = \oint \vec{E}_1 \bullet d\vec{A} = \oint \frac{1}{4\pi\epsilon_o}\frac{q_1}{r_{0-1}^2}\hat{r}_{0-1} \bullet d\vec{A} = \frac{q_1}{4\pi\epsilon_o}\oint \frac{1}{r_{0-1}^2}\hat{r}_{0-1} \bullet d\vec{A} = \frac{q_1}{4\pi\epsilon_o}\oint \frac{1}{r_{0-1}^2}\hat{r}_{0-1} dA_\perp$$

Now, let's consider that dot-product. It simply calls for the projection of the patch of area perpendicular to r-hat. So, even if we're dealing with a surface like



we still only have to worry about the perpendicular projection of the patch of area. Let's parameterize the patch in terms of variables we can hope to integrate. Given the spherical symmetry of the point charge's electric field, a good choice of coordinate systems is spherical. In that system, a differential area is

$$dA_\perp = r d\phi r \sin\theta d\theta = r^2 d\phi \sin\theta d\theta \ .$$

Plugging that into our flux equation, we have

- **Flux through patch.**
  - Now, across just this little patch of surface area, the flux is then just

$$\Delta\Phi_{E1} = \frac{q_1}{4\pi\varepsilon_o}\frac{1}{r_{0-1}^2} r_{0-1}{}^2 \Delta\phi(\sin\theta)\Delta\theta = \frac{q_1}{4\pi\varepsilon_o}\Delta\phi(\sin\theta)\Delta\theta$$

Depends only on Angles. Remarkably, it only depends on the solid angle the area subtends.

> If "the solid angel of the area subtends" is a bit foreign to you, think of it this way. Say you're a charge, radiating in all directions, and this room, it's walls, ceiling, and floor define the surface area. How much of your radiation passes through a given surface depends on how much of your view it takes up. Looking at a tile above you, it measures about 10° (or π/18 radians) by about 10° (or π/18 radians) so its solid angle is roughly the product, $\pi^2/324$ rad^2, that's what determines how much of your radiation flows through it.

Flux through whole surface. Going ahead and integrating over the whole surface area then gives

$$\Phi_{E1} = \frac{q_1}{4\pi\varepsilon_o}\oint d\phi \sin\theta d\theta = \frac{q_1}{4\pi\varepsilon_o}\int_{\phi=-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_{\theta=0}^{2\pi} d\phi \sin\theta d\theta = \frac{q_1}{4\pi\varepsilon_o}4\pi = \frac{q_1}{\varepsilon_o}$$

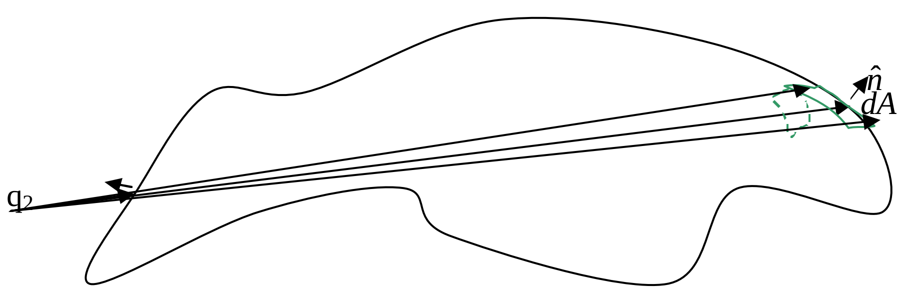
And there's a *very* simple result!

Note that, in the end, it didn't matter what *r* was – therefore, it doesn't really matter what shape the surface has or where the charges are within it.

Charge *outside* the area.

Now, before we go back to the total flux for a big charge distribution, we need to consider the other case – here we've considered a point *inside* the surface, what about one *outside* the surface?

Well, notice that the dot-product of r and n was positive, and that, as long as the charge is inside the surface, it r and n will always both point outward, so they'll always be positive.



If the charge is *outside,* on the near surface n and r will point in opposite directions, so the dot product will be negative while on the far side they'll point in the same direction, so they'll be positive. All that matters in the math is the angle subtended; perhaps you'll buy that, as you sweep through angles, for every patch through which the field flows *out* there's a canceling patch through which the field flows *in.* So summing over the whole surface, the flux comes to zero. (This is a very sketchy argument, the book is a little more thorough.)

The analogous situation would be if you had a shower head spraying water out in all directions (instead of a charge) and a loose-mesh bag (for the surface), then whatever water flows in from the left, flows out through the right, making for no net flux "into" or "out of" the bag.

So any charge *enclosed* by the surface contributes to the flux to the tune of $\frac{q_{enclosed}}{e_o}$ while all excluded charges contribute nothing.

