<table>
<thead>
<tr>
<th>Day</th>
<th>Schedule</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wed.</td>
<td>11.7 - .9 (.11) Motion With &amp; Without Torque Lab L11 Rotation Lab Evals</td>
</tr>
<tr>
<td>Fri.</td>
<td>11.10 Quantization, Quiz 11, Lect Evals</td>
</tr>
<tr>
<td>Mon.</td>
<td>Review for Final (1-11)</td>
</tr>
<tr>
<td>Sat.</td>
<td>9 a.m.</td>
</tr>
</tbody>
</table>
Using Angular Momentum
The measure of motion about a point

Magnitude
$$L = \|p_\perp\| r = |p|r_\perp = |p||r|\sin(\theta)$$

Direction
Orient Right hand so fingers curl from the axis and with motion, then thump points in direction of angular momentum.
Angular Momentum Principle

\[
\frac{d}{dt} \vec{L}_{(about)A} = \sum_{\text{net}} \vec{\tau}_{(about)A}
\]

Torque
Interaction that changes motion about a point

where,
\[
\vec{\tau}_{(about)A} \equiv \vec{r}_{(from)A} \times \vec{F}
\]

Magnitude
(yet another cross product)

\[
|\vec{\tau}_A| = |r_A||F_\perp| = |r_{A\perp}||F|
\]

\[
\vec{\tau}_A = |r_A|(|F| \sin \theta) = (|r_A| \sin \theta)F = |r_A||F|\sin \theta
\]

Direction
(yet another cross product)
Multi-Particle Angular Momentum Principle

With Multi-Particle Angular Momentum

\[
\frac{d\vec{L}_{c-s}}{dt} = \vec{\tau}_{1\leftarrow ext} + \vec{\tau}_{2\leftarrow ext} + \vec{\tau}_{3\leftarrow ext} + \ldots
\]

where \( \vec{\tau}_{i\leftarrow ext} = \vec{r}_{i-s} \times \vec{F}_{i\leftarrow ext} \), etc.

\[
\vec{L}_{c-s} = \vec{L}_{cm-s} + \sum_i \vec{L}_{i-cm}
\]

\[
\vec{L}_{c-s} = \vec{L}_{trans-s} + \vec{L}_{rot-cm}
\]

for rigid objects

\[
\vec{L}_{rot-cm} = \vec{I}_{cm} \vec{\omega}_{cm}
\]
Example: A uniform solid disk with radius 9 cm has mass 0.9 kg (moment of inertia $I = \frac{1}{2}MR^2$). A constant force 12 N is applied as shown. At the instant shown, the angular velocity of the disk is 45 radians/s in the $-z$ direction. The length of the string $d$ is 18 cm.

At this instant, what are the magnitude and direction of the angular momentum about the center of the disk?

What are the magnitude and direction of the torque on the disk, about the center of mass of the disk?

The string is pulled for 0.2 s. What are the magnitude and direction of the angular impulse applied to the disk during this time?

After the torque has been applied for 0.2 s, what are the magnitude and direction of the angular momentum about the center of the disk?

At this later time, what are the magnitude and direction of the angular velocity of the disk?
Zero-Torque Systems

**Spinning Skater**

**Initial**

\[ \vec{L}_{\text{rot}, f} - \vec{L}_{\text{rot}, i} = \bar{\tau}_{\text{ave}} \Delta t \]

\[ \vec{L}_{\text{rot}, f} - \vec{L}_{\text{rot}, i} \approx 0 \]

\[ \vec{L}_{\text{rot}, i} \approx \vec{L}_{\text{rot}, f} \]

**Final**

**What about the energy?**

\[ \Delta E \approx 0 \]

\[ \Delta K_{\text{rot}} + \Delta E_{\text{int}, \text{other}} \approx 0 \]

\[ \Delta K_{\text{rot}} \approx -\Delta E_{\text{int}, \text{other}} \]

\[ \frac{L_f^2}{2I_f} - \frac{L_i^2}{2I_i} \approx -\Delta E_{\text{int}, \text{other}} \]

Increased kinetic energy fueled by change in internal energy (Wheaties)

Increased rotational kinetic energy fueled by change in internal energy (gravitational potential)

Also consider: diver, Sit-spin & flip spinning wheel

---

**Solar system formation**

http://lifeng.lamost.org/courses/astrotoday/CHAISSON/AT315/HTML/AT31502.HTM
Child runs and jumps on playground merry-go-round. For the system of the child + disk (excluding the axle and the Earth), which statement is true from just before to just after impact?

K is total (macroscopic) kinetic energy
\( \vec{P} \) is total (linear) momentum
\( \vec{L} \) is total angular momentum (about the axle)

What is the initial angular momentum of the child + disk about the axle?

<table>
<thead>
<tr>
<th></th>
<th>a) (&lt; 0, 0, 0 &gt;)</th>
<th>b) (&lt; 0, \text{-Rmv}, 0 &gt;)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>K, ( \vec{P} ), and ( \vec{L} ) do not change</td>
<td>( \vec{P} ), and ( \vec{L} ) do not change</td>
</tr>
<tr>
<td>b.</td>
<td></td>
<td>( \vec{L} ) does not change</td>
</tr>
<tr>
<td>c.</td>
<td></td>
<td>K and ( \vec{P} ) do not change</td>
</tr>
<tr>
<td>d.</td>
<td></td>
<td>K and ( \vec{L} ) do not change</td>
</tr>
<tr>
<td>e.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\( R \) = radius of the disk
\( m \) = mass of the child
\( M \) = mass of the disk
\( v \) = velocity of the child
\( z \) - axis points out of page

Completely Inelastic Collision & Angular Motion
The disk has moment of inertia $I$, and after the collision it is rotating with angular speed $\omega$. The rotational angular momentum of the disk alone (not counting the child) is

<table>
<thead>
<tr>
<th>Option</th>
<th>Vector Components</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>$&lt; 0, 0, 0 &gt;$</td>
</tr>
<tr>
<td>b.</td>
<td>$&lt; 0, -I\omega, 0 &gt;$</td>
</tr>
<tr>
<td>c.</td>
<td>$&lt; 0, I\omega, 0 &gt;$</td>
</tr>
<tr>
<td>d.</td>
<td>$&lt; 0, 0, -I\omega &gt;$</td>
</tr>
<tr>
<td>e.</td>
<td>$&lt; 0, 0, I\omega &gt;$</td>
</tr>
</tbody>
</table>

After the collision, what is the speed (in m/s) of the child?

<table>
<thead>
<tr>
<th>Option</th>
<th>Speed</th>
</tr>
</thead>
<tbody>
<tr>
<td>a.</td>
<td>$\omega R$</td>
</tr>
<tr>
<td>b.</td>
<td>$\omega$</td>
</tr>
<tr>
<td>c.</td>
<td>$\omega R^2$</td>
</tr>
<tr>
<td>d.</td>
<td>$\omega/R$</td>
</tr>
<tr>
<td>e.</td>
<td>$\omega^2 R$</td>
</tr>
</tbody>
</table>

After the collision, what is translational angular momentum of the child about the axle?

<table>
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<tr>
<td>a.</td>
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</tr>
<tr>
<td>b.</td>
<td>$&lt; 0, -Rm\omega, 0 &gt;$</td>
</tr>
<tr>
<td>c.</td>
<td>$&lt; 0, Rm\omega, 0 &gt;$</td>
</tr>
<tr>
<td>d.</td>
<td>$&lt; 0, -Rm(\omega R), 0 &gt;$</td>
</tr>
<tr>
<td>e.</td>
<td>$&lt; 0, Rm(\omega R), 0 &gt;$</td>
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</tbody>
</table>
**Completely Inelastic Collision & Angular Motion**

What’s the final angular speed of the merry-go-round after the kid jumps on?