Thus we have

$$\Phi_E \equiv \oint \vec{E} \bullet d\vec{A} = \frac{q_1}{e_o} + \frac{q_3}{e_o} + \ldots \frac{q_N}{e_o}$$

$$\boxed{\Phi_E = \oint \vec{E} \bullet d\vec{A} = \frac{Q_{enclosed}}{e_o}}$$

regardless of the shape of the surface.

This is the integral form of **Gauss's Law.** (later we'll describe this same relation in differential form)

Julie's Slide 4

Next time, we'll put this to some use.

All the rest of this is redundant with the above – it roughly follows the book's development.

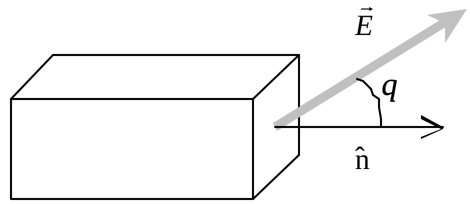
Definition of Electric Flux:

This depends on three factors:

- Direction of the electric field

  Does the electric field point out of, into, or along the surface? These cases will correspond to positive, negative, and zero flux.

  What if the electric field points outward, but not perpendicular to the surface? Define an "outward-going normal," a unit vector $\hat{n}$ perpendicular to the surface pointing outward (as shown below). The electric flux should depend on $\cos q$. This works regardless of direction of the electric field.



- Magnitude of the electric field

  If the direction of the electric field remains the same, but its size gets larger, the flux should increase. It should depend on $E\cos q = \vec{E}\cdot\hat{n}$.

- Size of the surface area

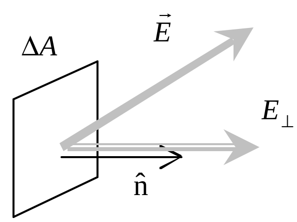
  Suppose the electric field is the same size and direction on two flat surfaces (e.g. a side of a box). A larger surface area should have a larger flux, so it should be defined as $\vec{E}\cdot\hat{n}\,\Delta A$.

For a closed surface, the magnitude and direction of the electric field may vary with location. The direction of the outward-going normal will vary over the closed surface. To find the electric flux, the surface must be divided into small enough patches of area that these factors are approximately constant for each patch. Sum (integrate) over all of the patches to get:

$$\text{electric flux on a surface} = \sum \vec{E}\cdot\hat{n}\,\Delta A = \int \vec{E}\cdot\hat{n}\,dA$$

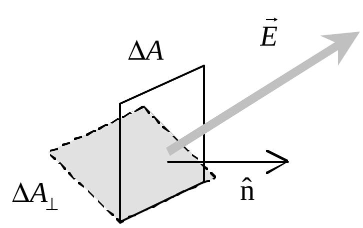
There are two ways of thinking of the flux for a small surface:

1. The area times the component of the electric field perpendicular to it.



$$\vec{E}\cdot\hat{n}\,\Delta A = E_{\perp}\,\Delta A$$

2. The size of the electric field times the projection of the surface area perpendicular to the electric field. The perpendicular projection of the surface area $\Delta A_{\perp}$ can be thought of as the area of a shadow cast by the surface on a paper perpendicular to the electric field if a light shines along the electric field.



$$\vec{E}\cdot\hat{n}\,\Delta A = E\,\Delta A_{\perp}$$

I prefer the first of these!

Gauss's Law:

We want to show that the electric flux on a closed surface is proportional to the amount of charge inside the surface:

$$\sum \vec{E} \cdot \hat{n}\, \Delta A = \frac{\sum q_{\text{inside}}}{e_0}.$$

1. Find the proportionality constant– use a single positive charge $+Q$ in the center of a sphere of radius r.

   The electric field is the same size everywhere on the surface:

   $$E = \frac{1}{4pe_0}\frac{Q}{r^2}$$

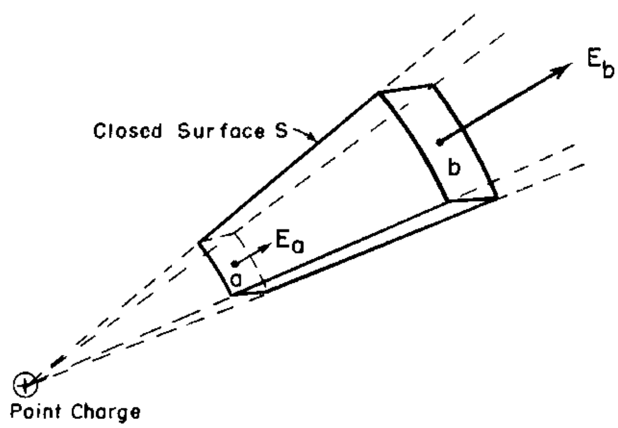
   The electric field is perpendicular and outward to each patch of the surface, so $q = 0$. The electric flux is easy to calculate:

   $$\sum \vec{E} \cdot \hat{n}\, \Delta A = \left(\frac{1}{4pe_0}\frac{Q}{r^2}\right)\left(4pr^2\right) = \frac{Q}{e_0},$$