**System:** child + merry-go-round

**Active environment:** none that change angular momentum

**Approximations:** negligible frictional torque at axel

**Axis:** Axle

\[ \vec{L}_{A.f} = \vec{L}_{A,i} + \vec{r}_A \Delta t \]

\[ \vec{L}_{m.g,r-A.f} + \vec{L}_{\text{child-A.f}} = \vec{L}_{\text{child-A.i}} \]

\[ \langle 0, -I_{\text{disk.cm}} \omega, 0 \rangle + \langle 0, -(mR \omega)R, 0 \rangle = \langle 0, -mvR, 0 \rangle \]

\[ I_{\text{disk.cm}} = \frac{1}{2} MR^2 \]

\[ \langle 0, -\frac{1}{2} MR^2 \omega, 0 \rangle + \langle 0, -mR^2 \omega, 0 \rangle = \langle 0, -mvR, 0 \rangle \]

\[ \langle 0, -(m + \frac{1}{2} M)R^2 \omega, 0 \rangle = \langle 0, -mvR, 0 \rangle \]

\[ -(m + \frac{1}{2} M)R^2 \omega = -mvR \]

\[ \omega = \frac{mv}{(m + \frac{1}{2} M)R} = \frac{1}{(1 + \frac{1}{2} \frac{M}{m})} \frac{v}{R} \]

Reasonable?
Two-Step Example: A blob of clay (mass $m$) is dropped a distance $h$ to land on and stick to a wheel (mass $M$, radius $R$) horizontally $\frac{1}{2}R$ off axis. What’s the wheel’s angular speed just after the collision?

**Step 1:** Ball falls to wheel; use energy

$\Delta E_{E&B} = 0$

$\Delta K_B + \Delta E_{B,\text{int}} + \Delta E_E + \Delta U_{E,B} = 0$

$\left(\frac{1}{2}mv_{B,1}^2 - \frac{1}{2}mv_{B,0}^2\right) - mgh = 0 \quad \Rightarrow \quad v_{B,1} = \sqrt{2gh}$

**Step 2:** Ball & wheel collide; use angular momentum

Axis: C  
System: Ball + Wheel  
Active environment: no torques

Approximations: time interval small enough axel friction and Earth’s gravitation have negligible effect

By right-hand-rule, all in the $+z$ direction

$\vec{L}_{W&B-C,f} = \vec{L}_{B-C,i}$

$I_{W&B,C}\omega_2 = mv_B r_\perp$

$I_{W&B,C}\omega_2 = m\left(\sqrt{2gh}\right)\left(\frac{1}{2}R\right)$

$\left(M + m\right)R\omega_2 = m\left(\sqrt{2gh}\right)\left(\frac{1}{2}R\right)$

$\left(M + m\right)R\omega_2 = m\frac{1}{2}\sqrt{2gh} \quad \Rightarrow \quad \omega_2 = \frac{m\sqrt{2gh}}{2(M + m)R} = \frac{\sqrt{2gh}}{2(\frac{M}{m} + 1)R}$

Reasonable?

System: Ball + Earth
Active environment: none
Approximations: negligible drag

Moment 0

Moment 1

Moment 2
Three Fundamental Principles

Angular Momentum: \( \frac{d}{dt} \vec{L}_{(about) A} = \sum_{\text{net}} \vec{\tau}_{(about) A} \)

(linear) Momentum: \( \frac{d}{dt} \vec{p} = \sum_{\text{net}} \vec{F} \)

Energy: \( \Delta E = \sum_{\text{net}} W \) \( \Delta K_{cm} = \sum_{\text{net}} "W_{cm}" \)

Example all three together! Say we have a uniform 0.4 kg puck with an 8 cm radius. A 24 cm string is initially wrapped around its circumference. If it’s on a frictionless surface and a 10 N force is applied to the end of the string until it’s unwound…

\[ R = 0.08 \text{m} \]
\[ l = 0.24 \text{ m} \]
\[ I_{puck} = \frac{1}{2} mR^2 \quad d = ? \quad \Delta t = ? \]

a. What will be its rate of rotation when the string is fully unwound?