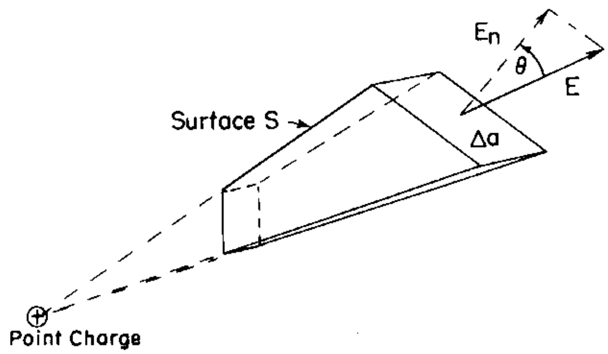
   so $1/e_0$ is the proportionality constant that relates electric flux and charge inside.

2. Show that the shape of the surface doesn't matter for a single charge

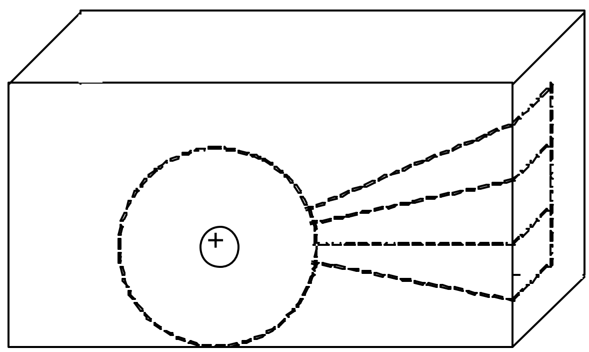
   The flux through a "truncated cone" (charge is at point of the cone) is zero. Easy to see if the ends are parts of spheres – E decreases as $1/r^2$ and ΔA increases as r^2, flux is in one end and out of the other.



   This is also true if the ends are cut off at other angles because only the perpendicular projection of the surface area matters.

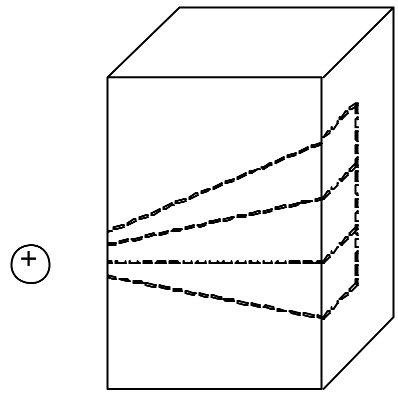


Any surface containing a charge can be constructed from a sphere centered on the charge and truncated cones (as shown below). The addition of the cones does not change the electric flux, so the relation from step 1 still holds.



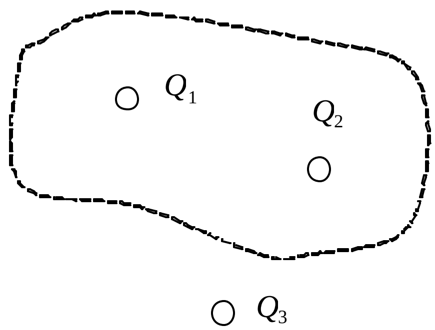
3. The electric flux is zero for a single charge outside of a closed surface.

   Any such closed surface can be constructed from truncated cones centered on the charge, so this is true.



4. Gauss's Law holds for multiple charges – use the Superposition Principle

   Ex: three charges, two inside and one outside of a surface



The law holds for each individual charge, so:

$$\sum_{\text{closed}} \vec{E}_1 \cdot \hat{n}\, \Delta A = \frac{Q_1}{e_0}$$

$$\sum_{\text{closed}} \vec{E}_2 \cdot \hat{n}\, \Delta A = \frac{Q_2}{e_0}$$

$$\sum_{\text{closed}} \vec{E}_3 \cdot \hat{n}\, \Delta A = 0 \text{ (since } Q_3 \text{ is outside)}$$

We can add these equations up to get (the charges could be positive or negative):

$$\sum_{\text{closed}} \left(\vec{E}_1 + \vec{E}_2 + \vec{E}_3\right) \hat{n}\, \Delta A = \frac{Q_1 + Q_2}{e_0}$$

In general, the rule (Gauss's Law) for the net electric flux (from the net electric field) is:

$$\boxed{\sum \vec{E} \cdot \hat{n}\, \Delta A = \frac{\sum q_{\text{inside}}}{e_0}}$$