Energy Principle

\[ \Delta E_{total} = W \]
\[ \Delta K_{trans} + \Delta E_{int} = \vec{F} \cdot \Delta \vec{r}_F \]
\[ \vec{F} \cdot \Delta \vec{r}_{cm} + \Delta K_{rot} = F(d + l) \]

\[ Fd + \left( \frac{1}{2} I \omega_f^2 - \frac{1}{2} I \omega_i^2 \right) = Fd + Fl \]

\[ \frac{1}{2} I \omega_f^2 = Fl \]
Example all three together! Say we have a uniform 0.4 kg puck with an 8 cm radius. A 24 cm string is initially wrapped around its circumference. If it’s on a frictionless surface and a 10 N force is applied to the end of the string until it’s unwound…

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### a. What will be its rate of rotation when the string is fully unwound?

\[ I_{puck} = \frac{1}{2} mR^2 \]
\[ \omega_f = \frac{2}{R} \sqrt{\frac{Fl}{m}} \]

\[ d = ? \quad \Delta t = ? \]

### b. How long was the force applied?

Angular Momentum Principle
(\text{axis through final location of cm})

\[ \Delta \vec{L} = \int \vec{\tau} dt \]
\[ \vec{L}_f - \vec{L}_i = \vec{\tau} \Delta t \]

\[ (\vec{L}_{f,\text{trans}} + \vec{L}_{f,\text{rot}}) - (\vec{L}_{i,\text{trans}} + \vec{L}_{i,\text{rot}}) = \vec{\tau} \Delta t \]

\[ I \vec{\omega}_f = \left( \vec{r}_{F-a} \times \vec{F} \right) \Delta t \]

Torque and final angular velocity in \(-z\) direction

\[ I \omega_f = (RF) \Delta t \quad \Rightarrow \quad \Delta t = \frac{I \omega_f}{RF} = \frac{1}{2} \frac{R^2 m \omega_f}{RF} = \frac{Rm \omega_f}{2F} = \frac{Rm \frac{2}{R} \sqrt{\frac{Fl}{m}}}{2F} = \sqrt{\frac{ml}{F}} \]
Example all three together! Say we have a uniform 0.4 kg puck with an 8 cm radius. A 24 cm string is initially wrapped around its circumference. If it’s on a frictionless surface and a 10 N force is applied to the end of the string until it’s unwound…

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\[ I_{puck} = \frac{1}{2} mR^2 \]

\[ d = ? \quad \Delta t = ? \]

a. What will be its rate of rotation when the string is fully unwound?

\[ \omega_f = \frac{2}{R} \sqrt{\frac{Fl}{m}} \]

b. How long was the force applied?

\[ \Delta t = \sqrt{\frac{ml}{F}} \]

You try:

c. How quickly is the puck finally sliding, \( v_{cm_f} \)?

d. How far has the puck moved, \( d \)?
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<thead>
<tr>
<th>Day</th>
<th>Topic</th>
<th>Evaluation</th>
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<tr>
<td>Mon</td>
<td>11.4-.6, (.13) Angular Momentum Principle &amp; Torque</td>
<td>RE 11.c</td>
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<tr>
<td>Tues.</td>
<td>11.7 - .9, (.11) Motion With &amp; Without Torque</td>
<td>EP11</td>
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<tr>
<td>Wed.</td>
<td>Lab</td>
<td>RE 11.d</td>
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<td>Fri.</td>
<td>L11 Rotation Course Evals</td>
<td>RE 11.e</td>
</tr>
<tr>
<td>Fri.</td>
<td>11.10 Quantization, Quiz 11</td>
<td></td>
</tr>
<tr>
<td>Mon.</td>
<td>Review for Final (1-11)</td>
<td>HW11: Pr’s 39, 57, 64, 74, 78</td>
</tr>
<tr>
<td>Sat.</td>
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**Final Exam (Ch. 1-11)**